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A LOCAL GEOMETRIC CHARACTERIZATION OF THE BOCHNER-MARTINELLI KERNEL

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Abstract. In this paper it is shown that a connected smooth local hypersurface in $\mathbb{C}^n$ for which the skew-hermitian part of the Bochner-Martinelli kernel has a weak singularity must lie on a surface having one of the following forms: $S^{2m+1} \times \mathbb{C}^{n-m-1}$ for some $1 \leq m < n$, or $C \times \mathbb{C}^{n-1}$ where C is a one-dimensional curve. This strengthens results of Boas about the Bochner-Martinelli kernel and it generalizes a result of Kerzman and Stein about the Cauchy kernel.

1. Introduction

For a smooth hypersurface M in $\mathbb{C}^n$ write ν_w for the unit normal at w. (If M is the boundary of a domain, then ν should point outward.) We make the identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ so that a point $w = (w_1, \dots, w_n) \in \mathbb{C}^n$ corresponds to $(x_1, \dots, x_{2n}) \in \mathbb{R}^{2n}$ where $w_j = x_j + ix_{j+n}$. In general, the space to which a vector belongs is to be inferred from the context. Then the Bochner-Martinelli kernel is the function

$$K(z,w) = \frac{1}{c_n} \frac{(\overline{w} - \overline{z}) \cdot \nu_w}{|w-z|^{2n}}\,,\ w \in M, z \neq w.$$

Here c_n is the area of a unit sphere in $\mathbb{C}^n$ and the product is the complex dot product: $z \cdot w = \sum z_j\, w_j$ for $z, w \in \mathbb{C}^n$. $K(z,w)$ is a generalization of the Cauchy kernel, for if M is the boundary of a domain Ω and $f \in O(\Omega) \cap C(\overline{\Omega})$, then $Kf(z) = f(z)$ where, as an operator, K is defined by integration against its kernel,

$$Kf(z) = \int_{b\Omega} K(z,w) f(w)\, d\sigma_w\,,\ z \in \Omega.$$

Here $d\sigma$ is Euclidean surface measure. In one dimension, $K(z,w)$ is holomorphic in the z variable and in fact equals the Cauchy kernel. For higher dimensions, however, $K(z,w)$ is not holomorphic.

In [6], Kerzman and Stein showed that if $K(z,w)$ is the Cauchy kernel, then its skew-hermitian part $A(z,w) = K(z,w) - \overline{K}(w,z)$ has a strictly weaker singularity at the boundary diagonal, so $A(z,w)$ is smooth. This means that the operator A, which has kernel $A(z,w)$, is compact on $L^2(b\Omega)$, so A and K can together be used to represent the Szegő projector via $S = K(I+A)^{-1}$. (See Bell's book [1] for a nice treatment of one-variable complex analysis based on this idea.) Kerzman and

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Stein showed also that $A(z,w)$ vanishes identically precisely when $b\Omega$ is a circle or a line.

In [2], Boas generalized this last result to the Bochner-Martinelli kernel in higher dimensions, thereby answering a question of Kerzman from [5]. Boas showed that $K(z,w) \equiv \overline{K}(w,z)$ in the boundary of a smooth bounded domain precisely when that boundary is a sphere. Later he extended the result in [3] to give a local characterization of hypersurfaces for which $K(z,w) \equiv \overline{K}(w,z)$. Namely, $K(z,w) \equiv \overline{K}(w,z)$ for all w,z in a hypersurface if and only if that hypersurface lies on a surface of the form $S^{2m+1} \times \mathbb{C}^{n-m-1}$ where $0 \leq m < n$. In [8], Wegner gave a differential geometric proof of the same result.

In this paper we weaken the assumption on the Bochner-Martinelli kernel but recover essentially the same geometric condition for the hypersurface. Notice that $K(z,w)$ has a singularity like $|w-z|^{1-2n}$ at the diagonal in $M \times M$. Its skew-hermitian part,

$$A(z,w) = \frac{1}{c_n} \frac{(\overline{w}-\overline{z}) \cdot \nu_w - (z-w) \cdot \overline{\nu}_z}{|w-z|^{2n}}, \tag{1}$$

has a weaker singularity, since rewriting the numerator gives

$$\begin{aligned} (\overline{w}-\overline{z}) \cdot \nu_w - &(z-w) \cdot \overline{\nu}_z \\ &= -2\,\mathrm{Re}\left[(z-w) \cdot \overline{\nu}_w\right] + (z-w) \cdot \left[\overline{\nu}_w - \overline{\nu}_z\right]. \end{aligned} \tag{2}$$

Then the second term on the right side vanishes to second order at the diagonal, and so does the first term since it is a multiple of the Euclidean inner product of $z-w$ with the normal ν_w.[1] So if $|a(z,w)| \lesssim |w-z|^j$ means there is a $k \in \mathbb{R}$ so that $|a(z,w)| \leq k\,|w-z|^j$ for $z \neq w$, then it follows generally that $|A(z,w)| \lesssim |w-z|^{2-2n}$. Our result is the following.

Theorem. *A connected smooth (local) hypersurface $M \subset \mathbb{C}^n$ satisfies $|A(z,w)| \lesssim |w-z|^{3-2n}$ at each $w \in M$ $(z \neq w)$ if and only if it lies on a surface of the form $S^{2m+1} \times \mathbb{C}^{n-m-1}$ for some $1 \leq m < n$, or of the form $C \times \mathbb{C}^{n-1}$ where C is a one-dimensional curve.*

It is easy to check that the condition on $A(z,w)$ is satisfied for the two kinds of surfaces, so one direction is trivial. The first kind of surface is the kind for which $A(z,w) \equiv 0$, and the second comes from the fact that if $n=1$, then there is always the extra cancelation of singularities, so $A(z,w)$ is bounded and it even vanishes at the diagonal. This appeared in [6] as a remark by Robert Jackson.

We remark that since derivatives of curvatures are involved in the proof of the theorem, M must be assumed at least C^3 smooth. Boas' method only requires that M be C^1. For more about the Bochner-Martinelli kernel and its applications, see the book by Kytmanov [7].

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[1] If $a=(x_1,\dots,x_{2n})$ and $b=(y_1,\dots,y_{2n})$, then $\mathrm{Re}\,[a \cdot \overline{b}] = \mathrm{Re}\,\sum_{j=1}^{n}(x_j+ix_{j+n})(y_j-iy_{j+n}) = \sum_{j=1}^{2n} x_j y_j$.

2. The main singularity

Consider first the numerator of $A(z,w)$ as given by (2). Suppose that $z : t = (t_1, \dots, t_{2n-1}) \in \mathbb{R}^{2n-1} \to z_t \in M$ parameterizes M, and write

$$z = z(s) \in M,\ \nu_z = \nu(s) \ \text{ and } \ w = z(t) \in M,\ \nu_w = \nu(t).$$

Using the expansions

$$z(s) - z(t) = \sum_{j=1}^{2n-1} \partial_j z(t)(s_j - t_j) + \frac{1}{2}\sum_{i,j=1}^{2n-1} \partial_{ij} z(t)(s_j - t_j)(s_i - t_i) + O(s-t)^3$$

and

$$\nu(s) - \nu(t) = \sum_{i=1}^{2n-1} \partial_i \nu(t)(s_i - t_i) + O(s-t)^2$$

we have

$$\begin{aligned}
&-2\operatorname{Re}\left[(z-w)\cdot\overline{\nu}_w\right] + (z-w)\cdot\left[\overline{\nu}_w - \overline{\nu}_z\right] \\
&\qquad = -\sum_{i,j=1}^{2n-1} \operatorname{Re}\left[\partial_{ij} z(t)\cdot\overline{\nu}(t)\right](s_j - t_j)(s_i - t_i) \\
&\qquad\qquad - \sum_{i,j=1}^{2n-1} \partial_j z(t)\cdot\partial_i\overline{\nu}(t)\,(s_j - t_j)(s_i - t_i) + O(s-t)^3 \\
&\qquad = -\sum_{i,j=1}^{2n-1} i\operatorname{Im}\left[\partial_j z(t)\cdot\partial_i\overline{\nu}(t)\right](s_j - t_j)(s_i - t_i) + O(s-t)^3,
\end{aligned}$$

with the last step coming from

$$\operatorname{Re}\left[\partial_{ij} z\cdot\overline{\nu} + \partial_j z\cdot\partial_i\overline{\nu}\right] = \partial_i \operatorname{Re}\left[\partial_j z\cdot\overline{\nu}\right] = 0.$$

By setting $v_j = s_j - t_j$ we then have $|A(z,w)| \lesssim |w-z|^{3-2n}$ if and only if

$$\text{(C}_1\text{)} \qquad \sum_{i,j} \operatorname{Im}\left[v_j\partial_j z(t)\cdot v_i\partial_i\overline{\nu}(t)\right] = 0 \ \text{ for all } \ (v_1, \dots, v_{2n-1}) \in \mathbb{R}^{2n-1}.$$

Next we interpret this condition independently of the choice of coordinates.

3. Notation

We use the following notation, much of which can be found in the book by Hicks [4], for instance. As before, let $w_1, \dots, w_n$ be coordinates in $\mathbb{C}^n$ with $w_j = x_j + ix_{j+n}$, so that corresponding real coordinates in $\mathbb{R}^{2n}$ are $x_1, \dots, x_{2n}$. The complex structure $J : T\mathbb{R}^{2n} \to T\mathbb{R}^{2n}$ corresponds to multiplication by $\sqrt{-1}$ in $\mathbb{C}^n$ and is given by $J(\partial_{x_j}) = \partial_{x_{j+n}}$, $J(\partial_{x_{j+n}}) = -\partial_{x_j}$. Then J preserves the Euclidean inner product $\langle\cdot,\cdot\rangle$ and $J^2 = -I$, $J^* = -J$. The complex tangent space of M is $HM = TM \cap J(TM)$. Let N be a unit normal vector on M; then the direction orthogonal to HM in TM is JN. If Y is any vector field on M and $X \in TM$, then $d_X Y$ is the derivative of Y in the X direction. So $d_X Y$ is a vector field that might not be tangent to M even if $Y \in TM$. It is not hard to check that d and J commute.

The map $L : TM \to TM$ defined by $L(X) = d_X N$ is called the Weingarten map. Its image lies in TM for if $X \in TM$, then

$$0 = X\langle N, N\rangle = 2\langle L(X), N\rangle.$$

For $X, Y \in TM$ one has $\langle L(X), Y\rangle = \langle X, L(Y)\rangle$. The (orthogonal) eigenvectors of L are the principal directions of M and the corresponding eigenvalues are the principal curvatures. The relationship between the induced metric on $M \subset \mathbb{R}^{2n}$ and its curvature form is given in the Codazzi equation

$$\langle d_X L(Y) - d_Y L(X), Z\rangle = \langle L[X, Y], Z\rangle,$$

valid for $X, Y, Z \in TM$. The brackets indicate taking the commutator, $[X, Y] = d_X Y - d_Y X$.

So consider again the sum $\sum v_j \partial_j z(t)$ from condition (C_1). If $(v_1, \ldots, v_{2n-1})$ ranges over $\mathbb{R}^{2n-1}$, then this sum ranges over $X \in T_pM$ where $p = z(t)$. Moreover, the sum $\sum v_i \partial_i \overline{\nu}(t)$ is $d_X \overline{N}$. So (C_1) can be rewritten

$$\operatorname{Im}(X \cdot d_X \overline{N}) = 0 \ \text{ for every } \ X \in T_pM,$$

and this is equivalent to

$$\langle X, J\, d_X N\rangle = 0 \ \text{ for every } \ X \in T_pM \tag{C_2}$$

since $\operatorname{Im}(X \cdot \overline{Y}) = \langle X, J\, Y\rangle$ for all X, Y. (On the left side of this equation, X and Y are viewed as vectors in $\mathbb{C}^n$ with the complex dot product; on the right side they are in $\mathbb{R}^{2n}$ with the Euclidean product.) In what remains we assume that (C_2) is satisfied at all $p \in M$. This is the hypothesis of the Theorem.

4. Proof of the Theorem

The main idea, as in [8], is to show that there are at most two distinct values for the principal curvatures at different points of the surface, and if there are two, then one of them must be equal to zero.

Lemma 1. *For all $X, Y \in T_pM$, $\langle X, J\, d_Y N\rangle + \langle Y, J\, d_X N\rangle = 0$.*

Proof. Since also $X + Y \in T_pM$,

$$\begin{aligned} 0 = \langle X + Y,\, J\, d_{X+Y} N\rangle &= \langle X, J\, d_X N\rangle + \langle X, J\, d_Y N\rangle \\ &\quad + \langle Y, J\, d_X N\rangle + \langle Y, J\, d_Y N\rangle = \langle X, J\, d_Y N\rangle + \langle Y, J\, d_X N\rangle. \end{aligned}$$

□

Lemma 2. *$JN \in T_pM$ is a principal direction. The remaining principal directions span H_pM.*

Proof. Since $d_{JN} N \in T_pM$ we need only show $d_{JN} N$ is orthogonal to the complex tangent space. So take $X \in H_pM$ and apply Lemma 1 to $JX, JN \in T_pM$. Then

$$\begin{aligned} 0 = \langle JX,\, J\, d_{JN} N\rangle &+ \langle JN, J\, d_{JX} N\rangle \\ &= \langle X, d_{JN} N\rangle + \langle N,\, d_{JX} N\rangle = \langle X, d_{JN} N\rangle. \end{aligned}$$

□

Lemma 3. *Suppose $X \in H_pM$ is a principal direction associated to k^X. Then JX is another principal direction associated to k^X.*

Proof. Note that $d_{JX}N$ is orthogonal to JN since JX can be written as a combination of the principal directions which span H_pM.

Now let Y range over the principal directions in H_pM, so the set of JY gives a basis for H_pM. Applying the first lemma to $JX, Y \in T_pM$ we have

$$\begin{aligned} 0 = \langle JX, J\, d_Y N\rangle + \langle Y, J\, d_{JX} N\rangle &= \langle X, d_Y N\rangle - \langle JY, d_{JX} N\rangle \\ &= k^Y \langle X, Y\rangle - \langle JY, d_{JX} N\rangle. \end{aligned}$$

If $Y \neq X$ this gives $\langle JY, d_{JX}N\rangle = 0$, and if $Y = X$, then $\langle JX, d_{JX}N\rangle = k^X$. Since the JY form a basis for H_pM it follows that $d_{JX}N = k^X JX$. □

Lemma 4. *Suppose X is a principal direction in the complex tangent space associated to k^X. Then $k^X = k^{JN}$ or $k^X = 0$.*

Proof. We apply the Codazzi equation to X, JN, and $JX \in T_pM$,

$$\langle d_X\, L(JN) - d_{JN}\, L(X), JX\rangle = \langle L(d_X JN - d_{JN} X), JX\rangle.$$

Then on the left side we have

$$\begin{aligned} &\langle d_X(k^{JN} JN), JX\rangle - \langle d_{JN}(k^X X), JX\rangle \\ &\qquad = X(k^{JN})\langle JN, JX\rangle + k^{JN}\langle d_X JN, JX\rangle \\ &\qquad\quad - (JN)(k^X)\langle X, JX\rangle - k^X\langle d_{JN} X, JX\rangle \\ &\qquad = k^{JN}\langle d_X JN, JX\rangle - k^X\langle d_{JN} X, JX\rangle, \end{aligned}$$

and on the right we have

$$\begin{aligned} \langle d_X JN - d_{JN} X\,,\, L(JX)\rangle &= \langle d_X JN, k^X JX\rangle - \langle d_{JN} X, k^X JX\rangle \\ &= k^X\langle d_X JN, JX\rangle - k^X\langle d_{JN} X, JX\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} 0 = (k^{JN} - k^X)\,\langle d_X JN, JX\rangle &= (k^{JN} - k^X)\,\langle J\, d_X N, JX\rangle \\ &= (k^{JN} - k^X)\,\langle d_X N, X\rangle = (k^{JN} - k^X)\, k^X. \end{aligned}$$

□

Following the last two lemmas we denote orthonormal principal directions corresponding to curvature $k = k^{JN}$ by $X_0, \ldots, X_{2m}$, with $X_0 = JN$, and the orthonormal principal directions corresponding to curvature zero by $Y_{2m+1}, \ldots, Y_{2n-2}$.

Lemma 5. *$Y_i(k) = 0$ for all i.*

Proof. Apply the Codazzi equation to JN, Y_i, and JN,

$$\langle d_{JN}\, L(Y_i) - d_{Y_i}\, L(JN), JN\rangle = \langle L(d_{JN} Y_i - d_{Y_i} JN), JN\rangle.$$

Then on the left side we have

$$-\langle d_{Y_i}(kJN), JN\rangle = -Y_i(k)\,\langle JN, JN\rangle - k\,\langle d_{Y_i} JN, JN\rangle = -Y_i(k),$$

and on the right we have

$$\begin{aligned} &\langle d_{JN} Y_i - d_{Y_i} JN\,,\, kJN\rangle \\ &\qquad = k\,\langle d_{JN} Y_i, JN\rangle - k\,\langle d_{Y_i} JN, JN\rangle = k\,\langle d_{JN} Y_i, JN\rangle. \end{aligned}$$

Now using Lemma 1 with JN and Y_i we have

$$0 = \langle JN, J\, d_{Y_i} N\rangle + \langle Y_i, J\, d_{JN} N\rangle = \langle Y_i, d_{JN} JN\rangle.$$

It follows that $Y_i(k) = -k\,\langle d_{JN} Y_i, JN\rangle = k\,\langle Y_i, d_{JN} JN\rangle = 0$. □

Lemma 6. *If $m \neq 0$, then $X_i(k) = 0$ for all i.*

Proof. For $i \neq j$, apply the Codazzi equation to X_i, X_j, and X_j,

$$\langle d_{X_i} L(X_j) - d_{X_j} L(X_i), X_j \rangle = \langle L(d_{X_i} X_j - d_{X_j} X_i), X_j \rangle.$$

Then on the left side we have

$$\begin{aligned}\langle d_{X_i}(kX_j), X_j \rangle - \langle d_{X_j}(kX_i), X_j \rangle &= X_i(k) \langle X_j, X_j \rangle + k \langle d_{X_i} X_j, X_j \rangle \\ &\quad - X_j(k) \langle X_i, X_j \rangle - k \langle d_{X_j} X_i, X_j \rangle = X_i(k) - k \langle d_{X_j} X_i, X_j \rangle,\end{aligned}$$

and on the right we have

$$\begin{aligned}&\langle d_{X_i} X_j - d_{X_j} X_i \, , \, kX_j \rangle \\ &\qquad = k \langle d_{X_i} X_j, X_j \rangle - k \langle d_{X_j} X_i, X_j \rangle = -k \langle d_{X_j} X_i, X_j \rangle.\end{aligned}$$

It follows that $X_i(k) = 0$. □

Proof of the Theorem. If $m \neq 0$, then k is constant and the second fundamental form is completely determined. In particular, M is part of a surface of the form $S^{2m+1} \times \mathbb{C}^{n-m-1}$, where S is a sphere of radius k^{-1} inside $\mathbb{C}^{m+1} \cong \mathbb{R}^{2m+2}$. This is the same condition that Boas determined for when $A(z, w)$ vanishes identically.

If $m = 0$, then the curvature is arbitrary in the JN direction but is constant in every Y direction. In particular, M is part of a $C \times \mathbb{C}^{n-1}$ where C is a curve in a one-dimensional complex plane. □

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