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RANDOM WALKS ON ABELIAN-BY-CYCLIC GROUPS

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Abstract. We describe the large time asymptotic behaviors of the probabilities $p_{2t}(e,e)$ of return to the origin associated to finite symmetric generating sets of abelian-by-cyclic groups. We characterize the different asymptotic behaviors by simple algebraic properties of the groups.

1. Introduction

Let G be a finitely generated group and let

$$p : G \times G \to [0,1]$$

be a kernel which is symmetric, left-invariant, and has support within bounded distance of the diagonal with respect to a word metric on G. Assume also that p is Markovian, that is,

$$\forall x,\ \Sigma_y p(x,y) = 1.$$

The probability of going from x to y in time t for the random walk on G associated to p is defined as

$$p_t(x,y) = \Sigma p(z_0,z_1)p(z_1,z_2)...p(z_{t-1},z_t)$$

where the sum is over $t+1$-uples with $z_0 = x, z_t = y$. In case $x = y = e$, we simply write p_t for $p_t(e,e)$ and we will assume that t is even to avoid usual parity problems. The kernel p is called irreducible if for any two points x, y, there exists a finite set of points $z_1, ..., z_n$ such that $z_1 = x, z_n = y$ and $p(z_i, z_{i+1}) > 0$.

If f, g are two monotone right-continuous non-negative functions defined on positive numbers, we use the notation $f \prec g$ if there exist constants $a, b > 0$ such that, for x big enough, $\bar{f}(x) \leq a\bar{g}(bx)$. If the symmetric relation also holds, we write $f \sim g$. When a function is defined on the integers, we extend it by linear interpolation and use the same name for its extension.

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If p and p' are irreducible kernels on quasi-isometric groups G and G', then $p_{2t} \sim p'_{2t}$. In particular, in order to compute the equivalence class of p_{2t}, we can choose a finite symmetric generating set S of G and its associated kernel

$$p(x,y) = \frac{1}{|S|}$$

if $y = xs, s \in S$, and $p(x,y) = 0$ if not. We call the equivalence class of p_{2t} the on-diagonal heat decay type of G or simply the heat decay of G. If G' is a quotient of G or a finitely generated subgroup of G, then

$$p_{2t} \prec p'_{2t}. \tag{1.1}$$

See [12].

The group G is non-amenable if and only if $p_t \prec \exp(-t)$. See [8]. The group G is virtually nilpotent of growth degree d if and only if $p_{2t} \sim t^{-d/2}$. See [15]. If G has exponential growth, then $p_t \prec \exp(-t^{1/3})$. If G is polycyclic, then $p_{2t} \succ \exp(-t^{1/3})$. See [2] or [10] and [15]. If $G = \mathbb{Z} \wr \mathbb{Z}$ is the wreath product (associated to the left-regular representation) of $\mathbb{Z}$ with itself, then

$$p_{2t} \sim \exp(-t^{1/3}(\log t)^{2/3}) \tag{1.2}$$

and if $G = F \wr \mathbb{Z}$ where F is a non-trivial finite group, then

$$p_{2t} \sim \exp(-t^{1/3}). \tag{1.3}$$

See [9] and [14].

The aim of this paper is to prove the following.

Theorem 1.1. *Let G be a finitely generated abelian-by-cyclic group with exponential growth. If there exists a subgroup of G which is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$, then $p_{2t} \sim \exp(-t^{1/3}(\log t)^{2/3})$. If not, then $p_{2t} \sim \exp(-t^{1/3})$.*

The Theorem applies to the affine transformations groups G_a of the real line generated by $x \mapsto x+1$ and $x \mapsto ax$ where a is different from $0; 1; -1$. Compare with [2, Section 7.4]. In those cases, $G_a = \mathbb{Z} \wr \mathbb{Z}$ if and only if a is transcendental.

Corollary 1.2. *Let G be a finitely presented abelian-by-cyclic group with exponential growth. Then $p_{2t} \sim \exp(-t^{1/3})$.*

To deduce this Corollary from Theorem 1.1, it is enough to show that a group G as in the Corollary has no subgroup isomorphic to $\mathbb{Z} \wr \mathbb{Z}$. This follows from [1, Theorem C (b)], which implies that the abelian kernel A of G satisfies $dim_{\mathbb{Q}} A \otimes \mathbb{Q} < \infty$, combined with the second sentence in the proof of Lemma 2.2 below. Finitely presented abelian-by-cyclic groups are classified, up to quasi-isometries in [5]. An interesting open problem is to extend this classification to the class of finitely generated abelian-by-cyclic groups. The above Theorem implies for example that $\mathbb{Z} \wr \mathbb{Z}$ and $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ are not quasi-isometric.

Some illustrative examples and geometric descriptions of abelian-by-cyclic groups are given in [2, Sections 5.1, 5.2], [3, Chapter VII 3, 4, 5, 8], [4], [11, Theorem 3.5] and [13, Theorem 4.7.13].

In order to elucidate the content of Theorem 1.1 and to put the result in perspective, let us comment further on four key cases which are easy to describe and characterize, using the language of modules. If the group G is an extension of the infinite cyclic group $\mathbb{Z}$ by an abelian group A, the conjugation action of G on the abelian kernel A induces a module structure on A over the group ring $\mathbb{Z}[t,t^{-1}]$ of $\mathbb{Z}$. In other words, the action of the unit t on A defines the semi-direct product $0 \to A \to G \to \mathbb{Z} \to 0$. Now we describe four examples of groups to illustrate each of the four key cases in the proof of Theorem 1.1.

1) The so-called lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$. On this group, the heat decay satisfies $p_{2t} \sim \exp(-t^{1/3})$. This was proved in [14]. The abelian kernel A of this group is the direct sum of countably many copies of $\mathbb{Z}/2\mathbb{Z}$. Hence, according to [1, Theorem C (a)], the group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$ is not finitely presented because its torsion subgroup is infinite.
2) The Baumslag-Solitar group with two generators and the relation $aba^{-1} = b^2$. Its heat decay satisfies $p_{2t} \sim \exp(-t^{1/3})$. This is explained in [11, Section 7]. See also [16, Theorem 15.14 p. 168]. It is isomorphic to the group of affine transformations of the real line generated by the translation $x \mapsto x+1$ and the homothety $x \mapsto 2x$. The abelian kernel A is isomorphic to the group $\mathbb{Z}[1/2]$.
3) The group of affine transformations of the real line generated by the translation $x \mapsto x+1$ and the homothety $x \mapsto (2/3)x$. Its heat decay satisfies $p_{2t} \sim \exp(-t^{1/3})$. This is proved in [2, Theorem 7.8]. The abelian kernel A is isomorphic to the group $\mathbb{Z}[1/6]$. In contrast with the previous example, this group is not finitely presented. This follows from [1, Theorem C (c)] because in this case, $A \otimes \mathbb{Q} = \mathbb{Q}$ and the characteristic polynomial of the endomorphism $t \otimes 1$ is either $2/3 - x$ or $3/2 - x$, depending on the choice of the generator of $\mathbb{Z}$. In any case, it is not integral.
4) The wreath product $\mathbb{Z} \wr \mathbb{Z}$, which is isomorphic to the group of affine transformations of the real line generated by the translation $x \mapsto x+1$ and the homothety $x \mapsto ax$ where a is any transcendental real. It is probably the easiest example of a group with "exotic" heat decay, that is, not equivalent to $t^{-d/2}$, $\exp(-t^{1/3})$, or $\exp(-t)$. Its heat decay is $p_{2t} \sim \exp(-t^{1/3}(\log t)^{2/3})$. This is mentioned in [11, Theorem 7.3] and proved in [9, Theorem 3.11]. The abelian kernel A of this group is the direct sum of countably many copies of $\mathbb{Z}$. Hence, as already mentioned, it follows from [1, Theorem C (b)] that this group is not finitely presented.

With those examples in mind, let us explain the main steps in the proof of Theorem 1.1.

The first step is to split the problem into two cases. The first case is when the abelian group A is a torsion group, as in example 1) above. The second case is when the group A and hence the group G has no torsion element, as in examples 2), 3), and 4) above. This is easily achieved in Section 3 by applying a result of Ph. Hall.

The second step, achieved in Section 4, is to reduce the problem to the case when the module A is cyclic, as in all the above examples. The classical decomposition theorem for finitely generated modules assumes that the ring is a principal ideal domain. The ring $\mathbb{Z}[t,t^{-1}]$ is not principal. Hence we replace the module A with the module $A \otimes \mathbb{Q}$ in order to work with the principal ideal ring $\mathbb{Q}[t,t^{-1}]$. We prove the following lemma.

Lemma 1.3. *Let $0 \to A \to G \to \mathbb{Z} \to 0$ be an exact sequence with G finitely generated, torsion free and A abelian. There is a finite family of groups H_i with the following properties*:

1) *Each H_i is a quotient of $\mathbb{Z} \wr \mathbb{Z}$. More precisely, there exists a cyclic module B_i over $\mathbb{Z}[t, t^{-1}]$ and H_i is the semi-direct product*

$$0 \to B_i \to H_i \to \mathbb{Z} \to 0$$

defined by the action of the unit t on B_i.

2) *The group G embeds in the direct product $\oplus H_i$.*
3) *If G does not contains $\mathbb{Z} \wr \mathbb{Z}$, then no H_i is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$.*

This statement is crucial because, together with the splitting result of Section 3, it enables us to reduce the computation of the heat decay type of an arbitrary finitely generated abelian-by-cyclic group to those of the quotients of $\mathbb{Z} \wr \mathbb{Z}$. As a consequence—and it is the goal of this paper to explain how—the analytic and probabilistic results from [12], [9] and [2] imply Theorem 1.1. Indeed, after those two reductions, we are left with a cyclic module A, let us say with generator a, over $\mathbb{Z}[t, t^{-1}]$. If a has finite order, we are essentially in the case of example 1) above. If G has no torsion, we consider the natural map from $\mathbb{Z}[t, t^{-1}]$ to A sending a Laurent polynomial $p(t)$ to $p(t)a$. If its kernel is trivial, then G is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$, that is, we are in the case of example 4) above. If its kernel is non-trivial, as in examples 2) and 3) above, we show in Lemma 7.1 below that G embeds in a finitely generated group H with special properties. The relevant properties are that it contains a finitely generated abelian group B and two commuting elements y and z, such that $B \cup \{y, z\}$ generates H and such that the conjugation by any element in the semi-group generated by y and z preserves B. See Lemma 7.1 below for the exact properties we require for the group H. It is shown in [2, Corollary 7.6] that a group H with those properties admits a lower bound $\exp(-t^{1/3})$ for its heat decay. Let us explain how the construction of the group H specializes in the case of example 3) above. We choose H to be equal to the subgroup of affine transformations of the real line generated by $b : x \mapsto x + 1$, $y : x \mapsto 2x$, $z : x \mapsto 3x$. It contains G because $yz^{-1} : x \mapsto (2/3)x$. We choose B to be equal to the subgroup generated by b. The group B is isomorphic to the group $\mathbb{Z}$ of translations with integral displacement. The conjugation by y doubles the displacement of the translation, the conjugation by z multiplies the displacement of the translation by a factor of three, hence the semi-group generated by y and z preserves B. Notice that there is no finitely generated subgroup B in G invariant under conjugation by $a : x \mapsto (2/3)x$ (respectively a^{-1}) such that $B \cup \{a\}$ (respectively $B \cup \{a^{-1}\}$) generates G.

2. The wreath product $\mathbb{Z} \wr \mathbb{Z}$

Let $C_0(\mathbb{Z}, \mathbb{Z})$ be the additive group of finitely supported functions on $\mathbb{Z}$ with values in $\mathbb{Z}$. The group $\mathbb{Z}$ acts on $C_0(\mathbb{Z}, \mathbb{Z})$ as

$$\lambda(z)f(x) = f(x - z).$$

The group $\mathbb{Z} \wr \mathbb{Z}$ is defined as the semi-direct product of $\mathbb{Z}$ with $C_0(\mathbb{Z}, \mathbb{Z})$ associated to this action. Let $\mathbb{Z}[t, t^{-1}]$ be the integral group ring of $\mathbb{Z}$. The formula $t(f) = \lambda(1)f$ induces an action of $\mathbb{Z}[t, t^{-1}]$ on $C_0(\mathbb{Z}, \mathbb{Z})$. We denote by δ_x the element of $C_0(\mathbb{Z}, \mathbb{Z})$ which is the characteristic function of $x \in \mathbb{Z}$.

Lemma 2.1. 1) *The group $\mathbb{Z} \wr \mathbb{Z}$ is generated by δ_0 and any element z in the preimage of $1 \in \mathbb{Z}$.*

2) *Let $f \in C_0(\mathbb{Z}, \mathbb{Z})$ be non-zero and let $z \in \mathbb{Z} \wr \mathbb{Z}$ not in $C_0(\mathbb{Z}, \mathbb{Z})$. Then the subgroup generated by f and z is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$.*

Proof. Recall that a module over a ring R is called free if it is isomorphic to a direct sum of copies of R. The first statement follows from the fact that the group $C_0(\mathbb{Z}, \mathbb{Z})$ is a free cyclic module over $\mathbb{Z}[t, t^{-1}]$ generated by the characteristic function δ_0. For the second statement, let $k \in \mathbb{Z}$ be the projection of z. By hypothesis, $k \neq 0$ hence the subring $\mathbb{Z}[t^k, t^{-k}]$ is isomorphic to $\mathbb{Z}[t, t^{-1}]$. The cyclic $\mathbb{Z}[t^k, t^{-k}]$-submodule generated by any non-zero element $f \in C_0(\mathbb{Z}, \mathbb{Z})$ is also free. $\square$

Lemma 2.2. *Let G be a group which fits into an exact sequence $0 \to A \to G \to \mathbb{Z} \to 0$ with A abelian. Any quotient Q of G is again abelian-by-cyclic and if Q contains $\mathbb{Z} \wr \mathbb{Z}$, then G also contains $\mathbb{Z} \wr \mathbb{Z}$.*

Proof. Let $c \in \mathbb{Z} \wr \mathbb{Z} \subset Q$ be a non-trivial commutator (for example $c = \delta_1 - \delta_0$) and let $z \in \mathbb{Z} \wr \mathbb{Z}$ in the preimage of $1 \in \mathbb{Z}$. According to the second statement in Lemma 2.1, c and z generate a group isomorphic to $\mathbb{Z} \wr \mathbb{Z}$. Let $\pi : G \to Q = \pi(G)$ be the projection. We consider the following two exact sequences with commutative squares:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi(A) & \longrightarrow & \pi(G) & \longrightarrow & \pi(G)/\pi(A) & \longrightarrow & 0. \end{array}$$

As the right vertical arrow is a surjection from $\mathbb{Z}$ onto $\pi(G)/\pi(A)$, this quotient is cyclic and as c is a commutator, there exists $a \in A$ such that $\pi(a) = c$. Let $y \in G$ such that $\pi(y) = z$. Let M be the cyclic module generated by a for the action of $\mathbb{Z}[t, t^{-1}]$ defined by $t(a) = yay^{-1}$ and let M' be the cyclic module generated by c for the action of $\mathbb{Z}[t, t^{-1}]$ defined by $t(c) = zcz^{-1}$. The restriction of π to M is a morphism of modules with image M'. As M' is free and cyclic, so is M. This implies that the subgroup generated by a and y is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$. $\square$

3. Splitting mixed groups

By definition, an abelian-by-cyclic group G fits into a short exact sequence

$$0 \to A \to G \to C \to 0$$

with A abelian and C cyclic. The hypothesis on the growth implies that C is infinite (otherwise G would be a finite extension of a free abelian group of finite rank).

We now consider an exact sequence

$$0 \to A \to G \to \mathbb{Z} \to 0.$$

Let T be the torsion subgroup of A. According to [6, Lemma 8], there is an integer n such that $\forall x \in T,\ nx = 0$. Let

$$B = \{x \in A : \exists\ y \in A : ny = x\}.$$

This is a characteristic subgroup of A, hence a normal subgroup of G. The obvious map

$$G \to G/T \times G/B$$

is an embedding because $T \cap B$ is trivial. Indeed, if $x = ny$ is of finite order, then y is of finite order. Hence, by definition of n, we obtain $ny = 0$. According to Lemma 2.2, each factor group is again finitely generated abelian-by-cyclic. What we have gained is that the abelian kernel A/B of the projection onto $\mathbb{Z}$ in the group G/B is a torsion group while the group G/T is torsion-free.

4. Cyclic modules. Proof of Lemma 1.3

We prove Lemma 1.3. The conjugation induces an action of $\mathbb{Z}[t, t^{-1}]$ on A and as G is finitely generated, the module A is finitely generated over this ring. Let $A \otimes \mathbb{Q}$ be the tensor product of A with the field of rational numbers. The unit $t \in \mathbb{Z}[t, t^{-1}]$ acts on $A \otimes \mathbb{Q}$. Let

$$0 \to A \otimes \mathbb{Q} \to L \to \mathbb{Z} \to 0$$

be the corresponding semi-direct product. The action of $\mathbb{Z}[t, t^{-1}]$ on A induces an action of $\mathbb{Q}[t, t^{-1}]$ on $A \otimes \mathbb{Q}$. The module $A \otimes \mathbb{Q}$ is finitely generated over the principal ideal ring $\mathbb{Q}[t, t^{-1}]$. Hence, according to [7, Section 15 Theorem 16], it decomposes as a direct sum of finitely many cyclic modules M_i. For each M_i we choose a generator and represent it as $x_i \otimes 1/m_i$ where $x_i \in A$ and $m_i \in \mathbb{Z}$. We choose an element $z \in G$ in the preimage of $1 \in \mathbb{Z}$ and a finite number of elements a_j in A so that, together with z, they generate the group G. As $A \otimes \mathbb{Q} = \bigoplus M_i$, there is a finite number of Laurent polynomials $p_{ji}(t) \in \mathbb{Q}[t, t^{-1}]$ such that

$$a_j \otimes 1 = \sum_i p_{ji}(t) x_i \otimes 1/m_i.$$

Let $d \in \mathbb{Z}$ be the product of all the m_i and all the denominators of the coefficients of the polynomials $p_{ji}(t)$.

Let B_i be the cyclic $\mathbb{Z}[t, t^{-1}]$-submodule of $A \otimes \mathbb{Q}$ generated by $x_i \otimes 1/d$. Notice that

$$B_i \cap B_j \subset M_i \cap M_j = 0$$

if $i \neq j$. Each group B_i is a normal subgroup of L. Let H_i be the subgroup of L generated by B_i and z. It is a quotient of the wreath product $\mathbb{Z} \wr \mathbb{Z}$. This proves the first statement in the Lemma.

Let $B = \bigoplus B_i$ and let H be the subgroup of L generated by B and z. It is finitely generated. As A is torsion free, the map sending a to $a \otimes 1$ is an injection of A into $A \otimes \mathbb{Q}$. We identify A with its image in $A \otimes \mathbb{Q}$. The integer d has been chosen so that each a_j lies in B. We deduce that $A \subset B$. Hence $G \subset H$. The diagonal embedding of $\mathbb{Z}$ into $\bigoplus_i \mathbb{Z}$ induces an embedding of H into $\bigoplus H_i$. This proves the second statement in the Lemma.

Suppose some H_i is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$. Recall that the generator $x_i \otimes 1/d$ of B_i has been chosen so that $x_i \in A$. Hence $d(x_i \otimes 1/d)$ and z both belong to the intersection $H_i \cap G$. According to the second statement in Lemma 2.1 they generate a group isomorphic to $\mathbb{Z} \wr \mathbb{Z}$. This proves the third statement in Lemma 1.3.

5. Lower bounds

According to inequality (1.1) from the Introduction, a lower bound for the heat decay in a finitely generated group Γ implies the same lower bound on the heat decay in any quotient G of Γ. Recall from (1.2) and (1.3) in the Introduction that we know the asymptotics of the heat decay in the groups $\mathbb{Z} \wr \mathbb{Z}$ and $F \wr \mathbb{Z}$ where F is a finite group.

Lemma 5.1. *Let $0 \to A \to G \to \mathbb{Z} \to 0$ be an exact sequence with G finitely generated and A abelian. Let $z \in G$ be in the preimage of $1 \in \mathbb{Z}$. Then:*

1) *If A is a torsion group, then G is a quotient of a wreath product $F \wr \mathbb{Z}$ where F is a finite group. The heat decay in G satisfies*
$$p_{2t} \succ \exp(-t^{1/3}).$$
2) *If G is generated by z and a single element $a \in A$, then G is a quotient of $\mathbb{Z} \wr \mathbb{Z}$. The heat decay in G satisfies*
$$p_{2t} \succ \exp(-t^{1/3}(\log t)^{2/3}).$$

Proof. In the case A is a torsion group, we choose a finite number of elements $a_i \in A$ so that, together with z, they generate G. Let m_i be the order of a_i. Let F be the direct product of the cyclic groups $\mathbb{Z}/m_i\mathbb{Z}$. The relations
$$a_i^{m_i}, [z^m a_i z^{-m}, z^n a_j z^{-n}], m, n \in \mathbb{Z},$$
form a complete set for the wreath product $F \wr \mathbb{Z}$ and these relations hold in G. In the second case, the relations
$$[z^i a z^{-i}, z^j a z^{-j}], i, j \in \mathbb{Z},$$
form a complete set for the wreath product $\mathbb{Z} \wr \mathbb{Z}$ and these relations hold in G. □

The lower bounds on the heat decays follow, as explained in the first paragraph of this section.

6. Strategy for the proof

The heat decay in a direct product $G \times G'$ is the product of the heat decays of the factors. (To prove this, choose generator sets S, S', SS' containing the identity element of $G, G', G \times G'$ respectively, and consider the associated kernels.) Notice that $\exp(-t^{1/3})$ is equivalent to its square:
$$[\exp(-t^{1/3})]^2 \sim \exp(-t^{1/3}).$$
The same is true for $\exp(-t^{1/3}(\log t)^{2/3})$.

We now explain why the heat decay in our group G satisfies
$$\exp(-t^{1/3}(\log t)^{2/3}) \prec p_{2t} \prec \exp(-t^{1/3}).$$
Recall from the Introduction that this upper bound holds for all groups with exponential growth. For the lower bound, we embed G in the product $G/T \times G/B$ as explained in Section 3. We obtain the lower bound $\exp(-t^{1/3})$ for the heat decay in the group G/B by applying the first statement in Lemma 5.1. As G/T is torsion free, we can apply Lemma 1.3 and embed G/T in a direct product $\bigoplus H_i$ with finitely many factors, each of them isomorphic to a quotient of $\mathbb{Z} \wr \mathbb{Z}$, and conclude by applying Lemma 5.1 again.

Let us explain how to conclude the proof of Theorem 1.1. In the case G does not contain $\mathbb{Z} \wr \mathbb{Z}$, Lemma 2.2 implies that the quotient G/T does not contain $\mathbb{Z} \wr \mathbb{Z}$. According to the third statement in Lemma 1.3, no H_i is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$. Applying the following Proposition to each group H_i and module B_i finishes the proof of the Theorem.

Proposition 6.1. *Let A be a cyclic module over $\mathbb{Z}[t,t^{-1}]$ and assume that the abelian group underlying A is torsion free. Let G be the semi-direct product*

$$0 \to A \to G \to \mathbb{Z} \to 0$$

defined by the action of the unit t on A. Let $a \in A$ be a generator of A and let I be the kernel of the map

$$\mathbb{Z}[t,t^{-1}] \to A$$

sending a Laurent polynomial $\sum_{i\in\mathbb{Z}} n_i t^i$, $n_i \in \mathbb{Z}$, to $\sum_{i\in\mathbb{Z}} n_i t^i a$. If $I = \{0\}$, then G is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$. If $I \neq \{0\}$, then the heat decay in G is bounded below by $\exp(-t^{1/3})$.

The aim of the next section is to prove the above Proposition.

7. Quotients of $\mathbb{Z}[t,t^{-1}]$

Let G and A be as in Proposition 6.1. As we assume that A is cyclic with generator a, the map from $\mathbb{Z}[t,t^{-1}]$ to A which sends a Laurent polynomial $\sum_{i\in\mathbb{Z}} n_i t^i$, $n_i \in \mathbb{Z}$, to $\sum_{i\in\mathbb{Z}} n_i t^i a$ is surjective. If it is also injective, then G is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$. The action of $\mathbb{Z}[t,t^{-1}]$ on A induces an action of $\mathbb{Q}[t,t^{-1}]$ on $A \otimes \mathbb{Q}$. As $\mathbb{Q}[t,t^{-1}]$ is a principal ideal ring, the kernel I of the map defined by

$$\sum_{i\in\mathbb{Z}} r_i t^i \mapsto \sum_{i\in\mathbb{Z}} r_i t^i a \otimes 1$$

($r_i \in \mathbb{Q}$) is generated by a single element $p(t)$. Up to multiplication by a unit, we may assume that $p(t)$ is a polynomial in t with integer coefficients. Moreover, if the kernel is non-trivial, we may assume that the constant coefficient of $p(t)$ is non-zero:

$$p(t) = n_d t^d + n_{d-1} t^{d-1} + ... + n_0,$$

$d \in \mathbb{N} \cup 0, n_i \in \mathbb{Z}, n_d \neq 0$, and $n_0 \neq 0$. In the case $A \otimes \mathbb{Q}$ is non-trivial, as a non-zero constant $r \in \mathbb{Q}$ acts by automorphism, $p(t)$ cannot be constant, hence $d > 0$.

Lemma 7.1 (compare with [2, Theorem 7.8])**.** *If I is non-trivial and G is torsion free, then G imbeds in a finitely generated group H which has the following properties*:

1) *There is a split exact sequence $0 \to N \to H \to \mathbb{Z}^2 \to 0$ with N abelian.*
2) *There exist $y, z \in H$ which project onto a basis of $\mathbb{Z}^2$ and a finitely generated subgroup B of N such that the smallest subgroup containing $B \cup \{y, z\}$ is the whole group H.*
3) *The conjugations by y and z (but not necessary by y^{-1} and z^{-1}) preserve H.*

Proof. The leading coefficient n_d of $p(t)$ and $n_d t$ are two units of $\mathbb{Q}[t,t^{-1}]$ which define two commuting automorphisms of $A \otimes \mathbb{Q}$. Let

$$0 \to A \otimes \mathbb{Q} \to L \to \mathbb{Z}^2 \to 0$$

be the corresponding semi-direct product and let us denote by y and z two elements of L which project onto a basis of $\mathbb{Z}^2$ and such that

$$\forall x \in A \otimes \mathbb{Q} \;\; yxy^{-1} = n_d x, \;\; zxz^{-1} = n_d t x.$$

Let $H \subset L$ be the subgroup generated by the set $\{a \otimes 1, y, z\}$. As A is torsion free by hypothesis, and as $n_d^{-1}(n_d t) = t$, the group G embeds in H. If $A \otimes \mathbb{Q}$ is trivial, the Lemma is obviously true. If not, recall that $d > 0$. Let B be the subgroup of A generated by the elements $t^i a,\ 0 \le i \le d-1$. As $p(t)$ is in the kernel I, we deduce that $(n_d t)B \subset B$. Obviously, $n_d B \subset B$. □

We can deduce Proposition 6.1 from the above Lemma and from (1.1) because it follows from [2, Corollary 7.6] that a group H as in Lemma 7.1 has a lower bound $\exp(-t^{1/3})$ for its heat decay.

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