

THE QUASILINEARIZATION METHOD ON AN UNBOUNDED DOMAIN

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ABSTRACT. We apply a method of quasilinearization to a boundary value problem for an ordinary differential equation on an unbounded domain. A uniquely determined Green's function, which is integrable and of fixed sign, is employed. The hypotheses to apply the quasilinearization method imply uniqueness of solutions. The quasilinearization method generates a bilateral iteration scheme in which the iterates converge monotonically and quadratically to the unique solution.

In this paper, we shall apply a method of quasilinearization to the singular boundary value problem (BVP),

$$(1) \quad x''(t) + q(t)x(t) = f(t, x(t)), \quad t \in \mathbb{R}^+,$$

$$(2) \quad x(0) = x_0, \quad x(t) \text{ bounded on } \mathbb{R}^+,$$

where x_0 is real, $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $q : \mathbb{R}^+ \rightarrow \mathbb{R}^-$ is continuous, and $q(t) \leq -c^2 < 0$, $t \in \mathbb{R}^+$, for some $c^2 > 0$. We model the singular BVP based on the work of Bebernes and Jackson [1].

The method of quasilinearization has recently been studied and extended extensively. It is generating a rich history beginning with the works by Bellman [2, 3]. Lakshmikantham, Leela, Vatsala, and many co-authors have extensively developed the method and have applied the method to a wide range of problems. We refer the reader to the recent work by Lakshmikantham and Vatsala [12] and the extensive bibliography found there. The method we produce here is modeled by the method developed by Lakshmikantham, Leela and McRae [11]; this method is referred to as the improved generalized quasilinearization method. Analogous methods have been applied to two-point boundary value problems for ordinary differential equations and we refer the reader to the papers, [15, 13, 14, 10, 6, 7, 4].

To our knowledge, this paper provides the first application of the quasilinearization method to singular boundary value problems on unbounded domains. Devi and Vatsala [5] have recently applied the method to a singular BVP on a bounded domain.

The method of quasilinearization for BVPs employs a delicate balance of upper and lower solution methods with monotone methods. Recently, Eloë, Grimm and

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Mashburn [8] developed both the monotone methods and the upper and lower solution methods that provide the supporting material. Before we develop the quasilinearization method here, we shall prove one result, Theorem 1, not proved by Eloë, Grimm and Mashburn [8]; we shall state without proof the other results we require from [8]. In Theorem 2 we shall develop the method of quasilinearization.

Throughout most of this paper (Lemmas 1, 2, and 3 are exceptions) we shall assume that

$$(3) \quad x_1 \leq x_2 \implies f(t, x_1) \leq f(t, x_2), \quad t \in \mathbb{R}^+.$$

We also require the definitions of upper and lower solutions for the BVP, (1), (2).

β is an **upper solution** of the BVP, (1), (2), if β'' is continuous on \mathbb{R}^+ and

$$\beta''(t) + q(t)\beta(t) \leq f(t, \beta(t)), \quad t \in \mathbb{R}^+,$$

$$\beta(0) \geq x_0, \quad \beta(t) \text{ bounded on } \mathbb{R}^+,$$

and α is a **lower solution** of the BVP, (1), (2), if α'' is continuous on \mathbb{R}^+ and

$$\alpha''(t) + q(t)\alpha(t) \geq f(t, \alpha(t)), \quad t \in \mathbb{R}^+,$$

$$\alpha(0) \leq x_0, \quad \alpha(t) \text{ bounded on } \mathbb{R}^+.$$

Theorem 1. *Assume the monotonicity condition (3). Let α and β denote a lower and an upper solution of the BVP, (1), (2), respectively. Then*

$$(4) \quad \alpha(t) \leq \beta(t), \quad t \in \mathbb{R}^+.$$

Proof. Set $h = \alpha - \beta$ and note by the definition of upper and lower solutions that $h(0) \leq 0$. Assume for the sake of contradiction that h is not non-positive on \mathbb{R}^+ . Set

$$\tau = \sup\{t \geq 0 : h(t) \leq 0\}.$$

Then $h(\tau) = 0$ and $h'(\tau) \geq 0$.

As a first case, assume there is some $t_1 > \tau$ such that $h'(t_1) < 0$. Then there exists some $t_0 \in (\tau, t_1)$ such that $h(t_0)$ is a maximum value of h on $[\tau, t_1]$. Thus, $h(t_0) > 0$, $h'(t_0) = 0$, and $h''(t_0) \leq 0$. By the definition of upper and lower solutions, and an application of (3),

$$(5) \quad h''(t_0) \geq f(t_0, \alpha(t_0)) - f(t_0, \beta(t_0)) - q(t_0)h(t_0) > 0.$$

This is a contradiction and so, for the second case, assume $h'(t) \geq 0$, $t \geq \tau$. By the definition of τ , $h(\tau + 1) > 0$. Thus, there exists $c \in (\tau, \tau + 1)$ such that $h'(c) > 0$. It follows as in the development of (5) that if $h(t) \geq 0$, then $h''(t) \geq 0$. Thus, $h''(t) \geq 0$ on (τ, ∞) and so, h' is nondecreasing on (τ, ∞) . In particular,

$$h'(t) \geq h'(c) > 0, \quad t \in [c, \infty),$$

which implies that h is unbounded. This contradicts the definition of upper and lower solutions of the BVP, (1), (2) and completes the proof. \square

Corollary 1. *Under the hypotheses of Theorem 1, solutions of the BVP, (1), (2), are unique.*

Lemmas 1, 2, and 3 have all been obtained in [8]. Theorem 4.2 is stated incorrectly in [8] and is stated correctly here in Lemma 1.

Lemma 1. Assume $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and for each $M > 0$, there exists $f_M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which is bounded and continuous and such that

$$|f(s, x)| \leq f_M(s), \quad |x| \leq M.$$

Assume $q : \mathbb{R}^+ \rightarrow \mathbb{R}^-$ is continuous such that $q(t) \leq -c^2 < 0$ for some $c^2 > 0$. Assume there exist α and β , lower and upper solutions of the BVP, (1), (2), respectively, such that α and β satisfy (4). Then, there exists a solution, x , of the BVP, (1), (2), satisfying

$$(6) \quad \alpha(t) \leq x(t) \leq \beta(t), \quad t \in \mathbb{R}^+.$$

Lemma 2. If $-q(t) \geq c^2 > 0$, then there exists a unique solution, $p_0(t)$, of the BVP,

$$x''(t) + q(t)x(t) = 0, \quad t \in \mathbb{R}^+,$$

satisfying the boundary conditions, (2).

Lemma 3. Assume $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and for each $M > 0$, there exists $f_M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which is bounded and continuous and such that

$$|f(s, x)| \leq f_M(s), \quad |x| \leq M.$$

Assume $q : \mathbb{R}^+ \rightarrow \mathbb{R}^-$ is continuous such that $q(t) \leq -c^2 < 0$ for some $c^2 > 0$. Then there exists a uniquely defined Green's function, $G(t, s)$, defined on $\mathbb{R}^+ \times \mathbb{R}^+$ such that x is a solution of the BVP, (1), (2), if, and only if, x is bounded and continuous on \mathbb{R}^+ and

$$x(t) = p_0(t) + \int_0^\infty G(t, s)f(s, x(s))ds, \quad t \in \mathbb{R}^+.$$

The Green's function satisfies the sign property

$$G(t, s) < 0, (t, s) \in (0, \infty) \times (0, \infty)$$

and the integrability property

$$\int_0^\infty |G(t, s)|ds \leq 1/c^2.$$

We now develop the quasilinearization method for the BVP, (1), (2).

Theorem 2. Assume f satisfies the monotonicity condition (3). Assume f_x and f_{xx} exist and are continuous on $\mathbb{R}^+ \times \mathbb{R}$. Assume $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and for each $M > 0$, there exists $f_M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which is bounded and continuous and such that

$$|f(s, x)| \leq f_M(s), \quad |f_x(s, x)| \leq f_M(s), \quad |x| \leq M.$$

Assume $q : \mathbb{R}^+ \rightarrow \mathbb{R}^-$ is continuous such that $q(t) \leq -c^2 < 0$ for some $c^2 > 0$. Assume there exists $\phi(t, x)$, defined on $\mathbb{R}^+ \times \mathbb{R}$, such that ϕ, ϕ_x , and ϕ_{xx} are continuous on $\mathbb{R}^+ \times \mathbb{R}$ and such that

$$\phi_{xx}(t, x) > 0, \quad (t, s) \in \mathbb{R}^+ \times \mathbb{R}.$$

Moreover, assume there exists $\phi_M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which is bounded and continuous and such that

$$|\phi(s, x)| \leq \phi_M(s), |\phi_x(s, x)| \leq \phi_M(s), |\phi_{xx}(s, x)| \leq \phi_M(s), \quad |x| \leq M.$$

Set $\phi = F - f$; we shall require that

$$(7) \quad F_{xx}(t, x) \geq 0, \quad (t, s) \in \mathbb{R}^+ \times \mathbb{R}.$$

Assume α_0 and β_0 , are lower and upper solutions of the BVP, (1), (2), respectively. Then there exist monotone sequences, $\{\alpha_n\}$ and $\{\beta_n\}$, that are solutions of linear differential equations and that converge in the space of bounded continuous functions on \mathbb{R}^+ to the unique solution of the BVP, (1), (2). Moreover, the convergence is quadratic.

Proof. First apply Theorem 1 and then Lemma 1 to note that there exists a unique solution, x , of the BVP, (1), (2), satisfying

$$\alpha_0(t) \leq x(t) \leq \beta_0(t), \quad t \in \mathbb{R}^+.$$

By (7), it follows that

$$(8) \quad f(t, x) \geq f(t, y) + F_x(t, y)(x - y) - (\phi(t, x) - \phi(t, y)), \quad (t, x, y) \in \mathbb{R}^+ \times \mathbb{R}^2.$$

$t \in \mathbb{R}^+, x, y \in \mathbb{R}$.

Define two truncations for the nonlinear term f . Define

$$g(t, x; \alpha_0, \beta_0) = f(t, \alpha_0(t)) + F_x(t, \beta_0(t))(x - \alpha_0(t)) - (\phi(t, x) - \phi(t, \alpha_0(t)))$$

and

$$G(t, x; \beta_0) = f(t, \beta_0(t)) + F_x(t, \beta_0(t))(x - \beta_0(t)) - (\phi(t, x) - \phi(t, \beta_0(t))).$$

For the sake of exposition, we may suppress explicit dependence on t in some of the algebraic expressions below. Note that since $\phi_{xx}(t, x) > 0$, it follows that

$$g_{xx}(t, x; u, v) < 0, \quad G_{xx}(t, x; v) < 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Thus,

$$(9) \quad g(t, x; u, v) \geq g(t, y; u, v) + g_x(t, x; u, v)(x - y), \quad (t, x, y) \in \mathbb{R}^+ \times \mathbb{R}^2,$$

and

$$(10) \quad G(t, x; u) \geq G(t, y; u) + G_x(t, x; u)(x - y), \quad (t, x, y) \in \mathbb{R}^+ \times \mathbb{R}^2.$$

Define further linear truncations, h and H , by

$$h(t, x; \alpha_0, \beta_0) = g(t, \alpha_0(t); \alpha_0, \beta_0) + g_x(t, \alpha_0(t); \alpha_0, \beta_0)(x - \alpha_0),$$

$$H(t, x; \alpha_0, \beta_0) = G(t, \beta_0(t); \beta_0) + G_x(t, \alpha_0(t); \beta_0)(x - \beta_0),$$

and consider two additional linear BVPs,

$$(11) \quad x''(t) + q(t)x(t) = h(t, x; \alpha_0, \beta_0), \quad t \in \mathbb{R}^+,$$

with the boundary conditions, (2), and

$$(12) \quad x''(t) + q(t)x(t) = H(t, x; \alpha_0, \beta_0), \quad t \in \mathbb{R}^+,$$

with the boundary conditions, (2). Because of the boundedness assumptions on f , f_x , ϕ , and ϕ_x , Lemma 1 can apply to each of the BVPs, (11), (2), and (12), (2).

We shall outline the first portion of the proof before providing the technical details. We have already noted that $\alpha_0(t) \leq \beta_0(t)$, $t \in \mathbb{R}^+$. Show that α_0 and β_0 are lower and upper solutions for the truncated BVP, (11), (2); apply Lemma 1 and obtain a solution, α_1 , of the BVP, (11), (2), satisfying

$$(13) \quad \alpha_0(t) \leq \alpha_1(t) \leq \beta_0(t), \quad t \in \mathbb{R}^+.$$

Similarly, show that α_0 and β_0 are lower and upper solutions for the truncated BVP, (12), (2); apply Lemma 1 and obtain a solution, β_1 , of the BVP, (12), (2), satisfying

$$(14) \quad \alpha_0(t) \leq \beta_1(t) \leq \beta_0(t), \quad t \in \mathbb{R}^+.$$

Now, show that α_1 and β_1 are lower and upper solutions for the original BVP, (1), (2); apply Theorem 1 and obtain

$$(15) \quad \alpha_0(t) \leq \alpha_1(t) \leq \beta_1(t) \leq \beta_0(t), \quad t \in \mathbb{R}^+.$$

Let us first obtain (13). Since

$$h(t, \alpha_0; \alpha_0, \beta_0) = g(t, \alpha_0; \alpha_0, \beta_0) = f(t, \alpha_0),$$

α_0 is clearly a lower solution of the BVP, (11), (2). As for β_0 , apply (8) and (9) to obtain

$$\begin{aligned} f(t, \beta_0(t)) &\leq f(t, \alpha_0(t)) - F_x(t, \beta_0(t))(\alpha_0(t) - \beta_0(t)) \\ &\quad + (\phi(t, \alpha_0(t)) - \phi(t, \beta_0(t))) = g(t, \beta_0(t); \alpha_0, \beta_0) \\ &\leq g(t, \alpha_0(t); \alpha_0, \beta_0) + g_x(t, \alpha_0(t); \alpha_0, \beta_0)(\beta_0 - \alpha_0) \\ &= h(t, \beta_0; \alpha_0, \beta_0). \end{aligned}$$

Thus, β_0 is an upper solution of the BVP, (11), (2). Apply Lemma 1 and obtain the existence of a solution, α_1 , of the BVP, (11), (2), satisfying (13).

To obtain (14), β_0 is now clearly an upper solution of the BVP, (12), (2). As for α_0 , apply (8) and (10) to obtain

$$\begin{aligned} f(t, \alpha_0(t)) &\geq f(t, \beta_0(t)) + F_x(t, \beta_0(t))(\alpha_0(t) - \beta_0(t)) \\ &\quad - (\phi(t, \alpha_0) - \phi(t, \beta_0)) = G(t, \alpha_0(t); \beta_0) \\ &\geq G(t, \beta_0(t); \beta_0) + G_x(t, \alpha_0(t); \beta_0)(\alpha_0 - \beta_0) \\ &= H(t, \beta_0; \alpha_0, \beta_0). \end{aligned}$$

Thus, α_0 is a lower solution of the BVP, (12), (2). Apply Lemma 1 and obtain the existence of a solution, β_1 , of the BVP, (12), (2), satisfying (14).

To complete the first part of the argument, we show that α_1 and β_1 are lower and upper solutions, respectively of the original BVP, (1), (2). First, consider, α_1 . Apply (9), (8) and note that $F_{xx} \geq 0$ implies F_x is increasing in x to obtain

$$\begin{aligned} h(t, \alpha_1; \alpha_0, \beta_0) &= g(t, \alpha_0; \alpha_0, \beta_0) + g_x(t, \alpha_0; \alpha_0, \beta_0)(\alpha_1 - \alpha_0) \\ &\geq g(t, \alpha_1; \alpha_0, \beta_0) \\ &= f(t, \alpha_0) + F_x(t, \beta_0)(\alpha_1(t) - \alpha_0(t)) - (\phi(t, \alpha_1) - \phi(t, \alpha_0)) \\ &\geq f(t, \alpha_0) + F_x(t, \alpha_1)(\alpha_1(t) - \alpha_0(t)) - (\phi(t, \alpha_1) - \phi(t, \alpha_0)) \\ &\geq f(t, \alpha_1(t)). \end{aligned}$$

Thus, α_1 is a lower solution of the BVP, (1), (2). To see that β_1 is an upper solution of the BVP, (1), (2), employ (10) and G_x decreasing to obtain

$$\begin{aligned} H(t, \beta_1; \alpha_0, \beta_0) &= G(t, \beta_0; \beta_0) + G_x(t, \alpha_0; \beta_0)(\beta_1 - \beta_0) \\ &\leq G(t, \beta_1; \beta_0) + (G_x(t, \alpha_0; \beta_0) - G_x(t, \beta_1; \beta_0))(\beta_1 - \beta_0) \\ &\leq G(t, \beta_1; \beta_0) \leq f(t, \beta_1). \end{aligned}$$

The next step of the proof is inductive and we outline the details. Define inductively

$$\begin{aligned} g(t, x; \alpha_n, \beta_n) &= f(t, \alpha_n(t)) + F_x(t, \beta_n(t))(x - \alpha_n) - (\phi(t, x) - \phi(t, \alpha_n)), \\ G(t, x; \beta_n) &= f(t, \beta_n(t)) + F_x(t, \beta_n(t))(x - \beta_n) - (\phi(t, x) - \phi(t, \beta_n)), \\ h(t, x; \alpha_n, \beta_n) &= g(t, \alpha_n; \alpha_n, \beta_n) + g_x(t, \alpha_n; \alpha_n, \beta_n)(x - \alpha_n), \end{aligned}$$

and

$$H(t, x; \alpha_n, \beta_n) = G(t, \beta_n; \beta_n) + G_x(t, \alpha_n; \beta_n)(x - \beta_n).$$

At each step, we have that α_n and β_n are lower and upper solutions, respectively, of the BVP, (1), (2), so

$$\alpha_n(t) \leq \beta_n(t), \quad t \in \mathbb{R}^+,$$

by Theorem 1. Show that α_n and β_n are lower and upper solutions, respectively, of a BVP,

$$x''(t) + q(t)x(t) = h(t, x(t); \alpha_n, \beta_n), \quad t \in \mathbb{R}^+,$$

satisfying the boundary conditions, (2), and apply Lemma 1 to obtain the existence of a solution, α_{n+1} , of this BVP satisfying

$$\alpha_n(t) \leq \alpha_{n+1}(t) \leq \beta_n(t), \quad t \in \mathbb{R}^+.$$

Similarly, obtain the existence of a corresponding solution, β_{n+1} , of the BVP,

$$x''(t) + q(t)x(t) = H(t, x(t); \alpha_n, \beta_n), \quad t \in \mathbb{R}^+,$$

satisfying the boundary conditions, (2); in particular,

$$\alpha_n(t) \leq \beta_{n+1}(t) \leq \beta_n(t), \quad t \in \mathbb{R}^+.$$

Finally, show that α_{n+1} and β_{n+1} are lower and upper solutions, respectively, of the original BVP, (1), (2), and apply Theorem 1 to obtain

$$\alpha_n(t) \leq \alpha_{n+1}(t) \leq \beta_{n+1}(t) \leq \beta_n(t), \quad t \in \mathbb{R}^+.$$

Due to the monotonicity and boundedness, there exists α such that $\{\alpha_n\}$ converges uniformly to α on compact subsets of \mathbb{R}^+ . Moreover,

$$h(t, \alpha_{n+1}; \alpha_n, \beta_n) \rightarrow f(t, \alpha).$$

It follows by Lemma 3 and the dominated convergence theorem that

$$\alpha(t) = p_0(t) + \int_0^\infty G(t, s)f(s, \alpha(s))ds, \quad t \in \mathbb{R}^+;$$

in particular, by Lemma 3, $\alpha = x$, the unique solution of the BVP, (1), (2). Similarly, $\beta = x$, the unique solution of the BVP, (1), (2).

We close the proof by showing the convergence is quadratic. We show the details for the convergence in β_n ; the details for the the convergence in α_n are analogous. Let x denote the unique solution of the BVP, (1), (2). Set,

$$r_n = \beta_n - x \geq 0, \quad p_n = x - \alpha_n \geq 0, \quad e_n = \max\{\|r_n\|, \|p_n\|\},$$

where $\|\cdot\|$ denotes the supremum norm on bounded continuous functions on \mathbb{R}^+ .

With repeated applications of the mean value theorem, there exist

$$x(t) \leq c_n(t) \leq d_n(t) \leq \beta_n(t)$$

such that

$$\begin{aligned}
 r''_{n+1}(t) + q(t)r_{n+1}(t) &= (\beta''_{n+1}(t) + q(t)\beta_{n+1}(t)) - (x''(t) + q(t)x(t)) \\
 &= H(t, \beta_{n+1}; \alpha_n, \beta_n) - f(t, x) \\
 &= f(t, \beta_n) + (F_x(t, \beta_n) - \phi_x(t, \alpha_n))(\beta_{n+1} - \beta_n) - f(t, x) \\
 &= f_x(t, c_n)(\beta_n - x) + (F_x(t, \beta_n) - \phi_x(t, \alpha_n))(\beta_{n+1} - \beta_n) \\
 &\geq f_x(t, c_n)(\beta_n - x) + (F_x(t, \beta_n) - \phi_x(t, c_n))(\beta_{n+1} - \beta_n) \\
 &= f_x(t, c_n)(\beta_n - x) + (F_x(t, \beta_n) - \phi_x(t, c_n))(\beta_{n+1} - x) \\
 &\quad + (\phi_x(t, c_n(t)) - F_x(t, \beta_n(t)))(\beta_n - x) \\
 &= (F_x(t, c_n(t)) - F_x(t, \beta_n(t)))r_n(t) + (F_x(t, \beta_n) - \phi_x(t, c_n))r_{n+1}(t) \\
 &\geq (F_x(t, c_n(t)) - F_x(t, \beta_n(t)))r_n(t) + (F_x(t, c_n) - \phi_x(t, c_n))r_{n+1}(t) \\
 &\geq F_{xx}(t, d_n(t))(c_n(t) - \beta_n(t))r_n(t) + f_x(t, c_n)r_{n+1} \geq -M\|e_n\|^2
 \end{aligned}$$

for some $M > 0$.

Finally, note that since β_{n+1} and x are bounded and are solutions of $x''(t) + q(t)x(t) = H(t, x; \alpha_n, \beta_n)$ and (1), respectively, it follows that r''_{n+1} is bounded. Thus, r_{n+1} is the unique solution of the BVP,

$$x''(t) = r''_{n+1}(t) + q(t)r_{n+1}(t), \quad t \in \mathbb{R}^+,$$

satisfying the boundary conditions,

$$x(0) = 0, \quad x(t) \text{ bounded on } \mathbb{R}^+;$$

in particular, by Lemma 3

$$\begin{aligned}
 0 \leq r_{n+1}(t) &= \int_0^\infty G(t, s)(r''_{n+1}(s) + q(s)r_{n+1}(s))ds \\
 &\leq M \int_0^\infty |G(t, s)|ds \|r_n\|^2 \leq M \|r_n\|^2 / c^2.
 \end{aligned}$$

□

Remark 1. If $f_{xx} \geq 0$, and sufficiently bounded, then $F = f$ and $\phi = 0$ is sufficient. Conditions other than the convexity condition, (7), can be employed [14]. For example, if $f_{xx} \leq 0$, and sufficiently bounded, take $F = f$ and $\phi = 0$. Define truncations, g and G , by

$$g(t, x; \alpha_0) = f(t, \alpha_0(t)) + F_x(t, \alpha_0(t))(x - \alpha_0(t))$$

and

$$G(t, x; \beta_0, \alpha_0) = f(t, \beta_0(t)) + F_x(t, \alpha_0(t))(x - \beta_0(t)).$$

See [9].

Remark 2. An analogous theorem to Theorem 2 can be stated and proved where one obtains quadratic convergence but the iterates are solutions of nonlinear equations. One employs g and G in the truncated BVPs instead of h and H . Similar details are presented in [11], [10] or [12] for example. In the statement of the theorem, one would require F to be convex and satisfy all the boundedness hypotheses; one would not need to require these conditions on ϕ or f_x . For this case, $F(t, x) = e^{-x}/(1+t^2)$ suffices and the hypotheses of the theorem are weakened.

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