

ORDERED GROUP INVARIANTS FOR NONORIENTABLE ONE-DIMENSIONAL GENERALIZED SOLENOIDS

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ABSTRACT. Let $f: X \rightarrow X$ be an edge-wrapping rule which presents a one-dimensional generalized solenoid \overline{X} , and let M be the adjacency matrix of f . When X is a wedge of circles and f leaves the unique branch point fixed, we show that the stationary dimension group of M is an invariant of homeomorphism of \overline{X} even if \overline{X} is not orientable.

1. INTRODUCTION

This paper is part of a study of certain one-dimensional spaces which can be associated to invariant sets in dynamical systems. In recent developments, ordered group invariants in C^* -algebras and topological dynamics have been used to provide invariants of homeomorphisms of certain one-dimensional compact metrizable spaces ([1], [2], [3], [6], [9], [10], [11], [16]). We can interpret these results as follows: the first Čech cohomology group of the space is given a preorder, and the isomorphism class of the resulting preordered group becomes an invariant of the homeomorphism class of the space.

An important type of ordered group arising in this setting is the **dimension group** (Δ_M, Δ_M^+) defined by an $n \times n$ nonnegative integer matrix M . Here Δ_M is the direct limit group $\varinjlim \mathbb{Z}^n = \{(\mathbf{v}, k) \mid \mathbf{v} \in \mathbb{Z}^n, k \in \mathbb{N}\} / \sim$, with $(\mathbf{v}, k) \sim (\mathbf{v}', k')$ if there exist $i, j \in \mathbb{N}$ such that $i + k = j + k'$ and $M^i \mathbf{v} = M^j \mathbf{v}'$. And the positive set is $\Delta_M^+ = \{[(\mathbf{v}, k)] \mid M^i \mathbf{v} \geq 0 \text{ for some } i \in \mathbb{N}\}$. The order structure gives rise to additional number-theoretic invariants ([1], [5]). In the case that two nonnegative integer matrices M_1 and M_2 are primitive (the case relevant to this paper), there is a procedure to determine whether the dimension groups $(\Delta_{M_1}, \Delta_{M_1}^+)$ and $(\Delta_{M_2}, \Delta_{M_2}^+)$ are isomorphic ([5]).

One class of topological spaces studied here is the class of **one-dimensional generalized solenoids** (1-solenoids) introduced by R. F. Williams. The 1-solenoids were employed to describe a theory of one-dimensional hyperbolic expanding attractors ([13], [14]). A 1-solenoid is a space homeomorphic to the inverse limit space \overline{X} obtained from a map $f: X \rightarrow X$ where X is a graph and f satisfies certain axioms ([13], [15]). The map f has a transition matrix $M_{X,f} = M$

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where $M(i, j)$ is the number of times the i^{th} edge of X is covered by the j^{th} edge. The presentation (X, f) is *elementary* if X is a wedge of circles. Every 1-solenoid has an elementary presentation ([14], [15]). If (X, f) is an elementary presentation of a 1-solenoid \overline{X} and $M = M_{X,f}$ is the adjacency matrix of $f: X \rightarrow X$, then $\check{H}^1(\overline{X})$ is isomorphic to Δ_M ([16]). This invariant of the homeomorphism class of \overline{X} was refined by Jacklitch ([3]), who showed that if (X, f) and (Y, g) are elementary presentations of oriented 1-solenoids \overline{X} and \overline{Y} , then the dimension groups $(\Delta_{M_{X,f}}, \Delta_{M_{X,f}}^+)$ and $(\Delta_{M_{Y,g}}, \Delta_{M_{Y,g}}^+)$ are isomorphic. We can define a positive set \check{H}_{\oplus}^1 for the *winding order* on \check{H}^1 for a large class of one-dimensional spaces, and explain the Jacklitch result as the computation of $(\check{H}^1, \check{H}_{\oplus}^1)$ for 1-solenoids ([16]). This can also be done by analyzing the direct limit order on \check{H}^1 , and we can recast the arguments of [3] and [9] in this framework.

In this paper, we extend the Jacklitch result to the nonorientable case with the following proof scheme. Given a nonorientable solenoid \overline{X} , there is a canonical double covering $\overline{p}: \overline{X} \rightarrow \overline{X}$ where \overline{X} is an orientable 1-solenoid ([17]). Then the quotient group $\check{H}^1(\overline{X})/\overline{p}^*(\check{H}^1(\overline{X}))$ is a homeomorphism invariant of \overline{X} . When (X, f) is an elementary presentation, the quotient group is isomorphic to $\Delta_{M_{X,f}}$, and when (X, f) is an orientable elementary presentation, the quotient group with the canonical order structure is isomorphic to $(\Delta_{M_{X,f}}, \Delta_{M_{X,f}}^+)$. The assumption of an elementary presentation is necessary and natural. If \overline{X} is orientable and elementary presented, then $\Delta_{M_{X,f}}$ is order isomorphic to $\check{H}^1(\overline{X})$; but this may fail if the presentation is not elementary ([16]). Likewise, if \overline{X} is not orientable and (X, f) is not an elementary presentation, then $\Delta_{M_{X,f}}$ need not be isomorphic to $\check{H}^1(\overline{X})/\overline{p}^*(\check{H}^1(\overline{X}))$.

2. ONE-DIMENSIONAL GENERALIZED SOLENOIDS AND ORDERED GROUPS

We review the definitions of one-dimensional generalized solenoids of Williams and ordered groups which will be used in later sections. As general references for the notions of one-dimensional generalized solenoids and their ordered group invariants we refer to [13], [14], [15].

One-dimensional generalized solenoids. Let X be a directed graph with vertex set \mathcal{V} and edge set \mathcal{E} , and $f: X \rightarrow X$ a continuous map. We state some axioms which might be satisfied by (X, f) .

- Axiom 0. (*Indecomposability*) (X, f) is indecomposable.
- Axiom 1. (*Nonwandering*) All points of X are nonwandering under f .
- Axiom 2. (*Flattening*) There is $k \geq 1$ such that for every $x \in X$ there is an open neighborhood U of x such that $f^k(U)$ is homeomorphic to $(-\epsilon, \epsilon)$.
- Axiom 3. (*Expansion*) There is a metric d compatible with the topology and positive constants C and λ with $\lambda > 1$ such that for all $n > 0$ and all points x, y on a common edge of X , if f^n maps the interval $[x, y]$ into an edge, then $d(f^n x, f^n y) \geq C\lambda^n d(x, y)$.
- Axiom 4. (*Nonfolding*) $f^n|_{X-\mathcal{V}}$ is locally one-to-one for every positive integer n .
- Axiom 5. (*Markov*) $f(\mathcal{V}) \subseteq \mathcal{V}$.

Let \overline{X} be the inverse limit space

$$\overline{X} = X \xleftarrow{f} X \xleftarrow{f} \cdots = \left\{ (x_0, x_1, x_2, \dots) \in \prod_0^\infty X \mid f(x_{n+1}) = x_n \right\}.$$

Suppose that Y is a topological space. We call Y a *1-dimensional generalized solenoid* or *1-solenoid* if there exists a directed graph X and a continuous map $f: X \rightarrow X$ such that (X, f) satisfies all six axioms and \overline{X} is homeomorphic to Y . And (X, f) is called a *presentation* of Y . If we can choose the direction of each edge in X so that the connection map $f: X \rightarrow X$ is orientation preserving, then we call (X, f) an *oriented presentation*, and Y an *orientable solenoid*. We call a point $x \in X$ a *non-branch point* if x has an open neighborhood which is homeomorphic to an open interval, and *branch point* otherwise. An *elementary presentation* (X, f) of a 1-solenoid is such that X is a wedge of circles and f leaves the unique branch point of X fixed.

Proposition 2.1 ([14, 5.2]). *For every 1-solenoid \overline{X} , there is an elementary presentation (Y, g) such that \overline{X} is homeomorphic to \overline{Y} .*

Notation 2.2. Suppose that (X, f) is a presentation of a 1-solenoid, and that $\mathcal{E} = \{e_1, \dots, e_n\}$ is the edge set of the directed graph X . For each edge $e_i \in \mathcal{E}$, we can give e_i the *partition* $\{I_{i,j}\}$, $1 \leq j \leq l(i)$, such that

- (1) the initial point of $I_{i,1}$ is the initial point of e_i ,
- (2) the terminal point of $I_{i,j}$ is the initial point of $I_{i,j+1}$ for $1 \leq j < l(i)$,
- (3) the terminal point of $I_{i,l(i)}$ is the terminal point of e_i ,
- (4) $f|_{\text{Int}I_{i,j}}$ is injective, and
- (5) $f(I_{i,j}) = e_{i,j}^{s(i,j)}$ where $e_{i,j} \in \mathcal{E}$, $s(i, j) = 1$ if the direction of $f(I_{i,j})$ agrees with that of $e_{i,j}$, and $s(i, j) = -1$ if the direction of $f(I_{i,j})$ is reverse to that of $e_{i,j}$.

By a *path* in X we mean a finite sequence $e_1^{s(1)} \cdots e_n^{s(n)}$ of edges such that, for $1 \leq i < n$, $s(i) = \pm 1$ represents the direction of e_i and the terminal vertex of $e_i^{s(i)}$ is the initial vertex of $e_{i+1}^{s(i+1)}$. Let \mathcal{E}^* denote the path set of X . Then the *wrapping rule* $\check{f}: \mathcal{E} \rightarrow \mathcal{E}^*$ associated with f is given by

$$\check{f}: e_i \mapsto e_{i,1}^{s(i,1)} \cdots e_{i,l(i)}^{s(i,l(i))},$$

and the *adjacency matrix* $M = M_{X,f}$ of (\mathcal{E}, \check{f}) is given by

$$M(i, k) = \#\{I_{i,j} \mid f(I_{i,j}) = e_k^{\pm 1}\}.$$

Definition 2.3 ([7, §2]). A *preordered group* is a pair (G, G_+) where G is an abelian group and the *positive cone* G_+ is a submonoid of G which generates G . We write $g_1 \leq g_2$ if $g_1 - g_2 \in G_+$ for $g_1, g_2 \in G$. If (G, G_+) satisfies the additional condition that $G_+ \cap -G_+ = \{0\}$, then (G, G_+) is called an **ordered group**.

For an oriented 1-solenoid \overline{X} , we can define a positive cone $\check{H}_\oplus^1(\overline{X})$ for the winding order on the first Čech cohomology group so that $(\check{H}^1(\overline{X}), \check{H}_\oplus^1(\overline{X}))$ is an ordered group ([16]).

Proposition 2.4 ([16]). *Suppose that (X, f) is an oriented presentation of a 1-solenoid \overline{X} with the adjacency matrix $M = M_{X,f}$.*

- (1) $(\check{H}^1(\overline{X}), \check{H}_\oplus^1(\overline{X}))$ is a quotient of (Δ_M, Δ_M^+) .

- (2) If (X, f) is an elementary presentation, then $(\check{H}^1(\overline{X}), \check{H}^1_{\oplus}(\overline{X}))$ is isomorphic to (Δ_M, Δ^+_M) .
- (3) Suppose that \overline{Y} is an orientable 1-solenoid. If $h: \overline{X} \rightarrow \overline{Y}$ is a homeomorphism, then the induced map $h^*: \check{H}^1(\overline{X}) \rightarrow \check{H}^1(\overline{Y})$ defines an isomorphism of ordered groups $(\check{H}^1(\overline{X}), \check{H}^1_{\oplus}(\overline{X}))$ and $(\check{H}^1(\overline{Y}), \check{H}^1_{\oplus}(\overline{Y}))$.

3. ORIENTABLE DOUBLE COVER OF 1-SOLENOIDS

Suppose that (X, f) is a presentation of a 1-solenoid \overline{X} . We will make a new directed graph \tilde{X} , a lifting $\tilde{f}: \tilde{X} \rightarrow \tilde{X}$ of $f: X \rightarrow X$ and a double covering map $p: \tilde{X} \rightarrow X$ such that $f \circ p = p \circ \tilde{f}$. If (X, f) is not orientable, then (\tilde{X}, \tilde{f}) is an orientable presentation of a 1-solenoid $\tilde{\overline{X}} = \varprojlim \tilde{X}$, and the induced map $\tilde{p}: \tilde{X} \rightarrow \tilde{\overline{X}}$ is a double covering map. We call (\tilde{X}, \tilde{f}) ($\tilde{\overline{X}}$, respectively) an **orientable double cover** of (X, f) (\overline{X} , respectively). The existence of this orientable double cover is known ([8], [17]). But we need to specify a concrete version of (\tilde{X}, \tilde{f}) for our computation.

Notation 3.1. We regard a directed edge e of X as a locally one-to-one map from $[0, 1]$ to e such that $e(0)$ is the initial point and $e(1)$ is the terminal point of e . So we can represent each point $x \in e$ as $e(t)$ (possibly $e(0) = e(1)$).

New graph \tilde{X} . Let $k \geq 1$ be an integer such that for every $x \in X$ there is an open neighborhood U_x of x such that $f^k(U_x)$ is homeomorphic to an open interval. Each edge e_i of X is the union of two subedges $e_{i,1} = e_i([0, \frac{1}{2}])$ and $e_{i,2} = e_i([\frac{1}{2}, 1])$. For a branch point b of X , let U_b be a neighborhood of b given as above. Then we can choose U_b to be small enough so that there exist two different subedges $e^1_b, e^2_b \in \{e_{i,j} \mid e_i \in \mathcal{E} \text{ and } j = 1, 2\}$ such that $f^k(U_b) \subset e^1_b \cup e^2_b$.

We can divide the set $\{e_{i,j} \mid b \text{ is a boundary point of } e_{i,j}\}$ into two disjoint sets

$$U^1_b = \{e_{i,j} \mid f^k(e_{i,j} \cap U_b) \subset e^1_b\} \text{ and } U^2_b = \{e_{i,j} \mid f^k(e_{i,j} \cap U_b) \subset e^2_b\}.$$

As in [15, Appendix B], without loss of generality we can redraw the graph X so that, for every branch point b , every subedge in U^1_b comes to b from the left or goes out from b to the left, and every subedge in U^2_b comes to b from the right or goes out from b to the right.

To make the double cover graph \tilde{X} , for each branch point b of X choose two points b_1 and b_2 which will be the fiber of b in \tilde{X} . Draw copies of U^1_b and U^2_b at b_1 and b_2 , and give the direction to each subedge in the liftings such that

- (1) at b_1 , every subedge contained in the copy of U^1_b goes out from b_1 to the left and every subedge in the copy of U^2_b comes in to b_1 from the right, and
- (2) at b_2 , every subedge contained in the copy of U^1_b comes in to b_2 from the left and every subedge in the copy of U^2_b goes out from b_2 to the right.

Now each subedge $e_{i,j}$ has two liftings in \tilde{X} , one of which has the same direction as $e_{i,j}$ and the other of which has the opposite direction. Let e^+_i be the lifting which has the same direction, and e^-_i the lifting with the opposite direction. Now connect $e^+_{i,1}$ to $e^+_{i,2}$, and $e^-_{i,1}$ to $e^-_{i,2}$ so that

- (1) for $E_{i,1} = e^+_{i,1} \cup e^+_{i,2}$, $E_{i,1}([0, \frac{1}{2}]) = e^+_{i,1}$ and $E_{i,1}([\frac{1}{2}, 1]) = e^+_{i,2}$, and
- (2) for $E_{i,2} = e^-_{i,1} \cup e^-_{i,2}$, $E_{i,2}([0, \frac{1}{2}]) = e^-_{i,2}$ and $E_{i,2}([\frac{1}{2}, 1]) = e^-_{i,1}$.

Then $E_{i,1}, E_{i,2}$ are liftings of e_i in \tilde{X} such that $E_{i,1}$ corresponds to e_i and that $E_{i,2}$ corresponds to e_i^{-1} .

Example 3.2. Suppose that X is Figure 1. Then its double cover \tilde{X} is given by Figure 2.

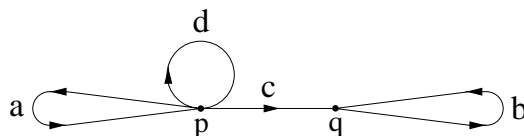


FIGURE 1. Directed graph X for a nonorientable solenoid

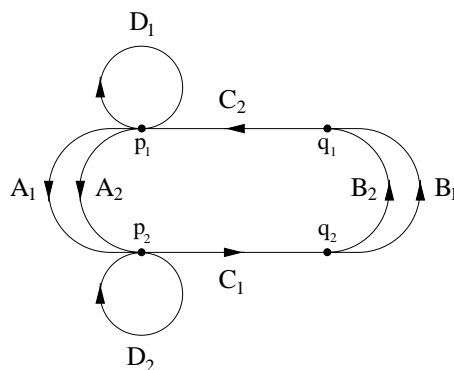


FIGURE 2. Double cover of X

Covering map. Define a map

$$p: \tilde{X} \rightarrow X \text{ by } \begin{cases} E_{i,1}(t) \mapsto e_i(t) \text{ and} \\ E_{i,2}(t) \mapsto e_i(1-t). \end{cases}$$

Then p is a 2-to-1 local homeomorphism. We call \tilde{X} a covering space of X and $p: \tilde{X} \rightarrow X$ a covering map.

The lifting $\tilde{f}: \tilde{X} \rightarrow \tilde{X}$. Suppose that (X, f) is a nonorientable presentation, that $e_i: [0, 1] \rightarrow X$ is an edge in X , that $E_{i,j}: [0, 1] \rightarrow \tilde{X}$, $j = 1, 2$, are the liftings of e_i in \tilde{X} , and $\tilde{f}(e_i) = e_{i(1)}^{s(1)} \cdots e_{i(n)}^{s(n)}$ where $s(j)$ represents the orientation. Then $p \circ E_{i,j}(0) = e_i(j-1)$ and $f \circ p \circ E_{i,j}(0)$ is the initial point of $e_{i(1)}^{s(1)}$ if $j = 1$ and the terminal point of $e_{i(n)}^{s(n)}$ if $j = 2$. Let

$$x_0 = \begin{cases} \text{the initial point of } E_{i(1),1} & \text{if } j = 1, s(1) = 1, \\ \text{the initial point of } E_{i(1),2} & \text{if } j = 1, s(1) = -1, \\ \text{the initial point of } E_{i(n),2} & \text{if } j = 2, s(n) = 1, \\ \text{the initial point of } E_{i(n),1} & \text{if } j = 2, s(n) = -1. \end{cases}$$

Then there exists a unique path $\Pi_{i,j}: [0, 1] \rightarrow \tilde{X}$ such that $p \circ \Pi_{i,j} = f \circ p \circ E_{i,j}$ and $\Pi_{i,j}(0) = x_0$. Define $\tilde{f}: \tilde{X} \rightarrow \tilde{X}$ by $\tilde{f}|_{E_{i,j}} = \Pi_{i,j} \circ E_{i,j}^{-1}$.

Proposition 3.3. *The map $\tilde{f}: \tilde{X} \rightarrow \tilde{X}$ is a well-defined orientation preserving continuous map such that $p \circ \tilde{f} = f \circ p$.*

Proof. Since a nonbranch point is contained in a unique edge, if we show that $E_{i,j}(0) = E_{k,l}(1) = E_{m,n}(0)$ implies $\Pi_{i,j}(0) = \Pi_{k,l}(1) = \Pi_{m,n}(0)$, then \tilde{f} is well defined and continuous. We only show that $E_{i,1}(0) = E_{k,1}(1)$ implies $\Pi_{i,1}(0) = \Pi_{k,1}(1)$.

Suppose $\tilde{f}(e_i) = e_{i(1)}^{s(1)} \cdots e_{i(n)}^{s(n)}$ and $\tilde{f}(e_k) = e_{k(1)}^{t(1)} \cdots e_{k(m)}^{t(m)}$. Let $[0, a]$ and $[b, 1]$ be subintervals of $[0, 1]$ such that $f \circ p \circ E_{i,1}([0, a]) = e_{i(1)}^{s(1)}([0, \frac{1}{2}])$ and $f \circ p \circ E_{k,1}([b, 1]) = e_{k(m)}^{t(m)}([\frac{1}{2}, 1])$. Then, by the construction of \tilde{X} and that p is a local homeomorphism, we have $e_{i(1)}^{s(1)}([0, \frac{1}{2}]) \in U_b^1$ if and only if $e_{k(m)}^{t(m)}([\frac{1}{2}, 1]) \in U_b^2$ where $b = f \circ p \circ E_{i,1}(0)$. If $\Pi_{i,1}(0) \neq \Pi_{k,1}(1)$, then the direction $\Pi_{i,1}([0, a])$ that goes out from $\Pi_{i,1}(0)$ and the direction $E_{k,1}([b, 1])$ that comes to $\Pi_{k,1}(1)$ are the same. Hence $p \circ \Pi_{i,1}([0, a]) \in U_b^1$ if and only if $E_{k,1}([b, 1]) \in U_b^1$, a contradiction. Therefore $\Pi_{i,1}(0) = \Pi_{k,1}(1)$, and \tilde{f} is well defined. That $\tilde{f}: \tilde{X} \rightarrow \tilde{X}$ is a continuous map and that p and \tilde{f} satisfy $p \circ \tilde{f} = f \circ p$ are obvious.

Suppose that $[a, b] \subset [0, 1]$ is an interval such that $f \circ p \circ E_{i,j}|_{[a,b]}$ is a local homeomorphism and $f \circ p \circ E_{i,j}([a, b]) = e_k^{-1}$. Then $p \circ \Pi_{i,j}([a, b]) = f \circ p \circ E_{i,j}([a, b]) = e_k^{-1}$ implies $\Pi_{i,j}([a, b]) = E_{k,2}$, and $p \circ \Pi_{i,j} = f \circ p \circ E_{i,j}$ is a local homeomorphism on $[a, b]$ with $\Pi_{i,j}(a) = E_{k,2}(0)$ and $\Pi_{i,j}(b) = E_{k,2}(1)$. Therefore \tilde{f} is orientation preserving. □

Example 3.4. Suppose that X and \tilde{X} are as in Example 3.2. If $f: X \rightarrow X$ is given by

$$a \mapsto c^{-1}dac, \quad b \mapsto cbc^{-1}, \quad c \mapsto bc^{-1}a, \quad \text{and} \quad d \mapsto c^{-1}acb^{-1},$$

then (X, f) is a presentation of a nonorientable solenoid and the lifting $\tilde{f}: \tilde{X} \rightarrow \tilde{X}$ is given by

$$\begin{aligned} A_1 &\mapsto C_2D_1A_1C_1, & B_1 &\mapsto C_1B_1C_2, & C_1 &\mapsto B_1C_2A_1, & D_1 &\mapsto C_2A_1C_1B_2 \text{ and} \\ A_2 &\mapsto C_2A_2D_2C_1, & B_2 &\mapsto C_1B_2C_2, & C_2 &\mapsto A_2C_1B_2, & D_2 &\mapsto B_1C_2A_2C_1. \end{aligned}$$

Lemma 3.5. *Suppose that (X, f) is a presentation of a nonorientable 1-solenoid with the edge set \mathcal{E} . Then for all $e_1, e_2 \in \mathcal{E}$, there is a positive integer $n = n(e_1, e_2)$ such that $e_1, e_1^{-1} \in \tilde{f}^n(e_2)$ for every positive integer $n \geq n(e_1, e_2)$.*

Proof. If there exists an edge $e \in \mathcal{E}$ such that e and e^{-1} are factors of $\tilde{f}^n(a)$ for some positive integer n and an edge $a \in \mathcal{E}$, then by Theorem 1.2 of [14] for all $e_1, e_2 \in \mathcal{E}$, $e_1, e_1^{-1} \in \tilde{f}^n(e_2)$ for some positive integer n . So we need to show the existence of such edges e and a .

Assume that, for any two edges a and b , only one of b or b^{-1} is a factor of $\tilde{f}^n(a)$ for every positive integer $n \geq n(a, b)$. Then the edge set \mathcal{E} is divided into two disjoint sets \mathcal{P} and \mathcal{N} such that for any $a_1, a_2 \in \mathcal{P}$ and $b_1, b_2 \in \mathcal{N}$, a_2 and b_2^{-1} are factors of $\tilde{f}^n(a_1)$ and a_2^{-1} and b_2 are factors of $\tilde{f}^n(b_1)$ for every positive integer n . But if we give opposite direction to each edge $b \in \mathcal{N}$ so that $b^{-1} \in \mathcal{E}$, then the connection map $f: X \rightarrow X$ is orientable, which is a contradiction. Therefore b and b^{-1} are factors of $\tilde{f}^n(a)$ for all edges $a, b \in \mathcal{E}$ and every positive integer $n \geq n(a, b)$. □

Proposition 3.6. *Suppose that (X, f) is a presentation of a 1-solenoid. Then (\tilde{X}, \tilde{f}) is an orientable double cover of (X, f) .*

- (1) *If (X, f) is orientable, then (\tilde{X}, \tilde{f}) is the disjoint union of orientable presentations (X_1, f_1) and (X_2, f_2) of 1-solenoids such that $p|_{X_1}$ is an orientation preserving homeomorphism, $p|_{X_2}$ is an orientation reversing homeomorphism, and (X_i, f_i) is topologically conjugate to (X, f) , $i = 1, 2$.*
- (2) *If (X, f) is nonorientable, then (\tilde{X}, \tilde{f}) is an orientable presentation of a 1-solenoid.*

Proof. (1) follows from Lemma 1.5 of [8] and the constructions of (\tilde{X}, \tilde{f}) and $p: \tilde{X} \rightarrow X$.

(2) The Expansion and Markov Axioms come from the construction of (\tilde{X}, \tilde{f}) . The Indecomposability and Nonwandering Axioms are derived from Lemma 3.5.

Since p is a local homeomorphism with $p(\tilde{X} \setminus \tilde{V}) = X \setminus \mathcal{V}$ and $f|_{X \setminus \mathcal{V}}$ is a local homeomorphism, $\tilde{f}|_{\tilde{X} \setminus \tilde{V}}$ is a local homeomorphism, and \tilde{f} satisfies the Nonfolding Axiom.

Suppose that b is a branch point of \tilde{X} and that U is an open neighborhood of $p(b)$ such that $f^k(U)$ is an interval for some integer $k \geq 1$. Then there exists an open neighborhood V of b such that $p|_V: V \rightarrow U$ is a homeomorphism. Since $p \circ \tilde{f}^k(V) = f^k \circ p(V) = f^k(U)$ is an interval and p is a local homeomorphism, $\tilde{f}^k(V)$ is an interval. So \tilde{f} satisfies the Flattening Axiom. Therefore (\tilde{X}, \tilde{f}) is an orientable presentation of a 1-solenoid. □

Proposition 3.7 ([8, 1.6]). *Suppose that \overline{X} and \overline{Y} are 1-solenoids with orientable double covers $\tilde{\overline{X}}$ and $\tilde{\overline{Y}}$, respectively. Then a homeomorphism $h: \overline{X} \rightarrow \overline{Y}$ lifts to an orientation preserving homeomorphism $\tilde{h}: \tilde{\overline{X}} \rightarrow \tilde{\overline{Y}}$.*

Sketch of Proof. We consider an arc component ℓ of a 1-solenoid as a locally one-to-one map from \mathbb{R} to ℓ so that each point $x \in \ell$ is represented as $\ell(t)$ and that $\{\ell_1(t), \ell_2(-t)\}$ in its orientable cover is a fiber of $\ell(t)$.

Let ℓ be an arc component of \overline{X} and let ℓ_1 and ℓ_2 be liftings of ℓ in $\tilde{\overline{X}}$ such that ℓ_1 has the same direction as ℓ , and ℓ_2 has the opposite direction. Suppose $h(\ell) = \ell'$. Then ℓ' is an arc component of \overline{Y} , and we give the direction to ℓ' so that $h|_\ell$ is orientation preserving. Let ℓ'_1 and ℓ'_2 be liftings of ℓ' in $\tilde{\overline{Y}}$ such that ℓ'_1 has the same direction as ℓ' , and ℓ'_2 has the opposite direction. Define $\tilde{h}: \tilde{\overline{X}} \rightarrow \tilde{\overline{Y}}$ by $\ell_i(t) \mapsto \ell'_i(t)$. Then \tilde{h} is a well-defined homeomorphism satisfying $p_Y \circ \tilde{h} = h \circ p_X$. □

Invariants for orientable double covers. Suppose that \overline{X} is a 1-solenoid with a presentation (X, f) . Let $X_k = X$ be the k th coordinate space of \overline{X} , and \mathcal{E}_k the set of directed edges in X_k . Let $C(\mathcal{E}_k, \mathbb{Z})$ denote the set of integer-valued functions on \mathcal{E}_k . If e is an edge in \mathcal{E}_{k+1} and $f: X_{k+1} \rightarrow X_k$ is the connection map, then $f(e)$ is a path $e_1^{s(1)} \cdots e_n^{s(n)}$ in X_k where e_i is an element of \mathcal{E}_k and $s(i)$ is 1 or -1 according to whether the orientation is preserved or reversed. Hence f induces a map

$$f^*: C(\mathcal{E}_k, \mathbb{Z}) \rightarrow C(\mathcal{E}_{k+1}, \mathbb{Z}) \text{ defined by } g \mapsto g \circ f$$

where $g \circ f(e) = \sum_{i=1}^n s(i)g(e_i)$ such that $f(e) = e_1^{s(1)} \cdots e_n^{s(n)}$ in X_k .

Remark 3.8. By Proposition 2.1, a 1-solenoid \overline{X} is homeomorphic to an elementary presented 1-solenoid. So without loss of generality we can assume that \overline{X} is elementary presented.

Suppose that (X, f) is an elementary presentation of a *nonorientable* 1-solenoid with the edge set \mathcal{E} and that (\tilde{X}, \tilde{f}) is its orientable double cover with the edge set $\tilde{\mathcal{E}}$ and the covering map $p: \tilde{X} \rightarrow X$. Let M be the $n \times n$ adjacency matrix of (\mathcal{E}, \tilde{f}) , and \tilde{M} the $2n \times 2n$ adjacency matrix of $(\tilde{\mathcal{E}}, \tilde{f})$.

Proposition 3.9. *Let M and \tilde{M} be given as above. Then there are two $n \times n$ nonnegative integer matrices M_1 and M_2 such that*

$$M = M_1 + M_2, \quad \check{H}^1(\overline{X}) = \varinjlim C(\mathcal{E}_k, \mathbb{Z}) \cong \Delta_{M_1 - M_2}, \quad \text{and} \quad \tilde{M} = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix}.$$

Proof. Given presentation (X, f) , let $\{I_{i,j}\}$ be the partition of edge e_i given in Notation 2.2. If M_1 and M_2 are the matrices given by

$$M_1(i, k) = \#\{I_{i,j} \mid f(I_{i,j}) = e_k\} \text{ and } M_2(i, k) = \#\{I_{i,j} \mid f(I_{i,j}) = e_k^{-1}\},$$

then we have $M = M_1 + M_2$.

Let X_k be the k th coordinate space of \overline{X} and $g \in C(\mathcal{E}_k, \mathbb{Z})$. Let $f^*: C(\mathcal{E}_k, \mathbb{Z}) \rightarrow C(\mathcal{E}_{k+1}, \mathbb{Z})$ be the induced homomorphism from $f: X_{k+1} \rightarrow X_k$. For edges $e_i \in \mathcal{E}_{k+1}$ and $e_j \in \mathcal{E}_k$, if $f(e_i)$ covers e_j $n_{i,j}$ -times in an orientation preserving way and $m_{i,j}$ -times in an orientation reversing way, then

$$g \circ f(e_i) = \sum (n_{i,j} - m_{i,j})g(e_j).$$

It is not difficult to see that the matrix $(n_{i,j})$ is equal to M_1 and that $(m_{i,j})$ is equal to M_2 . Hence $f^*: C(\mathcal{E}_k, \mathbb{Z}) \rightarrow C(\mathcal{E}_{k+1}, \mathbb{Z})$ is denoted by the matrix $M_1 - M_2$, and

$$\check{H}^1(\overline{X}) = \varinjlim_{f^*} C(\mathcal{E}_k, \mathbb{Z}) \cong \varinjlim_{M_1 - M_2} \mathbb{Z}^n = \Delta_{M_1 - M_2}.$$

That $\tilde{M} = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix}$ is an immediate conclusion from the construction of (\tilde{X}, \tilde{f}) . □

Then we have the following commuting diagram:

$$\begin{array}{ccccc} \mathbb{Z}^n & \xrightarrow{M} & \mathbb{Z}^n & \longrightarrow & \dots \\ (I \ I) \uparrow & & \uparrow (I \ I) & & \\ \mathbb{Z}^{2n} & \xrightarrow{\tilde{M}} & \mathbb{Z}^{2n} & \longrightarrow & \dots \\ (-I \ I) \uparrow & & \uparrow (-I \ I) & & \\ \mathbb{Z}^n & \xrightarrow{M_1 - M_2} & \mathbb{Z}^n & \longrightarrow & \dots \end{array}$$

It is not difficult to see from the diagram that there is a well-defined homomorphism $p_D^*: \Delta_{M_1 - M_2} \rightarrow \Delta_{\tilde{M}}$ induced by $\begin{pmatrix} I \\ -I \end{pmatrix}: \mathbb{Z}^n \rightarrow \mathbb{Z}^{2n}$ such that Δ_M is isomorphic to $\Delta_{\tilde{M}}/p_D^*(\Delta_{M_1 - M_2})$. Remark that $\Delta_{\tilde{M}}/p_D^*(\Delta_{M_1 - M_2})$ is a preordered group with a positive cone $\Delta_{\tilde{M}}^+/p_D^*(\Delta_{M_1 - M_2})$. Then we have the following lemma.

Lemma 3.10. $\left(\Delta_{\tilde{M}}/p_D^*(\Delta_{M_1 - M_2}), \Delta_{\tilde{M}}^+/p_D^*(\Delta_{M_1 - M_2})\right) \cong (\Delta_M, \Delta_M^+)$.

Recall by Proposition 2.4 that there is an order preserving quotient map

$$q: (\Delta_{\tilde{M}}, \Delta_{\tilde{M}}^+) \rightarrow (\check{H}^1(\tilde{X}), \check{H}_{\oplus}^1(\tilde{X})).$$

Suppose that $\tilde{X}_k = \tilde{X}$ is the k th coordinate of \tilde{X} , that $\pi_k: \tilde{X} \rightarrow \tilde{X}_k$ is the projection map, and that $\tilde{\mathcal{E}}_k$ is the set of directed edges in \tilde{X}_k . By a *path* in \tilde{X} we mean a finite sequence $e_1^{s(1)} \cdots e_n^{s(n)}$ of edges such that, for $1 \leq i < n$, $s(i) = \pm 1$ represents the direction of e_i and the terminal vertex of $e_i^{s(i)}$ is the initial vertex of $e_{i+1}^{s(i+1)}$. We write $e^s \in \wp$ if \wp is a path and e is an edge such that e^s is a factor of \wp . A *cycle* is a path $e_1^{s(1)} \cdots e_n^{s(n)}$ such that the terminal vertex of $e_n^{s(n)}$ is the initial vertex of $e_1^{s(1)}$. We say that a function g in $C(\tilde{\mathcal{E}}_k, \mathbb{Z})$ is *zero on cycles* if for every cycle $e_1^{s(1)} \cdots e_n^{s(n)}$ the sum $\sum s(i)g(e_i)$ vanishes.

Lemma 3.11 ([4], [16]). *Suppose that $\pi_k: \tilde{X} \rightarrow \tilde{X}_k$ is the projection map. Then the kernel of $q: \Delta_{\tilde{M}} \rightarrow \check{H}^1(\tilde{X})$ is generated by $\{\pi_k \circ g \mid g \text{ is zero on cycles in } \tilde{X}_k\}$.*

Then it is not difficult to verify that $q \circ p_D^*: \check{H}^1(\tilde{X}) \rightarrow \check{H}^1(\tilde{X})$ is the same as $\bar{p}^*: \check{H}^1(\tilde{X}) \rightarrow \check{H}^1(\tilde{X})$ induced by the double covering map $\bar{p}: \tilde{X} \rightarrow \tilde{X}$. Since $\text{Ker}(q) \subset p_D^*(\check{H}^1(\tilde{X}))$ from the above commuting diagram and Lemma 3.11, we have from Lemma 3.10 that $\check{H}^1(\tilde{X})/q \circ p_D^*(\check{H}^1(\tilde{X}))$ is order isomorphic to Δ_M . Therefore we have the following proposition.

Proposition 3.12. $(\check{H}^1(\tilde{X})/\bar{p}^*(\check{H}^1(\tilde{X})), \check{H}_{\oplus}^1(\tilde{X})/\bar{p}^*(\check{H}^1(\tilde{X}))) \cong (\Delta_M, \Delta_M^+)$.

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