PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 4, Pages 1063–1070
S 0002-9939(02)06725-4

AVERAGE VALUES OF SYMMETRIC SQUARE L-FUNCTIONS AT THE EDGE OF THE CRITICAL STRIP

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(Communicated by Dennis A. Hejhal)

Abstract. Let $\mathcal{B}_2^*(N)$ be the set of all normalized newforms of weight 2 and level N, and let $L(\mathrm{sym}^2 f, 1)$ be the symmetric square L-function associated to $f \in \mathcal{B}_2^*(N)$. If N is a prime, then there is a positive constant B such that

$$\sum_{f\in\mathcal{B}_2^*(N)} L(1, \mathrm{sym}^2 f) = \frac{\pi^4}{432}N + O\big(N^{27/28}(\log N)^B\big).$$

This improves a recent result of Akbary, which requires 45/46 in place of 27/28.

§1. Introduction

Let $S_2(N)$ be the set of all cusp forms of weight 2 for the full modular group $\Gamma_0(N)$ with trivial character. This is a finite-dimensional Hilbert space with respect to the Petersson inner product defined by

$$\langle f, g\rangle := \int_{\Gamma_0(N)\smallsetminus\mathbb{H}} f(z)\overline{g(z)}\mathrm{d}x\mathrm{d}y,$$

where $\mathbb{H}$ is the upper half-plane. Let $\mathcal{B}_2^*(N)$ be the set of all normalized newforms in $S_2(N)$. Every $f \in \mathcal{B}_2^*(N)$ has the Fourier expansion of type

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{1/2} e(nz),$$

where $e(z) := e^{2\pi i z}$, $\lambda_f(1) = 1$ and $\lambda_f(n) \in \mathbb{R}$.

The symmetric square L-function associated to $f \in \mathcal{B}_2^*(N)$ is defined by

$$L(s, \mathrm{sym}^2 f) := \zeta_N(2s)\sum_{n=1}^{\infty}\frac{\lambda_f(n^2)}{n^s} \qquad (\Re e\, s > 1), \tag{1}$$

where

$$\zeta_N(s) := \prod_{p\nmid N}(1 - 1/p^s)^{-1}.$$

Define

$$L_\infty(s, \mathrm{sym}^2 f) := \left(\frac{N}{\pi^{3/2}}\right)^s \Gamma\Big(\frac{s+1}{2}\Big)^2\Gamma\Big(\frac{s+2}{2}\Big).$$

Received by the editors November 12, 2001.
2000 *Mathematics Subject Classification.* Primary 11F67.

Then $\Lambda(s,\mathrm{sym}^2 f) := L_\infty(s,\mathrm{sym}^2 f)L(s,\mathrm{sym}^2 f)$ is an entire function and satisfies the functional equation

$$\Lambda(s,\mathrm{sym}^2 f) = \Lambda(1-s,\mathrm{sym}^2 f). \tag{2}$$

Similar to the class number formula of Dirichlet, the value of $L(s,\mathrm{sym}^2 f)$ at the edge of the critical strip (in this case $s=1$) is of interest. When N is square free, one can show that (cf. [3], Lemma 2.5)

$$L(1,\mathrm{sym}^2 f) = \frac{8\pi^3\langle f,f\rangle}{N}. \tag{3}$$

Therefore to study the average values of the Petersson inner product when f varies in $\mathcal{B}_2^*(N)$, it is enough to find an asymptotic formula for the average values of $L(1,\mathrm{sym}^2 f)$.

The mean values of $L(1,\mathrm{sym}^2 f)$ were first investigated by R. Murty [6]. He proved that *if we assume that* $L(\frac{1}{2}+i\tau,\mathrm{sym}^2 f) \ll \big(N(|\tau|+1)\big)^\theta$ *for some* $\theta>0$, *then for prime* N *we have*

$$\sum_{f\in\mathcal{B}_2^*(N)} L(1,\mathrm{sym}^2 f) = \frac{\pi^4}{432}N + O\big(N^{7/10+4\theta/5}\mathcal{L}^3\big), \tag{4}$$

where here and in the sequel we systematically write $\mathcal{L} := \log N$. According to ([5], page 336]), the value $\theta = 3/4$ is actually permissible. Thus the estimate (4) is only a conditional asymptotic formula. Very recently Akbary [1] has obtained an unconditional result: *If* N *is a prime, there is a positive constant* B *such that*

$$\sum_{f\in\mathcal{B}_2^*(N)} L(1,\mathrm{sym}^2 f) = \frac{\pi^4}{432}N + O\big(N^{45/46}\mathcal{L}^B\big).$$

In order to prove this asymptotic formula, he has combined the method of Duke [2] and that of Kowalski and Michel [4].

In this paper, we shall propose a better result.

Theorem. *Let* N *be a prime. Then there is a positive constant* B *such that*

$$\sum_{f\in\mathcal{B}_2^*(N)} L(1,\mathrm{sym}^2 f) = \frac{\pi^4}{432}N + O\big(N^{27/28}\mathcal{L}^B\big).$$

For comparison, we have $\frac{45}{46}\approx 0.978$ and $\frac{27}{28}\approx 0.964$. Our improvement comes from two observations. First we find that only applying the method of [4] is more effective than the combined method of [2] and [4]. Secondly we give a straightforward improvement of the method of [4], which allows us to make use of a wider range of parameters (see Lemma 2 below). Without the second remark, we only have 29/30 in place of 27/28.

§2. Some lemmas

As in [4], we define

$$\omega_f(x,y) := \sum_{x<d^2m\le y} \frac{\varepsilon_N(d)\lambda_f(m^2)}{d^2 m}, \tag{5}$$

where

$$\varepsilon_N(d) := \begin{cases} 1 & \text{if } (d, N) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The next lemma is essentially Lemma 6 of [4], but the second term here is sharpened. This improvement does not need to add any new idea. In fact their proof can yield such a result. For completeness, we reproduce their proof here with some minor modifications.

Lemma 1. *Let $r \geq 1$ be a fixed integer and $x, y, z \geq 2$ with $y > x$. Then there exist a real number $M \in [x^r z^{-1}, y^r z]$ and real numbers $c(m)$ such that*

$$\sum_{f \in \mathcal{B}_2^*(N)} \omega_f(x,y)^{2r} \ll \Bigg(\sum_{f \in \mathcal{B}_2^*(N)} \bigg| \sum_{m \sim M} \lambda_f(m^2) \frac{c(m)}{m} \bigg|^2 + N z^{-1} \Bigg) \{\log(Nyz)\}^{B_1} \tag{6}$$

and

$$|c(m)| \ll \{\tau(m) \log z\}^{B_2},$$

where the $B_i = B_i(r)$ are some positive constants and $\tau(m)$ is the usual divisor function.

Proof. By Lemmas 4 and 5 of [4], we have

$$\omega_f(x,y)^r = \sum_{\substack{x^r < mn \leq y^r \\ n \leq z}} \lambda_f(m^2) \frac{c(m,n)}{mn} + O\big(z^{-1/2} \{\log(Nyz)\}^{B_3}\big),$$

where $c(m,n)$ is defined as in Lemma 4 of [4] and $B_3 = B_3(r) \geq 1$ is a constant. Introducing

$$c(m) := \sum_{x^r m^{-1} < n \leq z} \frac{c(m,n)}{n},$$

the preceding estimate can be written as

$$\omega_f(x,y)^r = \sum_{x^r z^{-1} < m \leq y^r z} \lambda_f(m^2) \frac{c(m)}{m} + O\big(z^{-1/2} \{\log(Nyz)\}^{B_3}\big).$$

After a classic dyadic splitting, we find for some $M \in [x^r z^{-1}, y^r z]$,

$$\omega_f(x,y)^r \ll \Bigg(\bigg| \sum_{m \sim M} \lambda_f(m^2) \frac{c(m)}{m} \bigg| + z^{-1/2} \Bigg) \{\log(Nyz)\}^{B_3}.$$

By using the inequality $(a+b)^2 \leq 2(a^2+b^2)$ $(a, b \in \mathbb{R})$ and the fact that $|\mathcal{B}_2^*(N)| \asymp N$, we deduce the inequality (6) with $B_1 = 2B_3$.

It remains to prove the upper bound for $c(m)$. Since $|c(m,n)| \leq \tau(mn)^{B_4}$ for some positive constant $B_4 = B_4(r)$ ([4], Lemma 4), we have

$$|c(m)| \leq \sum_{n \leq z} \frac{\tau(mn)^{B_4}}{n} \leq \tau(m)^{B_4} \sum_{n \leq z} \frac{\tau(n)^{B_4}}{n} \ll \{\tau(m) \log z\}^{B_2}.$$

This completes the proof of Lemma 1. □

With the help of Lemma 1, we can improve Lemma 3 of [4]. The next lemma enlarges the admissible range of parameters in Lemma 3 of [4]. This is one of two keys in the proof of our theorem.

Lemma 2. *Let $r \geq 1$ be a fixed integer, $x^r \geq N^{10}$ and $y > x$ with $\log y \ll \mathcal{L}$. There exists a positive constant $B_5 = B_5(r)$ such that*

$$\sum_{f\in\mathcal{B}_2^*(N)} \omega_f(x,y)^{2r} \ll \mathcal{L}^{B_5}.$$

Proof. We take $z = N$ in Lemma 1 (instead of $z = N^2$ as in [4]). Then the assumption $x^r \geq N^{10}$ implies that $M \geq N^9$, and we may appeal to the mean value estimate of Corollary 1 of [4] (with $a(m) = c(m)/m\mathcal{L}^{B_2}$) to bound the double sums on the right-hand side of (6). This completes the proof of Lemma 2. □

The following lemma is an improved version of Proposition 1 of [1], which is feasible for greater range of parameters.

Lemma 3. *Let $r \geq 1$ be a fixed integer and $x^r \geq N^{10}$. There exists a positive constant $B_6 = B_6(r)$ such that*

$$\sum_{f\in\mathcal{B}_2^*(N)} \lambda_f(n) = \frac{N}{12\sqrt{n}}\delta_{n=\square} + O\big(N^{1-1/2r}\tau(n)\mathcal{L}^{B_6} + N^{-1/2}x\sqrt{n}\tau(n)\big),$$

where $\delta_{n=\square} = 1$ if n is a square and $\delta_{n=\square} = 0$ otherwise.

Proof. Without lost of generality, we can assume that $x \leq N^{3/2}$ since $|\lambda_f(n)| \leq \tau(n)$ (Deligne's bound) and $|\mathcal{B}_2^*(N)| \asymp N$.

Let $y \in (x,\infty)\smallsetminus\mathbb{N}$ with $\log y \ll \mathcal{L}$. By the Perron formula, we can write

$$\sum_{d^2m<y} \frac{\varepsilon_N(d)\lambda_f(m^2)}{d^2m} = \frac{1}{2\pi i}\int_{(1)} L(\mathrm{sym}^2 f, s+1)\frac{y^s}{s}\,\mathrm{d}s,$$

where $(\sigma) := \{\sigma + i\tau : -\infty < \tau < \infty\}$. Upon moving the line of integration from (1) to (−2) and using the functional equation (2), the theorem of residues allows us to deduce

$$\sum_{d^2m<y} \frac{\varepsilon_N(d)\lambda_f(m^2)}{d^2m}$$
$$= L(1,\mathrm{sym}^2 f) + \frac{1}{2\pi i A}\int_{(-2)} L(-s,\mathrm{sym}^2 f)\frac{\Gamma(\frac{1-s}{2})^2\Gamma(\frac{2-s}{2})}{\Gamma(\frac{2+s}{2})^2\Gamma(\frac{s+3}{2})}\Big(\frac{y}{A^2}\Big)^s\frac{\mathrm{d}s}{s},$$

where $A = N/\pi^{3/2}$. Since $L(\mathrm{sym}^2 f, s)$ is absolutely convergent for $\Re e\, s > 1$, it is easy to see that the last integral is $\ll N^3y^{-2}$. Hence we obtain

$$L(1,\mathrm{sym}^2 f) = \sum_{d^2m<y} \frac{\varepsilon_N(d)\lambda_f(m^2)}{d^2m} + O\big(N^3y^{-2}\big).$$

As in [4], we write

$$L(1,\mathrm{sym}^2 f) = \sum_{d^2m\leq x} \frac{\varepsilon_N(d)\lambda_f(m^2)}{d^2m} + \omega_f(x,y) + O\big(N^3y^{-2}\big), \tag{7}$$

where $\omega_f(x,y)$ is defined by (5).

In view of (3), we can write

$$\begin{aligned}\sum_{f\in\mathcal{B}_2^*(N)}\lambda_f(n) &= \sum_{f\in\mathcal{B}_2^*(N)}\frac{L(1,\mathrm{sym}^2f)}{L(1,\mathrm{sym}^2f)}\lambda_f(n)\\ &= \frac{N}{2\pi^2}\sum_{f\in\mathcal{B}_2^*(N)}\frac{\lambda_f(n)}{4\pi\langle f,f\rangle}L(1,\mathrm{sym}^2f).\end{aligned}$$

By using (7), we deduce

$$\sum_{f\in\mathcal{B}_2^*(N)}\lambda_f(n) = M + R_1 + O(R_2), \tag{8}$$

where

$$\begin{aligned}M &:= \frac{N}{2\pi^2}\sum_{f\in\mathcal{B}_2^*(N)}\frac{\lambda_f(n)}{4\pi\langle f,f\rangle}\sum_{d^2m\le x}\frac{\varepsilon_N(d)\lambda_f(m^2)}{d^2m},\\ R_1 &:= \frac{N}{2\pi^2}\sum_{f\in\mathcal{B}_2^*(N)}\frac{\lambda_f(n)}{4\pi\langle f,f\rangle}\omega_f(x,y),\\ R_2 &:= \frac{N^4}{y^2}\sum_{f\in\mathcal{B}_2^*(N)}\frac{|\lambda_f(n)|}{\langle f,f\rangle}.\end{aligned}$$

In order to evaluate M, we recall the following well-known estimate ([6], Proposition 1):

$$\sum_{f\in\mathcal{B}_2^*(N)}\frac{\lambda_f(m)\lambda_f(n)}{4\pi\langle f,f\rangle} = \delta_{m,n} + O\big(N^{-3/2}(m,n)^{1/2}(mn)^{1/2}\big), \tag{9}$$

where $\delta_{m,n}$ is the diagonal Kronecker symbol. It follows that

$$\begin{aligned}M &= \frac{N}{2\pi^2}\sum_{d^2m\le x}\frac{\varepsilon_N(d)}{d^2m}\sum_{f\in\mathcal{B}_2^*(N)}\frac{\lambda_f(m^2)\lambda_f(n)}{4\pi\langle f,f\rangle}\\ &= \frac{N}{2\pi^2}\sum_{d^2m\le x}\frac{\varepsilon_N(d)}{d^2m}\delta_{m^2,n} + O(R_3),\end{aligned}$$

where

$$\begin{aligned}R_3 &= \Big(\frac{n}{N}\Big)^{1/2}\sum_{d^2m\le x}\frac{(m^2,n)^{1/2}}{d^2}\\ &= \Big(\frac{n}{N}\Big)^{1/2}\sum_{\ell\mid n}\ell^{1/2}\sum_{m\le x,\,\ell\mid m^2}\ \sum_{d^2\le x/m}\frac{1}{d^2}.\end{aligned}$$

Now ℓ can be written uniquely as $\ell = \ell_1\ell_2^2$ with ℓ_1 square free. Then we have $\ell \mid m^2$ if and only if $\ell_1\ell_2 \mid m$. Therefore we have $m = \ell_1\ell_2 m'$ and

$$\begin{aligned} R_3 &\ll \Big(\frac{n}{N}\Big)^{1/2} \sum_{\ell\mid n} \ell^{1/2} \sum_{m\le x,\, \ell\mid m^2} 1 \\ &\ll \Big(\frac{n}{N}\Big)^{1/2} \sum_{\ell\mid n} \ell^{1/2} \sum_{m'\le x/\ell_1\ell_2} 1 \\ &\ll x\Big(\frac{n}{N}\Big)^{1/2} \sum_{\ell\mid n} \frac{\ell^{1/2}}{\ell_1\ell_2} \\ &\ll N^{-1/2}x\sqrt{n}\tau(n) \end{aligned}$$

since $(\ell_1\ell_2)^2 \ge \ell_1\ell_2^2 = \ell$.

If $n = k^2$, then

$$\begin{aligned} \sum_{d^2m\le x} \frac{\varepsilon_N(d)}{d^2m}\delta_{m^2,k^2} &= \frac{1}{k}\sum_{d\le (x/k)^{1/2}} \frac{\varepsilon_N(d)}{d^2} \\ &= \frac{1}{k}\bigg(\zeta_N(2) - \sum_{d>(x/k)^{1/2}} \frac{\varepsilon_N(d)}{d^2}\bigg) \\ &= \frac{\pi^2}{6\sqrt{n}} + O\big(x^{-1/2}n^{-1/4} + N^{-2}n^{-1/2}\big). \end{aligned}$$

Thus we can write, for any positive integer n,

$$\sum_{d^2m\le x} \frac{\varepsilon_N(d)}{d^2m}\delta_{m^2,n} = \bigg\{\frac{\pi^2}{6\sqrt{n}} + O\big(x^{-1/2}n^{-1/4} + N^{-2}n^{-1/2}\big)\bigg\}\delta_{n=\square}.$$

Combining these estimates, we find that

$$M = \frac{N}{12\sqrt{n}}\delta_{n=\square} + O\big(N^{-1/2}x\sqrt{n}\tau(n) + Nx^{-1/2}n^{-1/4}\delta_{n=\square} + n^{-1/2}\delta_{n=\square}\big). \tag{10}$$

By using Deligne's inequality $|\lambda_f(n)| \le \tau(n)$ and (9) with $m = n = 1$, we have

$$R_2 \ll N^4y^{-2}\tau(n). \tag{11}$$

Let $r' > 1$ such that $1/r' + 1/2r = 1$. By using the inequality $|\lambda_f(n)| \le \tau(n)$, the estimate $\langle f, f\rangle \gg N/\mathcal{L}$ (see [2], Proposition 4), the Hölder inequality and Lemma 2, we deduce that there exists a positive constant $B_6 = B_6(r)$ such that

$$\begin{aligned} R_1 &\ll \tau(n)\mathcal{L}\big|\mathcal{B}_2^*(N)\big|^{1/r'}\bigg(\sum_{f\in\mathcal{B}_2^*(N)} \omega_f(x,y)^{2r}\bigg)^{1/2r} \\ &\ll N^{1-1/2r}\tau(n)\mathcal{L}^{B_6}. \end{aligned} \tag{12}$$

Inserting (10), (11) and (12) into (8) and taking $y = N^2 + \frac{1}{2}$, we obtain

$$\sum_{f\in\mathcal{B}_2^*(N)} \lambda_f(n) = \frac{N}{12\sqrt{n}}\delta_{n=\square} + O\big(N^{1-1/2r}\tau(n)\mathcal{L}^{B_6} + N^{-1/2}xn^{1/2}\tau(n)\big),$$

where we have used the fact that $(Nx^{-1/2}n^{-1/4} + n^{-1/2})\delta_{n=\square}$ can be absorbed by $N^{1-1/2r}\tau(n)$ (since $x^r \ge N^{10}$). This completes the proof of Lemma 3. □

§3. PROOF OF THE THEOREM

Now we are ready to prove our Theorem.

We start with the formula (7) instead of representing $L(\mathrm{sym}^2 f, 1)$ by a sum of two absolutely convergent series as done in [1] (cf. [1], Lemma 4).

Let $r_1 \geq 1$ be an integer and $x_1^{r_1} \geq N^{10}$. Define $r_1' > 1$ by $1/r_1' + 1/2r_1 = 1$. By the Hölder inequality, Lemma 2 and the fact that $|\mathcal{B}_2^*(N)| \asymp N$, there exists a positive constant $B_7 = B_7(r_1)$ such that, for $y_1 > x_1$ with $\log y_1 \ll \mathcal{L}$,

$$\begin{aligned}\Big|\sum_{f\in\mathcal{B}_2^*(N)} \omega_f(x_1,y_1)\Big| &\leq \big|\mathcal{B}_2^*(N)\big|^{1/r_1'}\Big(\sum_{f\in\mathcal{B}_2^*(N)} \omega_f(x_1,y_1)^{2r_1}\Big)^{1/2r_1} \\ &\ll N^{1-1/2r_1}\mathcal{L}^{B_7}.\end{aligned}$$

Thus we have

$$\begin{aligned}&\sum_{f\in\mathcal{B}_2^*(N)} L(\mathrm{sym}^2 f,1) \\ &\quad= \sum_{d^2m<x_1}\frac{\varepsilon_N(d)}{d^2m}\sum_{f\in\mathcal{B}_2^*(N)}\lambda_f(m^2) + O\big(N^{1-1/2r_1}\mathcal{L}^{B_7} + N^4 y_1^{-2}\big).\end{aligned} \tag{13}$$

It remains to evaluate the last double sum. According to Lemma 3, we have

$$\sum_{d^2m<x_1}\frac{\varepsilon_N(d)}{d^2m}\sum_{f\in\mathcal{B}_2^*(N)}\lambda_f(m^2) = \frac{N}{12}\sum_{d^2m<x_1}\frac{\varepsilon_N(d)}{d^2m^2} + R_4,$$

where

$$\begin{aligned}R_4 &\ll N^{1-1/2r}\mathcal{L}^{B_6}\sum_{d^2m<x_1}\frac{\tau(m^2)}{d^2m} + N^{-1/2}x\sum_{d^2m<x_1}\frac{\tau(m^2)}{d^2} \\ &\ll \big(N^{1-1/2r} + N^{-1/2}xx_1\big)\mathcal{L}^{B_8},\end{aligned}$$

$x^r \geq N^{10}$ and $B_8 = B_8(r)$ is a positive constant.

It is apparent that

$$\begin{aligned}\sum_{d^2m<x_1}\frac{\varepsilon_N(d)}{d^2m^2} &= \sum_{m<x_1}\frac{1}{m^2}\big\{\zeta_N(2) + O\big((x_1/m)^{-1/2}\big)\big\} \\ &= \zeta_N(2)\sum_{m<x_1}\frac{1}{m^2} + O\big(x_1^{-1/2}\big) \\ &= \frac{\pi^4}{36} + O\big(x_1^{-1/2} + N^{-2}\big).\end{aligned}$$

Combining these estimates yields

$$\begin{aligned}&\sum_{d^2m<x_1}\frac{\varepsilon_N(d)}{d^2m}\sum_{f\in\mathcal{B}_2^*(N)}\lambda_f(m^2) \\ &\quad= \frac{\pi^4}{432}N + O\big((N^{1-1/2r} + N^{-1/2}xx_1 + Nx_1^{-1/2})\mathcal{L}^{B_8}\big).\end{aligned} \tag{14}$$

Inserting (14) into (13), we find that

$$\begin{aligned}\sum_{f\in\mathcal{B}_2^*(N)} & L(1,\mathrm{sym}^2 f)\\ &= \frac{\pi^4}{432}N + O\big((N^{1-1/2r_0}+N^{-1/2}xx_1+Nx_1^{-1/2}+N^4y_1^{-2})\mathcal{L}^{B_8}\big),\end{aligned}$$

where $r_0 := \max\{r, r_1\}$. Now by taking $r = r_1 = 14$, $x = x_1 = N^{41/56}$ and $y_1 = N^2+\frac{1}{2}$, we obtain the desired result. This completes the proof of the Theorem. □

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