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EXISTENCE OF C^{∞} LOCAL SOLUTIONS OF THE COMPLEX MONGE-AMPÈRE EQUATION

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Abstract. We prove the C^{∞} local solvability of the n-dimensional complex Monge-Ampère equation $\det(u_{i\bar{j}}) = K(z) f(z, u, \nabla u)$, $f > 0$, in a neighborhood of any point z_0 where $K(z_0) = 0$ but $dK(z_0) \neq 0$.

1. Introduction

Our motivation to study equations of Monge-Ampère type comes from their close connection with problems in complex analysis (see [1], [5]), in differential geometry (see [2], [6], [7], [10], [11], [12]) and in physics (see [4]).

Here we consider the complex Monge-Ampère equation

$$\det\left(u_{i\bar{j}}\right) = K(z) f(z, u, \nabla u) \tag{1.1}$$

in a neighborhood V of $z_0 \in \mathbb{C}^n$, where $f(z, u, p)$ is a positive C^{∞} function for $(z, u, p) \in V \times \mathbb{R} \times \mathbb{R}^{2n}$ and $K \in C^{\infty}(V, \mathbb{R})$. For $z \in \mathbb{C}^n$, we denote $x = (\operatorname{Re} z, \operatorname{Im} z) = (x_1, x_2, ..., x_n, ..., x_{2n}) \in \mathbb{R}^{2n}$ and $u_{i\bar{j}} = \dfrac{\partial^2 u}{\partial z_i \partial \overline{z_j}}$.

We study the existence of a local smooth solution to the problem (1.1) when

$$K(z_0) = 0 \text{ and } dK(z_0) \neq 0. \tag{1.2}$$

The main result of this paper is the following:

Theorem 1. *Under condition* (1.2), *equation* (1.1) *in* $\mathbb{C}^n$ *has a* C^{∞} *local solution in a neighborhood of* z_0.

2. Proof of Theorem 1

Let us recall the result of G.Nakamura and Y.Maeda ([8], [9]).

Given $x_0 \in \mathbb{R}^m$, U a neighborhood of x_0, $u_0 \in C^{\infty}(U, \mathbb{R})$, a nonlinear partial differential equation of order q with C^{∞} coefficients in U and

$$\Phi(x, D^{\alpha} u)_{|\alpha| \leq q} = g, \tag{2.1}$$

we can state

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Theorem 2 ([8], Th. 4)**.** *Assume that the linearized equation of* (2.1) *is of real principal type at* u_0 *and* x_0*. Then there exist an open neighborhood* ω *of* x_0*,* $s_0 \in \mathbb{N}$ *and* $\eta > 0$ *such that for every* $g \in C^{\infty}(\omega)$ *satisfying*

$$\left\| g - \Phi\left(x, D^{\alpha} u_0\right)_{|\alpha| \leq q} \right\|_{s_0} \leq \eta,$$

there exists $u \in C^{\infty}(\omega, \mathbb{R})$*, as a solution to equation* (2.1)*.* (*Here* $\| \|_{s_0}$ *is the Sobolev norm. The neighborhood* ω *and the constants* s_0*,* η *are independent from* u_0*, where* ω *can be chosen small enough.*)

Recall that a differential operator P is said to be of real principal type at $x_0 \in V$ if its principal symbol p is real and has no null bicharacteristics trapped over x_0.

Using condition (1.2) and invariance of equation (1.1) under rotation, we may assume, without loss of generality, that $z_0 = 0$ and

$$K(z) = x_n + \bigcirc\left(|z|^2\right). \tag{2.2}$$

Step 1. Determination of u_0.
Set $g(x, u, \nabla u) = K(z) f(z, u, \nabla u)$, and let $x = (x', x_n) \in \mathbb{R}^{2n-1} \times \mathbb{R}$.
Given an integer $N \geq s_0 + n + 1$, we consider the analytic Cauchy problem

$$\begin{cases} \det\left(u_{i\bar{j}}\right) - \sum\limits_{|\alpha|+|\beta|+|\gamma|<N} \frac{1}{(\alpha,\beta,\gamma)!} \partial_x^{\alpha} \partial_u^{\beta} \partial_p^{\gamma} g(0)\, x^{\alpha} u^{\beta} (\nabla u)^{\gamma} = 0, \\ u_{|x_n=0} = \frac{1}{2} |x'|^2, \\ \dfrac{\partial u}{\partial x_n}_{|x_n=0} = h(x') \end{cases} \tag{2.3}$$

where h is an analytic function such that $h(0) = 0$.

The coefficient of $\partial_{x_n}^2 u$ in the left-hand side of equation (2.3) is equal to

$$\det\left(u_{i\bar{j}}\right)_{1 \leq i,j \leq n-1}$$

which is equal to 1 at the origin (from initial data). Then (2.3) is a noncharacteristic Cauchy problem in a neighborhood of zero, and the existence of an analytic solution u° in a neighborhood of $x_0 = 0$ follows from the Cauchy-Kowalevsky theorem.

From the Taylor formula and using (2.3), we get

$$\begin{aligned} \det\left(u^{\circ}_{i\bar{j}}\right) - g(x, u^{\circ}, \nabla u^{\circ}) &= \det\left(u^{\circ}_{i\bar{j}}\right) - \sum_{|\delta|<N} \frac{1}{\delta!} \partial^{\delta}_{(x,u,p)} g(0) (x, u^{\circ}, \nabla u^{\circ})^{\delta} \\ &- \left(g(x, u^{\circ}, \nabla u^{\circ}) - \sum_{|\delta|<N} \frac{1}{\delta!} \partial^{\delta}_{(x,u,p)} g(0) (x, u^{\circ}, \nabla u^{\circ})^{\delta} \right) = \circ\left(|x|^{N-1}\right). \end{aligned}$$

Then we can deduce for $|\alpha| < N$,

$$\partial_x^{\alpha}\left[\det\left(u^{\circ}_{i\bar{j}}\right) - g(x, u^{\circ}, \nabla u^{\circ})\right] = \circ(1), \ x \to 0. \tag{2.4}$$

Step 2. One technical lemma.
Let us denote

$$F[u] = F\left(x, u, \nabla u, D^2 u\right) = \det\left(u_{i\bar{j}}\right). \tag{2.5}$$

Lemma 1. *For* $1 \leq i, j, a, b \leq n$, *we have*

$$F\frac{\partial^2 F}{\partial u_{a\overline{b}}\partial u_{i\overline{j}}} = \frac{\partial F}{\partial u_{a\overline{b}}}\frac{\partial F}{\partial u_{i\overline{j}}} - \frac{\partial F}{\partial u_{i\overline{b}}}\frac{\partial F}{\partial u_{a\overline{j}}}. \tag{2.6}$$

Proof. The proof of this lemma is similar to that given by C.Zuily in [13].

If we denote by H the matrix $\left(u_{i\overline{j}}\right)_{1\leq i,j\leq n}$, the matrix of cofactors is then $\widetilde{H} = \left(\frac{\partial F}{\partial u_{i\overline{j}}}\right)$. By differentiating the identity

$${}^t\widetilde{H}H = \det H.Id = F.Id$$

with respect to $u_{a\overline{b}}$, we get

$$({}^t\widetilde{H})'H + {}^t\widetilde{H}H' = (F.Id)'\,.$$

Multiplying this equality on the right by ${}^t\widetilde{H}$ and using that $H{}^t\widetilde{H} = F.Id$, we find

$$\underbrace{F({}^t\widetilde{H})'}_{(1)} = \underbrace{(F.Id)'\,{}^t\widetilde{H}}_{(2)} - \underbrace{{}^t\widetilde{H}H'{}^t\widetilde{H}}_{(3)}.$$

The (i, j) term of the matrix (1) is given by

$$F\frac{\partial}{\partial u_{a\overline{b}}}\left(\frac{\partial F}{\partial u_{j\overline{i}}}\right) = F\frac{\partial^2 F}{\partial u_{j\overline{i}}\partial u_{a\overline{b}}}.$$

On the other hand, since $(F.Id)'$ is the diagonal matrix $\frac{\partial F}{\partial u_{a\overline{b}}}.Id$, the general term of the matrix (2) is given by $\frac{\partial F}{\partial u_{a\overline{b}}}.\frac{\partial F}{\partial u_{j\overline{i}}}$.

The general term of the matrix H' is $\alpha_{rs} = \delta^a_r.\delta^s_b$, where $\delta^i_j = \begin{cases} 1, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$. Then, the general term of $H'{}^t\widetilde{H}$ is

$$\beta_{kl} = \sum_{s=1}^{n}\alpha_{ks}\frac{\partial F}{\partial u_{l\overline{s}}} = \frac{\partial F}{\partial u_{l\overline{b}}}.\delta^a_k.$$

Finally, the (i, j) term of the matrix (3) is given by

$$\sum_{l=1}^{n}\frac{\partial F}{\partial u_{l\overline{i}}}\beta_{lj} = \frac{\partial F}{\partial u_{a\overline{i}}}.\frac{\partial F}{\partial u_{j\overline{b}}}.$$

□

Step 3. The linearized equation of (1.1) at u° is of real principal type at the origin.

The principal symbol of this linearized equation is

$$p\,(x, \xi) = \sum_{1\leq i,j\leq n}\frac{\partial F}{\partial u_{i\overline{j}}}\left[u^\circ\right]\xi_i\overline{\xi_j}. \tag{2.7}$$

When $\xi = (\xi', \xi'') = \eta + i\zeta$, $(\xi', \xi'') \in \mathbb{C}^{n-1} \times \mathbb{C}$, $(\eta, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n$. $p\,(x, \xi)$ is a real symbol, and we have seen in Step 1 that

$$\frac{\partial F}{\partial u_{n\overline{n}}}\left(0, u^\circ\left(0\right), \nabla u^\circ\left(0\right), D^2u^\circ\left(0\right)\right) = \det\left(u^\circ_{i\overline{j}}\right)_{1\leq i,j\leq n-1} = 1. \tag{2.8}$$

Thus, if we denote the matrix $\left(u^{\circ}_{i\overline{j}}\right)_{1\leq i,j\leq n-1}$ by A, the matrix of cofactors is then

$$\tilde{A}=\left(\frac{\partial^2 F}{\partial u_{n\overline{n}}\partial u_{i\overline{j}}}\right)_{1\leq i,j\leq n-1}.$$

Using the identity $\tilde{A}=(\det A)\,A^{-1}$ and the initial condition $A\,(0)=Id$, we can deduce

$$\sum_{1\leq i,j\leq n-1}\frac{\partial^2 F}{\partial u_{i\overline{j}}\partial u_{n\overline{n}}}\left(0,u^{\circ}\,(0)\,,\nabla u^{\circ}\,(0)\,,D^2u^{\circ}\,(0)\right)\xi_i\overline{\xi_j}>0,\ \forall\xi'\neq 0. \tag{2.9}$$

Then using (2.6) we get

$$p\,(x,\xi)=\frac{\partial F}{\partial u_{n\overline{n}}}\,[u^{\circ}]\left\{\left|\xi_n+\frac{\sum_{j=1}^{n-1}\frac{\partial F}{\partial u_{j\overline{n}}}\xi_j}{\frac{\partial F}{\partial u_{n\overline{n}}}[u^{\circ}]}\right|^2+\frac{F[u^{\circ}]}{\left(\frac{\partial F}{\partial u_{n\overline{n}}}[u^{\circ}]\right)^2}\sum_{i,j=1}^{n-1}\frac{\partial^2 F}{\partial u_{n\overline{n}}\partial u_{i\overline{j}}}\,[u^{\circ}]\,\xi_i\overline{\xi_j}\right\}. \tag{2.7'}$$

We can deduce from (2.9) that $p\,(x,\xi)=0$ if and only if one of the following two conditions is satisfied:

$$\left\{\begin{array}{l} F\,[u^{\circ}]\,(x)<0,\\ \left|\xi_n+\frac{\sum_{j=1}^{n-1}\frac{\partial F}{\partial u_{j\overline{n}}}\xi_j}{\frac{\partial F}{\partial u_{n\overline{n}}}[u^{\circ}]}\right|^2=\frac{-F[u^{\circ}]}{\left(\frac{\partial F}{\partial u_{n\overline{n}}}[u^{\circ}]\right)^2}\sum_{i,j=1}^{n-1}\frac{\partial^2 F}{\partial u_{n\overline{n}}\partial u_{i\overline{j}}}\,[u^{\circ}]\,\xi_i\overline{\xi}_j \end{array}\right. \tag{2.10}$$

or

$$F\,[u^{\circ}]\,(x)=0 \text{ and } \xi_n=-\left(\frac{\partial F}{\partial u_{n\overline{n}}}\,[u^{\circ}]\right)^{-1}\sum_{j=1}^{n-1}\frac{\partial F}{\partial u_{j\overline{n}}}\,[u^{\circ}]\,\xi_j. \tag{2.11}$$

* First case. When (2.10) holds, then we have $\frac{\partial p}{\partial\xi_n}\,(x,\xi)\neq 0.$

* Second case. Consider now the case when (2.11) holds.

For $1\leq k\leq n$, using (2.11), we get by differentiating (2.7′) with respect to x_k

$$\frac{\partial F}{\partial u_{n\overline{n}}}\,[u^{\circ}]\,(0)\,\frac{\partial p}{\partial x_k}\,(0,\xi)=\frac{\partial}{\partial x_k}\left[\det\left(u^{\circ}_{i\overline{j}}\right)\right]_{|x=0}\sum_{i,j=1}^{n-1}\frac{\partial^2 F}{\partial u_{n\overline{n}}\partial u_{i\overline{j}}}\,[u^{\circ}]\,(0)\,\xi_i\overline{\xi_j}.$$

From (2.9), $q\,(0,\xi)=\sum_{1\leq i,j\leq n-1}\frac{\partial^2 F}{\partial u_{n\overline{n}}\partial u_{i\overline{j}}}\,[u^{\circ}]\,\xi_i\overline{\xi_j}>0.$

Using (2.4) and the definition of $g\,(x,u,\nabla u)$, we have

$$\det\left(u^{\circ}_{i\overline{j}}\right)=K\,(z)\,f\,(z,u^{\circ},\nabla u^{\circ})+\bigcirc\left(|z|^2\right),z\to 0,$$

where $K\,(z)=x_n+\bigcirc\left(|z|^2\right)$, $f\,(z,u^{\circ},\nabla u^{\circ})>0$ and $K\,(0)=0$. Then

$$\left\{\begin{array}{l} \frac{\partial F}{\partial u_{n\overline{n}}}\,[u^{\circ}]\,(0)\,\frac{\partial p}{\partial x_k}\,(0,\xi)=0,\ \ \forall k\in[1,2n]\setminus\{n\}\cap\mathbb{N},\\ \frac{\partial F}{\partial u_{n\overline{n}}}\,[u^{\circ}]\,(0)\,\frac{\partial p}{\partial x_n}\,(0,\xi)=f\,(0,u^{\circ}\,(0)\,,\nabla u^{\circ}\,(0))\,q\,(0,\xi)\,. \end{array}\right. \tag{2.12}$$

Consider the bicharacteristic $(x(t), \eta(t), \zeta(t))$ descended from the point $(0, \xi_0)$. We have $x(0) = 0$, $\xi(0) = \xi_0$ and $\dot{x}(0) = 0$. Using (2.12), we get

$$\begin{cases} \dot{\zeta}_k(0) = 0, \ \forall\ 1 \leq k \leq n, \\ \dot{\eta}_k(0) = 0, \ \forall\ 1 \leq k \leq n-1, \\ \dot{\eta}_n(0) = -\left[\frac{\partial F}{\partial u_{n\overline{n}}}[u^\circ](0)\right]^{-1} f(0)\, q(0, \xi_0). \end{cases}$$

We can deduce that

$$\begin{aligned} \ddot{x}_n(0) &= \frac{d}{dt}\frac{\partial p}{\partial \eta_n}(x(t), \eta(t), \zeta(t))_{|t=0} \\ &= \dot{\eta}_n(0)\frac{\partial^2 p}{\partial \eta_n^2}(0, \xi_0) \\ &= 2\dot{\eta}_n(0)\frac{\partial F}{\partial u_{n\overline{n}}}[u^\circ](0). \end{aligned}$$

Finally

$$\ddot{x}_n(0) = -2f(0)\, q(0, \xi_0) < 0 \quad \text{when } \xi_0 \neq 0.$$

Thus, p is a real principal type symbol at 0.

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