PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 4, Pages 1169–1176
S 0002-9939(02)06887-9
Article electronically published on October 24, 2002

ON THE REGULARIZED WHITTAKER-KOTEL'NIKOV-SHANNON SAMPLING FORMULA

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(Communicated by David Sharp)

ABSTRACT. Error estimation is given for the regularized Whittaker-Kotel'nikov-Shannon (WKS) sampling formula, which was found to be accurate and robust for numerically solving partial differential equations. The result improves the convergence rate of existing results.

1. INTRODUCTION

The Whittaker-Kotel'nikov-Shannon sampling formula [1, 2, 3] is one of the fundamental theorems of information and communication theory. It states that for any bandlimited function with bandwidth πW,

$$f(t) = \sum_{k=-\infty}^{\infty} f(\frac{k}{W}) sinc(Wt-k) \ (t \in \mathcal{R}), \tag{1.1}$$

where $sinc(t) = \frac{sin \pi t}{\pi t}$ for $t \neq 0$ and $sinc(0) = 1$.

The WKS sampling theorem and its application have been widely studied [4, 5, 6]. It uses an infinite number of terms to exactly interpolate a bandlimited signal from its samples. However, only a finite number of terms can be used in practice. The truncation error of the WKS sampling formula is substantial due to the slow decay of sinc-function.

The main concern is how to use as few terms as possible to achieve the highest possible approximation accuracy. It is expected that a modification of the sinc-function will yield a better convergence rate of the WKS sampling formula. Many particular convergence factors have been considered. Define the discrete convolution operators

$$(S_W f)(t) = \sum_{k=-\infty}^{\infty} f(\frac{k}{W}) sinc(bWt-k)\theta(Wt-k) \ (t \in \mathcal{R}). \tag{1.2}$$

Jagerman [7] chose $b = 1$ and $\theta(t) = [sinc(at/m)]^m, \ 0 < a < 1, m \in \mathcal{N}$.

Received by the editors April 23, 2001 and, in revised form, November 8, 2001.

2000 *Mathematics Subject Classification.* Primary 41A80, 41A30; Secondary 65D25, 65G99, 94A24.

Key words and phrases. Whittaker-Kotel'nikov-Shannon's sampling formula, regularization, error estimate.

Butzer et al. [8] generalized it to $0 < b < 2$ and $\theta \in \beta^p_{a\pi}$ for some $a < b < 2-a$ with $\theta(0) = b$. Here for $a \geq 0$, $1 \leq p \leq \infty$, $\beta^p_{a\pi}$ denotes the class of all entire functions f of exponential type $a\pi$ which belong to $L^p(\mathcal{R})$ when restricted to $\mathcal{R}$.

Gervais et al. [9] constructed $\theta(t)$ that can behave like $\theta(t) = \mathcal{O}(e^{-w(|t|)})$ as $t \to \infty$ provided w, besides certain regularity conditions, satisfies $\int_a^\infty \frac{w(t)}{t^2}dx < \infty$ for some $a > 0$. This latter condition on w cannot be relaxed.

Wei et al. [10, 11] proposed a new regularization technique to construct a regularized WKS sampling formula. It can be formulated as follows:

Definition 1.1. The continuous form of the regularized WKS sampling formula is defined as

$$(R^{(s)}_{M,\sigma}f)(t) = \sum_{n=\lceil \frac{t}{\Delta}\rceil - M}^{n=\lceil \frac{t}{\Delta}\rceil + M} f(n\Delta)\left[\frac{\sin\frac{\pi}{\Delta}(t-n\Delta)}{\frac{\pi}{\Delta}(t-n\Delta)}\exp(-\frac{(t-n\Delta)^2}{2\sigma^2})\right]^{(s)}, \ t \in R, \tag{1.3}$$

where $s \in \mathcal{Z}^+$, M, σ are parameters, $\lceil x \rceil$ is the integral part of x and $\lfloor x \rfloor = x - \lceil x \rceil$.

In numerical application, $R^{(s)}_{M,\sigma}F(t)$ is used to represent $F^{(s)}(t)$. It was found very accurate and robust for resolving various challenging dynamical problems, such as the homoclinic orbit excitation of the Sine-Gordon equation [12], the molecular quantum system described by the Schrödinger equation [13], nonlinear pattern formation of the Cahn-Hilliard equation [14], etc.

The objective of the present note is to provide a theoretical base for these excellent numerical results. Rigorous error estimations of the regularized WKS sampling formula are given for their applications to interpolations and derivatives of a function.

The result explains the application performance of the regularized WKS sampling formula. It also shows why the regularized WKS sampling formula exceeds in accuracy those using the existed convergence factors, and how it can be used to achieve desired accuracy.

2. Theorem and proof

Theorem 2.1. *Let $f(x)$ be a function $f \in L^\infty(R) \cap L^2(R) \cap C^s(R)$ and bandlimited to B, ($B < \frac{\pi}{\Delta}$, Δ is the grid spacing), $s \in \mathcal{Z}^+$, $\sigma = r\Delta > 0$, $M \in \mathcal{N}$, $M \geq \frac{sr}{\sqrt{2}}$. Then*

$$\left\| f^{(s)} - R^{(s)}_{M,\sigma}f \right\|_{L^\infty(R)} \leq \beta \exp\left(-\frac{\alpha^2}{2r^2}\right) \tag{2.1}$$

where

$$\alpha = \min\{M, r^2(\pi - B\Delta)\},$$
$$\beta = \frac{e^\pi r(s+1)!}{\Delta^s \pi\alpha}\left(\sqrt{2B}\|F\|_{L^2(R)} + 2r\|F\|_{L^\infty(R)}\right). \tag{2.2}$$

Proof. We proceed with the proof in several steps.

Separation of the error. The error breaks naturally into a few components. Denote

$$E(t) = f^{(s)}(t) - \sum_{n=\lceil \frac{t}{\Delta}\rceil - M}^{n=\lceil \frac{t}{\Delta}\rceil + M} f(n\Delta)\left[\frac{\sin\left(\frac{\pi}{\Delta}(t-n\Delta)\right)}{\frac{\pi}{\Delta}(t-n\Delta)}\exp\left(-\frac{(t-n\Delta)^2}{2\sigma^2}\right)\right]^{(s)}, \tag{2.3}$$

$$E_1(t) = \sum_{n=-\infty}^{n=+\infty} f(n\Delta)\left[\frac{\sin\left(\frac{\pi}{\Delta}(t-n\Delta)\right)}{\frac{\pi}{\Delta}(t-n\Delta)} - \frac{\sin\left(\frac{\pi}{\Delta}(t-n\Delta)\right)}{\frac{\pi}{\Delta}(t-n\Delta)}\exp\left(-\frac{(t-n\Delta)^2}{2\sigma^2}\right)\right]^{(s)}, \tag{2.4}$$

$$E_2(t) = \sum_{|n-\lceil \frac{t}{\Delta}\rceil| > M} f(n\Delta)\left[\frac{\sin\left(\frac{\pi}{\Delta}(t-n\Delta)\right)}{\frac{\pi}{\Delta}(t-n\Delta)}\exp\left(-\frac{(t-n\Delta)^2}{2\sigma^2}\right)\right]^{(s)}. \tag{2.5}$$

Here, $E_1(t)$ is the regularization error and $E_2(t)$ is the truncation error. From the WKS sampling theorem

$$f(t) = \sum_{n=-\infty}^{n=+\infty} f(n\Delta)\frac{\sin\left(\frac{\pi}{\Delta}(t-n\Delta)\right)}{\frac{\pi}{\Delta}(t-n\Delta)}, \tag{2.6}$$

which can be differentiated term by term [6]; the total error can be written as

$$E(t) = E_1(t) + E_2(t). \tag{2.7}$$

The corresponding error norms satisfy

$$\|E(t)\|_{L^\infty(R)} \le \|E_1(t)\|_{L^\infty(R)} + \|E_2(t)\|_{L^\infty(R)}. \tag{2.8}$$

Estimation of $\|E_1\|_{L^\infty(R)}$. Let $\hat{f}(\omega)$ be the Fourier transform of $f(x)$, $\hat{f}(\omega) = \int_R f(x)\exp(ix\omega)dx$. Since

$$\left[\frac{\sin\left(\frac{\pi}{\Delta}(t-n\Delta)\right)}{\frac{\pi}{\Delta}(t-n\Delta)}\right]\hat{}(\omega) = \Delta\exp(in\Delta\omega)\chi_{[-\frac{\pi}{\Delta},\frac{\pi}{\Delta}]}(w) \tag{2.9}$$

and

$$\left[\exp\left(-\frac{(t-n\Delta)^2}{2\sigma^2}\right)\right]\hat{}(\omega) = \sqrt{2\pi}\sigma\exp(in\Delta\omega - \frac{\sigma^2\omega^2}{2}), \tag{2.10}$$

one writes

$$\begin{aligned}
&\left[\frac{\sin\left(\frac{\pi}{\Delta}(t-n\Delta)\right)}{\frac{\pi}{\Delta}(t-n\Delta)}\right]\hat{}(\omega) * \left[\exp\left(-\frac{(t-n\Delta)^2}{2\sigma^2}\right)\right]\hat{}(\omega)\\
&= \int_R \Delta\sqrt{2\pi}\sigma\exp(in\Delta(\omega-\theta))\chi_{[-\frac{\pi}{\Delta},\frac{\pi}{\Delta}]}(\omega-\theta)\exp(in\Delta\theta - \frac{\sigma^2\theta^2}{2})d\theta\\
&= \Delta\sqrt{2\pi}\sigma\exp(in\Delta\omega)\int_{\omega-\frac{\pi}{\Delta}}^{\omega+\frac{\pi}{\Delta}}\exp(-\frac{\sigma^2\theta^2}{2})d\theta.
\end{aligned} \tag{2.11}$$

Therefore we have

$$
\begin{aligned}
&\left[\sum_{n=-\infty}^{n=+\infty} f(n\Delta)\frac{\sin\left(\frac{\pi}{\Delta}(t-n\Delta)\right)}{\frac{\pi}{\Delta}(t-n\Delta)}\exp\left(-\frac{(t-n\Delta)^2}{2\sigma^2}\right)\right]\hat{}(\omega)\\
&=\sum_{n=-\infty}^{n=+\infty} f(n\Delta)\frac{1}{2\pi}\left[\frac{\sin\left(\frac{\pi}{\Delta}(t-n\Delta)\right)}{\frac{\pi}{\Delta}(t-n\Delta)}\right]\hat{}(\omega) * \left[\exp\left(-\frac{(t-n\Delta)^2}{2\sigma^2}\right)\right]\hat{}(\omega)\\
&=\sum_{n=-\infty}^{n=+\infty} f(n\Delta)\Delta\exp(in\Delta\omega)\frac{1}{\sqrt{\pi}}\int_{\frac{\sigma(\omega-\frac{\pi}{\Delta})}{\sqrt{2}}}^{\frac{\sigma(\omega+\frac{\pi}{\Delta})}{\sqrt{2}}}\exp(-t^2)dt. \qquad (2.12)
\end{aligned}
$$

Since function f satisfies

$$\hat{f}(\omega) \in L^2[-B,B] \subset L^2[-\frac{\pi}{\Delta},\frac{\pi}{\Delta}], \tag{2.13}$$

it has a Fourier series expansion

$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} c_n \exp(in\Delta\omega), \tag{2.14}$$

where coefficients are given by

$$c_n = \frac{\Delta}{2\pi}\int_{-\frac{\pi}{\Delta}}^{\frac{\pi}{\Delta}} \hat{f}(\omega)\exp(-in\Delta\omega)d\omega = \Delta f(n\Delta). \tag{2.15}$$

Equivalently, $\hat{f}(\omega)$ can be written as

$$\hat{f}(\omega) = \hat{f}(\omega)\chi_{[-B,B]}(\omega) = \sum_{n=-\infty}^{\infty} \Delta f(n\Delta)\exp(in\Delta\omega)\chi_{[-B,B]}(\omega). \tag{2.16}$$

Denote

$$\varepsilon(\omega) = \chi_{[-B,B]}(\omega) - \frac{1}{\sqrt{\pi}}\int_{\frac{\sigma(\omega-\frac{\pi}{\Delta})}{\sqrt{2}}}^{\frac{\sigma(\omega+\frac{\pi}{\Delta})}{\sqrt{2}}}\exp(-t^2)dt. \tag{2.17}$$

Then combining (2.4), (2.12), (2.14) and (2.16), one has

$$\hat{E_1}(\omega) = (i\omega)^s \hat{f}(\omega)\varepsilon(\omega). \tag{2.18}$$

For $\omega \in [-B,B]$, $\varepsilon(\omega)$ can be evaluated as

$$
\begin{aligned}
\varepsilon(\omega) &= \frac{1}{\sqrt{\pi}}\left[\int_{-\infty}^{\infty}\exp(-t^2)dt - \int_{\frac{\sigma(\omega-\frac{\pi}{\Delta})}{\sqrt{2}}}^{\frac{\sigma(\omega+\frac{\pi}{\Delta})}{\sqrt{2}}}\exp(-t^2)dt\right]\\
&= \frac{1}{\sqrt{\pi}}\left[\int_{\frac{\sigma(\frac{\pi}{\Delta}-\omega)}{\sqrt{2}}}^{\infty}\exp(-t^2)dt + \int_{\frac{\sigma(\omega+\frac{\pi}{\Delta})}{\sqrt{2}}}^{\infty}\exp(-t^2)dt\right]. \qquad (2.19)
\end{aligned}
$$

Moreover, for $x \geq 0$, the following inequality [15] is valid:

$$\frac{1}{x+\sqrt{x^2+2}} \leq \exp(x^2)\int_x^{\infty}\exp(-t^2)dt \leq \frac{1}{x+\sqrt{x^2+\frac{4}{\pi}}}. \tag{2.20}$$

Therefore, the estimation for $\varepsilon(\omega)$ is obtained as

$$\varepsilon(\omega) \leq \frac{1}{\sqrt{\pi}}\left(\frac{\exp\left(-\frac{\sigma^2(\frac{\pi}{\Delta}-\omega)^2}{2}\right)}{\sqrt{2}\sigma(\frac{\pi}{\Delta}-\omega)}+\frac{\exp\left(-\frac{\sigma^2(\frac{\pi}{\Delta}+\omega)^2}{2}\right)}{\sqrt{2}\sigma(\frac{\pi}{\Delta}+\omega)}\right). \tag{2.21}$$

Since $\exp(-\frac{\sigma^2x^2}{2})/x$ decreases, and

$$\frac{\pi}{\Delta}-\omega, \frac{\pi}{\Delta}+\omega \in [\frac{\pi}{\Delta}-B, \frac{\pi}{\Delta}+B], \tag{2.22}$$

we have

$$\varepsilon(\omega) \leq \frac{1}{\sigma(\frac{\pi}{\Delta}-B)\exp\left(\frac{\sigma^2(\frac{\pi}{\Delta}-B)^2}{2}\right)}. \tag{2.23}$$

It follows from (2.18) and (2.23) that

$$|E_1\hat{(}\omega)| \leq \frac{|f\hat{(}\omega)(i\omega)^s|}{\sigma(\frac{\pi}{\Delta}-B)\exp(\frac{\sigma^2(\frac{\pi}{\Delta}-B)^2}{2})}. \tag{2.24}$$

Therefore

$$\begin{aligned}
|E_1(x)| &= |\frac{1}{2\pi}\int_R E_1\hat{(}\omega)\exp(-ix\omega)d\omega| \\
&\leq \frac{1}{2\pi}\int_R |E_1\hat{(}\omega)|d\omega \\
&\leq \frac{1}{2\pi\sigma(\frac{\pi}{\Delta}-B)\exp\left(\frac{\sigma^2(\frac{\pi}{\Delta}-B)^2}{2}\right)}\int_{-B}^{B}|f\hat{(}\omega)(i\omega)^s|d\omega. \qquad (2.25)
\end{aligned}$$

By the Cauchy inequality and the Parseval identity, one has

$$\|E_1(x)\|_{L^\infty(R)} \leq \frac{B^{s+\frac{1}{2}}}{\sigma(\frac{\pi}{\Delta}-B)\exp\left(\frac{\sigma^2(\frac{\pi}{\Delta}-B)^2}{2}\right)}\sqrt{\frac{2}{2s+1}}\|f\|_{L^2(R)}. \tag{2.26}$$

Estimation of $\|E_2(t)\|_{L^\infty(R)}$. Differentiations can be written as

$$\begin{aligned}
E_2(t) = \sum_{|n-\lceil\frac{t}{\Delta}\rceil|>M} f(n\Delta)\Bigg[\sum_{i+j+k=s}\frac{s!}{i!k!}(\frac{\pi}{\Delta})^{i-1}\sin(\frac{\pi}{\Delta}t-n\pi+\frac{\pi i}{2}) \\
\frac{(-1)^j}{(t-n\Delta)^{j+1}}\frac{(-1)^k}{(\sqrt{2}\sigma)^k}H_k(\frac{t-n\Delta}{\sqrt{2}\sigma})\exp\left(-\frac{(t-n\Delta)^2}{2\sigma^2}\right)\Bigg], \qquad (2.27)
\end{aligned}$$

where $H_k(x)$ is the kth order Hermite polynomial

$$\exp(-x^2)H_k(x) = (-1)^k(\frac{d}{dx})^k\exp(-x^2). \tag{2.28}$$

Let $l = |n - \lceil \frac{t}{\Delta} \rceil|$. Then

$$\begin{aligned} E_2(t) \quad = \quad & \sum_{|l|>M} f(t + l\Delta - \lfloor \frac{t}{\Delta} \rfloor \Delta) \Bigg[\sum_{i+j+k=s} \frac{s!}{i!k!} (\frac{\pi}{\Delta})^{i-1} (-1)^{j+k+l} \\ & \sin\left(\lfloor \frac{t}{\Delta} \rfloor \pi + \frac{\pi i}{2} \right) \frac{1}{(\sqrt{2}\sigma)^k \Delta^{j+1} (-l + \lfloor \frac{t}{\Delta} \rfloor)^{j+1}} \\ & \left[H_k\left(\frac{-l\Delta + \lfloor \frac{t}{\Delta} \rfloor \Delta}{\sqrt{2}\sigma} \right) \right] \exp\left(-\frac{(l\Delta - \lfloor \frac{t}{\Delta} \rfloor \Delta)^2}{2\sigma^2} \right) \Bigg]. \end{aligned} \tag{2.29}$$

From [16] one has

$$H_k(x) = \sum_{i=0}^{[\frac{k}{2}]} \frac{(-1)^i k! (2x)^{k-2i}}{i!(k-2i)!}. \tag{2.30}$$

Denote it as $H_k(x) = \sum_{i=0}^{[\frac{k}{2}]} (-1)^i a_i$; then it is easily shown that $\{a_i\}_{i \geq 0}$ decrease for $|x| \geq \frac{k}{2}$. This leads to $|H_k(x)| \leq |a_0|$. Therefore for $M \geq \frac{sr}{\sqrt{2}}$, one has

$$|H_k(\frac{-M\Delta}{\sqrt{2}\sigma})| \leq (\frac{\sqrt{2}M}{r})^k. \tag{2.31}$$

Applying (2.31) to (2.29),

$$|E_2(t)| \leq \frac{\|F\|_{L^\infty(R)}}{\Delta^s} \sum_{|l|>M} \left[\sum_{i+j+k=s} \frac{s! \pi^{i-1} l^{k-j-1}}{i!k!r^{2k}} \exp\left(-\frac{(l-1)^2}{2r^2} \right) \right]. \tag{2.32}$$

Since $\frac{x^n}{n!} < e^x$, we have

$$\begin{aligned} |E_2(t)| \quad & \leq \quad \frac{\|F\|_{L^\infty(R)} e^\pi s!}{\Delta^s \pi} \sum_{|l|>M} \left[\sum_{i+j+k=s} \frac{l^k}{k!r^{2k}} \exp\left(-\frac{(l-1)^2}{2r^2} \right) \right] \\ & \leq \quad \frac{\|F\|_{L^\infty(R)} e^\pi (s+1)!}{\Delta^s \pi} \sum_{|l|>M} \exp\left(\frac{l}{r^2} \right) \exp\left(-\frac{(l-1)^2}{2r^2} \right) \\ & \leq \quad \frac{\|F\|_{L^\infty(R)} e^\pi (s+1)!}{\Delta^s \pi} \sum_{|l|>M} \exp\left(-\frac{l^2}{2r^2} \right). \end{aligned} \tag{2.33}$$

From the inequality (2.20),

$$\begin{aligned} \sum_{l \leq -M-1} \exp\left(-\frac{l^2}{2r^2} \right) \quad & < \quad \int_{-\infty}^{-M} \exp\left(-\frac{x^2}{2r^2} \right) dx \\ & = \quad \sqrt{2}r \int_{\frac{M}{\sqrt{2}r}}^{\infty} \exp(-x^2) dx \\ & \leq \quad \frac{r^2}{M} \exp\left(-\frac{M^2}{2r^2} \right). \end{aligned} \tag{2.34}$$

Similarly

$$\begin{aligned}\sum_{l\geq M+1}\exp\left(-\frac{k^2}{2r^2}\right) &< \int_M^\infty \exp\left(-\frac{x^2}{2r^2}\right)dx\\ &\leq \frac{r^2}{M}\exp\left(-\frac{M^2}{2r^2}\right). \qquad (2.35)\end{aligned}$$

So, one has

$$|E_2(t)| \leq \frac{2\|F\|_{L^\infty(R)}e^\pi(s+1)!r^2}{\Delta^s\pi M}\exp\left(-\frac{M^2}{2r^2}\right). \qquad (2.36)$$

The end of the proof. By combining (2.8), (2.26) and (2.36), one obtains

$$\left\|F^{(s)}(t)-R^{(s)}_{M,\sigma}F(t)\right\|_{L^\infty(R)} \leq \beta\exp\left(-\frac{\alpha^2}{2r^2}\right), \qquad (2.37)$$

where

$$\alpha = \min\{M, r^2(\pi - B\Delta)\} \qquad (2.38)$$

and

$$\begin{aligned}\beta &= \frac{B^s\sqrt{2B}\|F(t)\|_{L^2(R)}}{\sqrt{2s+1}\sigma(\frac{\pi}{\Delta}-B)}+\frac{2\|F\|_{L^\infty(R)}e^\pi(s+1)!r^2}{\Delta^s\pi M}\\ &\leq \frac{\pi^s\sqrt{2B}\|F(t)\|_{L^2(R)}}{\Delta^s r(\pi-B\Delta)}+\frac{2\|F\|_{L^\infty(R)}e^\pi(s+1)!r^2}{\Delta^s\pi M}\\ &\leq \frac{r}{\Delta^s\alpha\pi}\left(\pi^{s+1}\sqrt{2B}\|F(t)\|_{L^2(R)}+2\|F\|_{L^\infty(R)}e^\pi(s+1)!r\right)\\ &\leq \frac{e^\pi r(s+1)!}{\Delta^s\pi\alpha}\left(\sqrt{2B}\|F\|_{L^2(R)}+2r\|F\|_{L^\infty(R)}\right). \qquad (2.39)\end{aligned}$$

This completes the proof. ☐

Equation (2.1) is the rigorous error statement for the formula widely used in the aforementioned numerical computations. Assume that all non-exponential quantities are combined to give unit, i.e. $\beta = 1$; then we have the following corollary.

Corollary. *If the approximation error for function f is set to be $10^{-\eta}$ $(\eta > 0)$, the following relations are to be satisfied:*

$$r(\pi - B\Delta) > \sqrt{\eta 2\ln 10} \qquad (2.40)$$

and

$$\frac{M}{r} > \sqrt{\eta 2\ln 10}. \qquad (2.41)$$

Proof. The proof is obvious from (2.1) and (2.2). ☐

By using (2.40) and (2.41), one may choose r, Δ and M appropriately to attain the desired accuracy. Roughly speaking, when Δ is small enough, if one chooses the ratio $r = 3$, then $M \sim 30$ can be used to ensure the highest accuracy in a double precision computation ($\eta = 15$). This theoretical estimation is in good agreement with a previous numerical test [11].

Remark 2.2. The rate of convergence in [7, 8] is of polynomial order; the convergence in (2.1) is much faster. The Gaussian regulator used here is not covered in [9]; it is also simpler than the construction in [9] where no explicit error estimate was given.

Remark 2.3. By using the Parseval theorem on (2.18) and the Schwarz inequality on (2.29), it is easy to obtain the L^2-norm error estimate of $E(t)$ where the dominate factor is still $\beta \exp\left(-\frac{\alpha^2}{2r^2}\right)$, $\alpha = \min\{M, r^2(\pi - B\Delta)\}$. The upper and lower bounds of $\|E_1(t)\|_{L^2(\mathcal{R})}$ are determined by $\varepsilon(\omega)$ at (2.20). If we add some decay conditions on $f(t)$, the estimate of $|E_2(t)|$ can be sharpened by using Abel inequality. We omit the details, as the estimate here is enough to verify the numerical results.

Open problem. How about the error estimate when $f(x)$ is a duration-limited function?

Acknowledgment

The author is very grateful to Associate Professor G. W. Wei, Professor C. A. Micchelli, the editor and the referee for their many valuable suggestions, which greatly improved the main result in this paper.

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