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HARDY'S INEQUALITY AND THE BOUNDARY SIZE

PEKKA KOSKELA AND XIAO ZHONG

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Abstract. We establish a self-improving property of the Hardy inequality and an estimate on the size of the boundary of a domain supporting a Hardy inequality.

1. Introduction

Let us consider the Hardy inequality

$$\int_\Omega |u(x)|^p d(x,\partial\Omega)^{-p}\,dx \le C\int_\Omega |\nabla u(x)|^p\,dx. \tag{1}$$

This inequality holds whenever $u\in C_0^1(\Omega)$ and Ω is a sufficiently nice domain; (1) holds (cf. [9]) for all $p>1$ if Ω is a Lipschitz domain, and by results of Ancona [2], Lewis [11] and Wannebo [15] inequality (1) is valid for a given $p>1$ provided the complement of Ω is p-fat, that is, provided it satisfies a uniform p-capacity density condition. Moreover, such a density condition is also necessary when $p=n$, the dimension of the underlying Euclidean space; see [2], [11]. This density condition holds for any proper subdomain Ω of $\mathbf{R}^n$ when $p>n$ and it implies that the Hausdorff dimension of the boundary of Ω is (even locally) strictly larger than $n-p$ for the remaining values of p. On the other hand, (1) also holds for all $1\le p<n$ when $\Omega=\mathbf{R}^n\setminus\{0\}$. Notice the dichotomy in these examples: either the dimension of the boundary is strictly larger than $n-p$ or (substantially) smaller than $n-p$. Our first result shows that this phenomenon follows from a general rule.

Theorem 1.1. *Let $\Omega\subset\mathbf{R}^n$ be a domain and suppose that the Hardy inequality* (1) *holds with a fixed $p>1$ and constant C for all $u\in C_0^1(\Omega)$. There exists a constant $\epsilon=\epsilon(p,n,C)>0$ so that either*

(i) $\dim_H(\partial\Omega)\ge n-p+\epsilon$ *and* (1) *holds (with possibly a different constant) for all $p-\epsilon<q<p+\epsilon$, or*

(ii) $\dim_M(\partial\Omega)\le n-p-\epsilon$ *and* (1) *holds (with possibly a different constant) for all $1\le q<p+\epsilon$.*

In fact, we also obtain a localized version: the given dichotomy on the size of the boundary holds for $B(x,r)\cap\partial\Omega$ when $B(x,r)$ is given. This implies that, for $p\le n$,

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inequality (1) cannot hold when the boundary of Ω contains an $(n-p)$-dimensional part.

In the above theorem, $\dim_H$ refers to the usual Hausdorff dimension and $\dim_M$ to the Minkowski dimension arising from the use of covers with ball of equal radii. Notice that always $\dim_H(E) \leq \dim_M(E)$. When E is sufficiently self-similar, these dimensions are the same.

Theorem 1.1 includes a self-improving property of the Hardy inequality: if the inequality holds for a given $p > 1$, then it actually holds for exponents in an open interval containing p. The size of this interval naturally depends on the given data: for example, the value $q = n$ is only allowed when the boundary is thick enough to satisfy a uniform density condition for the n-capacity (and then a positive lower bound on the Hausdorff dimension), whereas (1) holds for all $p \neq n$ when $\Omega = \mathbf{R}^n \setminus \{0\}$. It is worth recalling here the following result in [11] on the self-improving property of the p-fatness of a set: if the complement of a domain Ω is p-fat, then it is also q-fat for some $q < p$. This also implies that the Hardy inequality (1) holds not only for p but also for some $q < p$ (see [11]), which gives a self-improving property of the Hardy inequality in this case.

The self-improving property of (1) turns out to hold in surprising generality: in doubling spaces that support a Poincaré inequality. These spaces cover the case of the Heisenberg and Carnot groups and the interesting settings arising from vector fields; see [5] and the references therein. Let us introduce the necessary terminology. We consider a proper metric space X equipped with a Borel regular doubling measure μ. Here doubling means that $\mu(2B) \leq C_d \mu(B)$ with a fixed constant C_d for all balls B, and the properness that each closed ball is compact. We call such a metric space a doubling space. A non-negative Borel function g is called an upper gradient (see [7]) of a real valued function u on an open set $U \subset X$ provided

$$|u(\gamma(1)) - u(\gamma(0))| \leq \int_\gamma g\, ds$$

whenever $\gamma : [0,1] \to U$ is a rectifiable path. Following Heinonen and Koskela [7] we say that X supports a p-Poincaré inequality, $1 \leq p < \infty$, provided there are constants C and λ so that

$$\mu(B)^{-1} \int_B |u - u_B|\, d\mu \leq C \operatorname{diam}(B)(\mu(\lambda B)^{-1} \int_{\lambda B} g^p\, d\mu)^{1/p} \tag{2}$$

whenever B is a ball, u is continuous on λB and g is an upper gradient of u on λB. Here $u_B = \mu(B)^{-1} \int_B u d\mu$ is the integral average of u. From now on, if a constant C depends on the doubling constant C_d and the constants C, λ, p in the p-Poincaré inequality, we write $C = C(C_X)$.

Notice at once that, by Hölder's inequality, (2) for a given p implies the corresponding inequality for all larger exponents. Also, as is easy to check, the pointwise Lipschitz constant of a Lipschitz function is always an upper gradient.

We say that X is Ahlfors Q-regular if there is a constant C_μ so that

$$\operatorname{diam}(B)^Q / C_\mu \leq \mu(B) \leq C_\mu \operatorname{diam}(B)^Q$$

for each ball B in X. For example, the Heisenberg and the more general Carnot groups are Ahlfors regular and support a 1-Poincaré inequality. Moreover, for any $Q \geq 1$ there are Q-regular spaces that support even a 1-Poincaré inequality. Here

Q need not be an integer. For this see the works of Bourdon and Pajot [4] and Laakso [10].

A natural substitute for a continuously differentiable, compactly supported function in this setting is a Lipschitz function with bounded support that vanishes on the complement of a proper subdomain Ω of X, and the Hardy inequality (1) takes the form

$$\int_\Omega |u(x)|^p d(x,\Omega^c)^{-p}\, d\mu \le C_p \int_\Omega g^p\, d\mu. \tag{3}$$

We then say that the Hardy inequality holds in Ω.

Theorem 1.2. *Suppose that X is a doubling space that supports a p_0-Poincaré inequality. Let $\Omega \subset X$ be a domain and suppose that the Hardy inequality* (3) *holds with a fixed $p > p_0$ in Ω. Then there exists a constant $\epsilon = \epsilon(p, p_0, C_X, C_p) > 0$ so that* (3) *holds (with possibly a different constant) for all $p - \epsilon < q < p + \epsilon$.*

Remark 1.3. We do not know if Theorem 1.2 is still true if we replace the p_0-Poincaré inequality with a p-Poincaré inequality. It is also not known if a doubling space supporting a p-Poincaré ($p > 1$) supports a q-Poincaré inequality for some $q < p$; see [7].

We also obtain a size estimate analogous to that of Theorem 1.1 on the complement of Ω when the measure is further assumed to be Ahlfors regular.

Added. We have recently found out that Björn, MacManus and Shanmugalingam [3] have extended the result of Lewis [11] on a sufficient condition for the p-Hardy inequality to the metric setting.

2. Proofs of Theorems 1.1 and 1.2

For the proofs of Theorem 1.1 and Theorem 1.2, we need the following lemma [6, Thm 3.2].

Lemma 2.1. *Let X be a doubling space. Assume the pair u, g satisfies a p_0-Poincaré inequality* (2)*, $p \ge 1$. Then*

$$|u(x) - u(y)| \le C d(x,y) \left((Mg^{p_0}(x))^{1/p_0} + (Mg^{p_0}(y))^{1/p_0} \right) \tag{4}$$

for almost every $x, y \in X$, where $C = C(C_X) > 0$ and

$$Mv(x) = \sup_{r>0} \mu(B(x,r))^{-1} \int_{B(x,r)} |v| d\mu$$

is the usual maximal function.

We divide the proof of Theorem 1.2 into two parts. The idea of the proof of the following theorem is inspired by [12]. Actually, we construct a Lipschitz function by using the pointwise inequality in Lemma 2.1.

Theorem 2.2. *Suppose that X is a doubling space that admits a p_0-Poincaré inequality. Let $\Omega \subset X$ be a domain and suppose that the Hardy inequality* (3) *holds with a fixed $p > p_0$ in Ω. Then there exist constants $\epsilon = \epsilon(p, p_0, C_X, C_p) > 0$ and $C_\epsilon = C_\epsilon(p, p_0, C_X, C_p)$ so that* (3) *holds for all $p - \epsilon < q \le p$ with constant C_ϵ.*

Proof. Let u be a Lipschitz function that vanishes in $X \setminus \Omega$ and extend u to be 0 on Ω^c. For $\lambda > 0$ we define

$$F_\lambda = \{x \in \Omega : |u(x)| \le \lambda d(x, \Omega^c), \quad (Mg^{p_0}(x))^{1/p_0} \le \lambda\}.$$

We claim that the restriction of u to $F_\lambda \cup \Omega^c$ is $C_1\lambda$-Lipschitz, where $C_1 = C_1(C_X) \ge 1$. Indeed, if $x \in F_\lambda, y \in F_\lambda$, we have by Lemma 2.1 that

$$\begin{aligned} |u(x) - u(y)| &\le Cd(x, y)\left((Mg^{p_0}(x))^{1/p_0} + (Mg^{p_0}(y))^{1/p_0}\right) \\ &\le 2C\lambda d(x, y). \end{aligned}$$

If $x \in F_\lambda, y \in \Omega^c$, then

$$\begin{aligned} |u(x) - u(y)| &= |u(x)| \le \lambda d(x, \Omega^c) \\ &\le \lambda d(x, y). \end{aligned}$$

Thus the claim is true. By the classical McShane extension

$$\tilde{u}(x) = \inf_{y \in F_\lambda \cup \Omega^c} \{u(y) + C_1\lambda|x - y|\}$$

we extend the restriction of u on $F_\lambda \cup \Omega^c$ to a $C_1\lambda$-Lipschitz function $\tilde{u}$ in X. Write

$$E_\lambda = \{x \in \Omega : |u(x)| \le \lambda d(x, \Omega^c)\}$$

and

$$G_\lambda = \{x \in \Omega : (Mg^{p_0}(x))^{1/p_0} \le \lambda\}$$

so that $F_\lambda = E_\lambda \cap G_\lambda$. Noticing that $\tilde{g} = g\chi_{F_\lambda} + C_1\lambda\chi_{\Omega\setminus F_\lambda}$ is an upper gradient of $\tilde{u}$, we conclude by applying (3) to the pair $\tilde{u}, \tilde{g}$ that

$$\int_{F_\lambda} |\tilde{u}(x)|^p d(x, \Omega^c)^{-p}\, d\mu \le C_p \int_{F_\lambda} g(x)^p\, d\mu + C_pC_1^p\lambda^p\mu(\Omega \setminus F_\lambda)$$

and, furthermore, that

$$\begin{aligned} \int_{E_\lambda} |u(x)|^p d(x, \Omega^c)^{-p}\, d\mu &\le C_p \int_{F_\lambda} g(x)^p\, d\mu + C_pC_1^p\lambda^p\mu(\Omega \setminus F_\lambda) \\ &\quad + \int_{E_\lambda \setminus G_\lambda} |u(x)|^p d(x, \Omega^c)^{-p}\, d\mu \\ &\le C_p \int_{G_\lambda} g(x)^p\, d\mu + C_pC_1^p(\lambda^p\mu(\Omega \setminus E_\lambda) + \lambda^p\mu(\Omega \setminus G_\lambda)). \end{aligned}$$

This inequality holds for all $\lambda > 0$, and by multiplying it by $\lambda^{-1-\epsilon}$ for some small $0 < \epsilon < p - 1$, integrating over $(0, \infty)$ and changing the order of the integration on the left side, we have

$$\begin{aligned} \epsilon^{-1} \int_\Omega |u(x)|^{p-\epsilon} d(x, \Omega^c)^{\epsilon-p}\, d\mu &\le C_p \int_0^\infty \lambda^{-1-\epsilon} \int_{G_\lambda} g^p\, d\mu d\lambda \\ &+ C_pC_1^p \int_0^\infty \lambda^{p-1-\epsilon}(\mu(\Omega \setminus E_\lambda) + \mu(\Omega \setminus G_\lambda))\, d\lambda. \end{aligned}$$

By the definition of G_λ we notice that the first integral on the right is no more than $C_p\epsilon^{-1}\int_\Omega g^{p-\epsilon} d\mu$. As for the remaining integral, we notice from the definitions of E_λ and G_λ that it can be estimated from above by

$$C_pC_1^p(p - \epsilon)^{-1}\Big(\int_\Omega |u(x)|^{p-\epsilon} d(x, \Omega^c)^{\epsilon-p} d\mu + \int_\Omega (Mg^{p_0}(x))^{(p-\epsilon)/p_0}\, d\mu\Big).$$

By the Hardy-Littlewood maximal theorem,

$$\int_\Omega (Mg^{p_0}(x))^{(p-\epsilon)/p_0}\,d\mu \le C_0 \int_\Omega g^{p-\epsilon} d\mu$$

where C_0 depends only on the doubling constant C_d and p, p_0 provided $\epsilon < (p-p_0)/2$. Assume further that $C_p C_1^p \epsilon/(p-\epsilon) \le 1/2$. Combining these estimates, we have

$$\int_\Omega |u(x)|^{p-\epsilon} d(x,\Omega^c)^{\epsilon-p}\,d\mu \le C_\epsilon \int_\Omega g(x)^{p-\epsilon}\,d\mu,$$

where $C_\epsilon = 2C_p C_1^p(1 + C_0(p-1))$. The proof is complete.

Theorem 2.3. *Suppose that X is a doubling space and that the Hardy inequality* (3) *holds in $\Omega \subset X$. Then there exist constants $\epsilon = \epsilon(p, C_p) > 0$ and $C_1 = C_1(p, C_p)$ so that* (3) *holds for all $p \le q < p + \epsilon$ with constant C_1.*

Proof. Let u be a Lipschitz function that vanishes in $X \setminus \Omega$ and g an upper gradient of u. We may assume that $u \ge 0$. Let $\delta > 0$ be fixed later. Define a function $\tilde{u}$ by the formula $\tilde{u}(x) = u(x)^{1+\delta} d(x,\Omega^c)^{-\delta}$ when $x \in \Omega$ and set $\tilde{u}(x) = 0$ when $x \in \Omega^c$. Then it rather easily follows that $\tilde{u}$ is Lipschitz. Also, a straightforward argument shows that the zero extension of

$$\tilde{g}(x) = (1+\delta) g(x) u(x)^\delta d(x,\Omega^c)^{-\delta} + \delta u(x)^{1+\delta} d(x,\Omega^c)^{-\delta-1}$$

from Ω to the rest of X is an upper gradient of $\tilde{u}$, where $g(x)$ is an upper gradient of u. The Hardy inequality (3) applied to the pair $\tilde{u}, \tilde{g}$ gives

$$\int_\Omega |u|^{p(1+\delta)} d(x,\Omega^c)^{-p(1+\delta)}\,d\mu = \int_\Omega |\tilde{u}|^p d(x,\Omega^c)^{-p}\,d\mu \le C_p \int_\Omega \tilde{g}^p\,d\mu \tag{5}$$

$$\le C_p(1+\delta)^p \int_\Omega g^p |u(x)|^{p\delta} d(x,\Omega^c)^{-p\delta}\,d\mu + C_p \delta^p \int_\Omega |u(x)|^{p(1+\delta)} d(x,\Omega^c)^{-p(1+\delta)}\,d\mu.$$

By the Hölder inequality, the first integral on the right is not more than

$$\left(\int_\Omega g(x)^{p(1+\delta)}\,d\mu\right)^{1/(1+\delta)} \left(\int_\Omega |u(x)|^{p(1+\delta)} d(x,\Omega^c)^{-p(1+\delta)} d\mu\right)^{\delta/(1+\delta)}.$$

Now let $\delta^p \le 1/2C_p$. Then the second term on the right of inequality (5) can be controlled by one half of the term on the left, assuming, as we may, that this term is finite: replace u with $u_j = \max\{0, u - 1/j\}$ and let j tend to infinity. The claim now easily follows with $\epsilon = \delta p$ from inequality (5).

We need the concept of capacity. Given an open set G and a compact set $E \subset G$, we define $\mathrm{cap}_p(E; G)$ to be the infimum of the integrals

$$\int_X g^p d\mu,$$

where the infimum is taken over all upper gradients g of Lipschitz functions u of bounded support that vanish in $X \setminus G$ and take on the value 1 in E.

Lemma 2.4. *Suppose that the Hardy inequality* (3) *holds in Ω. Then there is a constant $\delta = \delta(p, C_d, C_p) > 0$ and a constant $C' = C'(p, C_d, C_p)$ so that the following holds: if $cap_p(\overline{B}(x_0, 2r_0) \setminus \Omega; B(x_0, 4r)) = 0$ and $\mu(B(x_0, 2r_0) \setminus \Omega) = 0$, then, for each ball $B \subset B(x_0, r_0)$,*

$$\int_B d(x,\Omega^c)^{-p-\delta}\,d\mu \le C'\,diam\,(B)^{-p-\delta}\mu(B).$$

Proof. Because $\mathrm{cap}_p(\overline{B}(x_0,2r_0)\setminus\Omega; B(x_0,4r))=0$, a standard approximation result shows that (3) holds for each Lipschitz function that vanishes in $\Omega^c\setminus B(x_0,2r_0)$. Given $B(x_1,r)\subset B(x_0,r_0)$, define

$$u(x)=d(x,X\setminus B(x_1,2r))/r.$$

Then $g(x)=r^{-1}\chi_{B(x_1,2r)}(x)$ is an upper gradient of u. Thus (3) applied to the pair u,g gives

(6)
$$\int_{B(x_1,r)} d(x,\partial\Omega)^{-p}\,d\mu\le\int_\Omega |u(x)|^p d(x,\partial\Omega)^{-p}\,d\mu\le C_p\mu(B(x_1,2r))r^{-p}.$$

On the other hand,

$$d(x,\Omega^c)\le 4r$$

in $B(x_1,2r)$ if $B(x_1,2r)\cap\Omega^c\neq\emptyset$, and in this case

$$\int_{B(x_1,r)} d(x,\Omega^c)^{-1}d\mu\ge(4r)^{-1}\mu(B(x_1,r)).$$

Thus, when $B(x_1,2r)\cap\Omega^c\neq\emptyset$ and $B(x_1,r)\subset B(x_0,r_0)$,

$$\begin{aligned}&(\mu(B(x_1,r))^{-1}\int_{B(x_1,r)} d(x,\Omega^c)^{-p}d\mu)^{1/p}\\ &\le C'\mu(B(x_1,r))^{-1}\int_{B(x_1,r)} d(x,\Omega^c)^{-1}d\mu,\end{aligned}\tag{7}$$

where $C'=C'(p,C_d,C_p)$. Noticing that $d(x,\Omega^c)\le\frac{3}{2}d(x_1,\Omega^c)\le 3d(x,\Omega^c)$ for $x\in B(x_1,r)$ whenever $B(x_1,2r)\setminus\Omega=\emptyset$, we conclude that (7) holds for each ball $B(x_1,r)\subset B(x_0,r_0)$.

The claim follows using (6), (7), and the self-improving property of reverse Hölder inequalities on metric doubling spaces (see, e.g. [14]). □

Remark 2.5. If p varies between p_1 and p_2, $1<p_1<p_2<\infty$, and the Hardy inequality holds with a uniform constant C, we may choose $\delta=\delta(p_1,p_2,c_d,C)>0$ independent of p.

Lemma 2.6. *Let E be compact and B_0 be a ball. Assume that $\mu(B(x,r))\ge Cr^Q$ for each ball $B(x,r)\subset 2B_0$ and that*

$$\int_{2B_0} d(x,E)^{-q}d\mu<\infty,$$

where $0<q<Q$. Then, for each small $\delta>0$, $E\cap B_0$ can be covered with $\sharp_\delta$ balls, all of radii δ, so that

$$\lim_{\delta\to 0}\sharp_\delta\delta^{Q-q}=0.$$

In particular, the Hausdorff dimension of $E\cap B_0$ is at most $Q-q$.

Proof. Notice first that $\mu(E\cap B_0)=0$ by the convergence of the above integral. Consider the collection $\{B(x,\delta): x\in E\cap B_0\}$ of balls. By the Vitali covering theorem we find a countable subcover of E of this collection of balls so that the corresponding dilated balls $\frac{1}{5}B(x_j,\delta) = B(x_j,\frac{1}{5}\delta)$ are pairwise disjoint. The doubling condition then shows that this subcollection must be finite. Let us denote these balls $B_1,...,B_k$. We have that

$$\int_{\frac{1}{5}B_j} d(x,E)^{-q}d\mu \geq C''\delta^{Q-q}.$$

The claim follows from the assumption

$$\int_{2B_0} d(x,E)^{-q}d\mu < \infty,$$

and the pairwise disjointness of the balls $\frac{1}{5}B_j$; notice that the measure of the union of the balls $B_1,...,B_k$ tends to zero when $\delta \to 0$.

Corollary 2.7. *Suppose that the Hardy inequality* (3) *holds on* $\Omega \subset X$, *where* X *is an Ahlfors* Q*-regular space that supports a* p_0*-Poincaré inequality for some* $p_0 < p$. *There is a constant* $\epsilon = \epsilon(p,p_0,C_X,C_p,Q)$ *so that for each ball* $B(x_0,r_0)$ *either* $\dim_H(B(x_0,2r_0)\setminus\Omega) > Q-p+\epsilon$ *or* $\dim_M(B(x_0,r_0)\setminus\Omega) < Q-p-\epsilon$.

Proof. By Theorem 2.2 there are constants ϵ and C_ϵ so that (3) holds on Ω for all $p-\epsilon < q \leq p$ with constant C_ϵ. Fix ϵ_0, $0 < \epsilon_0 < \epsilon/2$, to be chosen later and $B(x_0,2r_0)$. If $\dim_H(B(x_0,2r_0)\setminus\Omega) > Q-p+\epsilon_0$, we are done. Suppose that

$$\dim_H(B(x_0,2r_0)\setminus\Omega) \leq Q-p+\epsilon_0.$$

Then the upper volume bound $\mu(B(x,r)) \leq C_0r^Q$ implies that

$$\mathrm{cap}_{p-2\epsilon_0}(\overline{B}(x_0,2r_0)\setminus\Omega; B(x_0,4r_0)) = 0$$

[8] and $\mu(B(x_0,2r_0)\setminus\Omega) = 0$. Applying Lemma 2.4 (with $p = p-2\epsilon_0$) and Lemma 2.6, we have

$$\dim_M(B(x_0,r_0)\setminus\Omega) \leq Q-(p-2\epsilon_0)-\delta,$$

where $\delta = \delta(p-\epsilon,p,C_d,C_\epsilon) > 0$ is independent of the choice of ϵ_0. Now the claim follows by choosing $\epsilon_0 = \min(\epsilon/2,\delta/3)$.

Remark 2.8. We do not know if in the setting of Corollary 2.7 one could get a q-Hardy inequality for all $p_0 < q < p+\epsilon$ when $\dim_M(\Omega^c) < Q-p-\epsilon$. This question is clearly related to trace theorems [5].

Theorem 1.2 follows directly from Theorem 2.2 and Theorem 2.3. Regarding Theorem 1.1, based on Theorem 1.2 and Corollary 2.7, we only need to verify that the Hardy inequality holds for all $1 \leq q \leq p-\epsilon$ when the Minkowski dimension of the boundary of Ω does not exceed $n-p-2\epsilon$. It is easy to check that then, in fact, this dimension estimate holds for the complement of Ω. A more general result in [13, 2.3.3] implies that the Hardy inequality for the desired values of q holds if and only if

$$\int_K d(x,\Omega^c)^{-q}\,dx \leq C\mathrm{cap}_q(K;\Omega) \tag{8}$$

for each compact set $K \subset \Omega$. The weak type estimate (8) follows from Corollary 7.1.2 of [1] and our estimate on the Minkowski dimension, even though the notations are different. The proofs of Theorems 1.1 and 1.2 are complete.

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Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35, Fin-40351 Jyväskylä, Finland

E-mail address: pkoskela@math.jyu.fi

Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35, Fin-40351 Jyväskylä, Finland

E-mail address: zhong@math.jyu.fi