ON A CHARACTERIZATION OF MEASURES OF DISPERSION

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Abstract. Measures of dispersion are characterized by the set of all bounded random variables whose dispersion is minimized when taken around the origin.

1. Introduction

Let $\varphi$ be a real valued function on $\mathbb{R}$, $X$ a bounded random variable (b.r.v.), and $a$ a real number. The functional $E\varphi(X-a)$ may be used as a measure of dispersion of $X$ around $a$. The base of the measure is the set of all b.r.v. $X$ such that

$$
\min_a E\varphi(X-a) = E\varphi(X).
$$

For example, the base of the first absolute moment $E|X-a|$ is the set of all b.r.v. with zero median; the base of the second moment $E(X-a)^2$ is the set of all b.r.v. with zero mean value.

In this paper, we consider a characterization of the measures of dispersion by their bases. Kagan and Shepp [2] proved that if $\varphi$ is continuous and the base of the measure $E\varphi(X-a)$ contains all b.r.v. with $EX = 0$, then $\varphi(x) = a x^2 + \varphi(0)$ with some $\alpha \geq 0$, and they also obtained a multivariate version of the result.

In what follows all the functions are real valued; $f$ is a non-negative continuous function on $\mathbb{R}$ with $f(0) = 0$; $B_\varphi$ denotes the base of the measure $E\varphi(X-a)$ (so $B_0$ is the set of all b.r.v.).

**Theorem 1.** Let $f$ satisfy the following conditions:

1. $f(x)$ does not vanish identically on $(-\infty, 0)$ or on $(0, \infty)$
2. $y \int_0^z \{f(x+y) - f(x) - f(y)\} \, dx \geq 0$ for any $y, z \in \mathbb{R}$.

If

3. $\varphi$ is continuous on $\mathbb{R}$ and $B_f \subseteq B_\varphi$,

then

4. $\varphi(x) = \alpha f(x) + \varphi(0)$ with some $\alpha \geq 0$.

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In particular, if \( f \) is convex on \( \mathbb{R} \), then (2) is equivalent to \( f(\pm \infty) = \infty \). Moreover, in this case, the difference \( f(x + y) - f(x) \) is an increasing function of \( x \) for any fixed \( y > 0 \) (see, for example, [1, 3.18]). Therefore, (3) is fulfilled and we have the following:

**Corollary.** If \( f \) is convex on \( \mathbb{R} \) and \( f(\pm \infty) = \infty \), then (4) implies (5).

The bases of convex measures are described in the last section. Note that convexity of \( f \) on \( \mathbb{R} \) is not necessary for (3). For example, the function

\[
f(x) = x^2(x^2 - 3x + 3)
\]
satisfies (2) and (3) but is not convex on \( \mathbb{R} \).

**Theorem 2.** Let \( f \) be absolutely continuous on each finite interval and satisfy (2). Moreover, let \( g \) be defined on \( \mathbb{R} \), bounded on each finite interval, \( g(0) = 0 \) and \( g(x) = f'(x) \) at all the points of differentiability of \( f \) (hence, almost everywhere). If \( \varphi \) is continuous on \( \mathbb{R} \) and \( B_\varphi \) contains all \( x \in B_0 \) with \( E\varphi(X) = 0 \), then (5) holds.

Condition (2) is essential. The functions

\[
f(x) = (x + |x|)^2, g(x) = 4(x + |x|) \quad \text{and} \quad \varphi(x) = (x + |x|)^3
\]
satisfy all the conditions of Theorems 1 and 2 except (2). Moreover,

\[
B_f = B_\varphi = \{X \in B_0 : P(X > 0) = 0\},
\]
and \( E\varphi(X) = 0 \) is equivalent to \( X \in B_\varphi \). However, (5) is obviously not valid in this case.

The functions \( f(x) = |x| \) and \( g(x) = \text{sign} x \) satisfy all the conditions of Theorem 2. It follows from \( E\text{sign} X = 0 \) that \( X \) has zero median. So if \( B_\varphi \) contains all b.r.v. with zero median, then we have (5) with \( f(x) = |x| \) (this also follows from the Corollary). The result holds under more general conditions (in particular, the function \( \varphi \) may be a priori discontinuous).

**Theorem 3.** Let \( \varphi \) be a function on \( \mathbb{R} \) bounded from either above or below on some interval and let \( 0 < p < 1 \). If \( B_\varphi \) contains all binary r.v. \( X \) with \( \min X \leq 0 \leq \max X \) and \( P(X = \min X) = p \), then (5) holds with

\[
f(x) = |x| + (2p - 1)x.
\]

Note that in this case \( B_f = \{X \in B_0 : P(X < 0) \leq p \leq P(X \leq 0)\} \) (so that \( B_f \) consists of all bounded r.v. with zero quantile of order \( p \)).

2. Proof of Theorems 1 and 2

Let \( Y_w \) denote an r.v. equal to \( w \) with probability 1,

\[
M = \{x \in \mathbb{R} : f(x) > 0\}
\]
and \( \bar{M} \) is the closure of \( M \). Set, moreover, for \( u, v \in M \) and \( u < 0 < v \) (there exist the such \( u \) and \( v \) in view of (2))

\[
\lambda = \lambda(u, v) = \{v f(u) - uf(v)\}^{-1}.
\]

Let \( Y = Y(u, v) \) be an r.v. with the distribution function \( F(x) = F(x, u, v) \) and

\[
F(x) = \begin{cases} 
\lambda f(v)(x - u) & \text{for } x \in [u, 0], \\
\lambda f(u)x - uf(v) & \text{for } x \in [0, v].
\end{cases}
\]
Lemma 1. Let $f$ satisfy (2). If $B_{\phi}$ contains $Y_0, Y_w$ for $w \notin [M]$ and $Y(u, v)$ for $u, v \in M, u < 0 < v$, then (5) holds with some $\alpha \geq 0$.

Proof. It follows from $Y_w \in B_{\phi}$ that

$$\varphi(w) = E\varphi(Y_w) = \min_a E\varphi(Y_w - a) = \min_t \varphi(t).$$

Therefore,

$$\varphi(0) = \varphi(w) = \min_t \varphi(t)$$

for all $w \notin [M]$ and we obtain (5) for all $x \notin [M]$. Now let $u, v \in M, u < 0 < v$. Putting for any integrable function $r$

$$E_r(z) = E(r(Y + z) = \lambda \{ f(v) \int_{z+u}^z r(x) \, dx + f(u) \int_{z}^{z+v} r(x) \, dx \},$$

and taking into account that $Y(u, v) \in B_{\phi}$, we get $E_{\varphi}(0) = 0$, since $\varphi$ is continuous so $E_{\varphi}(z)$ is differentiable. Hence

$$s(u) = s(v) \quad \text{for } u, v \in M, u < 0 < v,$$

where

$$s(x) = \frac{\varphi(x) - \varphi(0)}{f(x)}.$$

It follows that $s(x)$ has the same value $\alpha$ for all $x \in M$, so we have (5) for all such $x$. Since $f$ and $\varphi$ are continuous, it implies (5) for all $x \in [M]$ and thus for all real $x$. It follows from (5) and (7) that

$$\min_x \alpha f(x) = 0$$

so $\alpha \geq 0$. \qed

To prove Theorem 1, it is enough now to show that

$$Y_0, Y_w, Y(u, v) \in B_f \quad \text{for any } u, v \in M, u < 0 < v, \quad \text{and any } w \notin [M].$$

Since

$$f(w) = f(0) = 0 = \min_t f(t),$$

we have $Y_0, Y_w \in B_f$. It follows from (3) and (8) that

$$\int_0^z \frac{f(x + u) - f(x)}{f(u)} \, dx \leq z \leq \int_0^z \frac{f(x + v) - f(x)}{f(v)} \, dx$$

and

$$E_f(z) \geq E_f(0) \quad \text{for all } z \in \mathbb{R},$$

so $Y(u, v) \in B_f$ for $u, v \in M, u < 0 < v$. \qed

Similarly, to prove Theorem 2, it is enough to show that $Eg(X) = 0$ for

$$X = Y_0, Y_w, Y(u, v), \quad \text{where } u, v \in M, u < 0 < v, \quad \text{and } w \notin [M].$$

Indeed, $Eg(Y_0) = g(0) = 0$. If $w \notin [M]$, then $f(x) = 0$ in some open interval containing $w$; therefore, also in this interval, $g(x) = f'(x) = 0$, so

$$Eg(Y_w) = g(w) = 0.$$
Moreover, it follows from (8) that
\[ E_g\{Y(u,v)\} = E_g(0) = E'_f(0) = 0 \]
because \( f \) is absolutely continuous and so \( f(x) = \int_0^x g(t) \, dt \) (see, for example, [4 11.7]).

3. Proof of Theorem 3

Continuity of \( \varphi \) is essential for the proof of Theorems 1 and 2. Therefore, we now use another approach.

Let \( U = U_p(u,v) \ (u < v) \) be a binary r.v. defined by
\[ P(U = u) = p, \quad P(U = v) = q = 1 - p. \] (9)

Let \( x > 0 \) and \( u \in [0, x] \). Then the r.v. \( U_p(0,x) \) and \( U_p(-u,x-u) \) satisfy the conditions of Theorem 3. It follows from (1) and (9) that
\[ p\varphi(-u) + q\varphi(x-u) \geq p\varphi(0) + q\varphi(x) \]
and
\[ p\varphi(0) + q\varphi(x) \geq p\varphi(-u) + q\varphi(x-u), \]
whence
\[ p\{\varphi(-u) - \varphi(0)\} = q\{\varphi(x) - \varphi(x-u)\}. \] (10)

In particular, we have by setting \( x = u \) that
\[ p\{\varphi(-u) - \varphi(0)\} = q\{\varphi(u) - \varphi(0)\}. \] (11)

It follows from (10) and (11) that
\[ \varphi(x) - \varphi(x-u) = \varphi(u) - \varphi(0) \quad \text{for} \ x \geq 0, u \in [0,x] \]
and (replacing \( x \) by \( u + v \))
\[ \psi(u + v) = \psi(u) + \psi(v) \quad \text{for any} \ u, v \geq 0, \] (12)
where \( \psi(x) = \varphi(x) - \varphi(0) \). So both the functions \( \psi \) and \( -\psi \) are convex on \([0, \infty) \) [1 3.20]. Since one of them is bounded from above on some interval, they are continuous [1 3.18] and therefore linear [1 3.19]. Thus \( \psi(x) = \beta x \), where \( \beta \) is a constant, and
\[ \psi(x) = \frac{\beta}{2p} \{ |x| + (2p - 1)x \} \quad \text{for} \ x \geq 0. \]

In view of (11), the last equality is also valid for \( x < 0 \). Setting \( \alpha = \beta/2p \), we obtain (5) with \( f \) defined by (6). Finally, it follows from (5) and (1) for \( X = U_p(0,1) \) that \( \alpha \geq 0 \).

Remark. According to the known Blumberg-Sierpinski theorem [3], every measurable convex function is continuous. So the proof shows that the condition on \( \varphi \) in Theorem 3 may be replaced by measurability of \( \varphi \).
4. Convex measures of dispersion

A convex measure of dispersion is a measure $E\varphi(X - a)$ generated by a convex continuous function $\varphi$. The bases of the such measures may be described as follows.

**Theorem 4.** If $\varphi$ is convex and continuous on $\mathbb{R}$, then

\begin{equation}
B_\varphi = \{ X \in B_0 : E\varphi'_-(X) \leq 0 \leq E\varphi'_+(X) \},
\end{equation}

where $\varphi'_-$ and $\varphi'_+$ denote the left and right derivatives of $\varphi$, respectively.

In particular, if $\varphi$ is convex and differentiable on $\mathbb{R}$, then

$$B_\varphi = \{ X \in B_0 : E\varphi'(X) = 0 \}.$$ 

**Proof.** The proof of Theorem 4 is based on the following lemmas.

**Lemma 2.** Let functions $\psi_n(x)$ ($n = 1, 2, \ldots$) and their variations be uniformly bounded on an interval $[a, b]$ and let

$$\lim_{n \to \infty} \psi_n(x) = \psi(x) \text{ for each } x \in [a, b].$$

If $K(x)$ is a function of bounded variation on $[a, b]$, then

$$\lim_{n \to \infty} \int_a^b \psi_n(x) dK(x) = \int_a^b \psi(x) dK(x).$$

It is enough to prove it for the cases in which $\psi(x) \equiv 0$ and $K(x)$ is either continuous or discrete on $[a, b]$. In the first case, it follows from the known Helly’s theorem by integration by parts. In the second case,

$$I_n = \int_a^b \psi_n(x) dK(x) = \sum_m \psi_n(x_m) h_m,$$

where $m = 1, 2, \ldots$, $x_m$ runs over all the points of discontinuity of $K(x)$ on $[a, b]$ and $h_m$ are the corresponding jumps, so that

$$\sum_m |h_m| < \infty.$$ 

Let $A > 0$, $|\psi_n(x)| \leq A$ for all $x \in [a, b]$, $n = 1, 2, \ldots$, and let $\varepsilon > 0$ and

$$\sum_{m > N} |h_m| \leq \varepsilon/A,$$

where $N = N(\varepsilon)$. Then

$$|I_n| \leq \sum_{m \leq N} |\psi_n(x_m) h_m| + \varepsilon,$$

whence it follows that

$$\limsup_{n \to \infty} |I_n| \leq \varepsilon,$$

so $\lim_{n \to \infty} I_n = 0$,

because $\psi_n(x) \to 0$ and $\varepsilon > 0$ is arbitrary. \qed
Lemma 3. Let the functions \( \psi_n(x) \) increase on \( R \) and be uniformly bounded on each finite interval. If
\[
\lim_{n \to \infty} \psi_n(x) = \psi(x) \quad \text{for all real } x,
\]
then
\[
\lim_{n \to \infty} E\psi_n(X) = E\psi(X) \quad \text{for all } X \in B_0.
\]
It follows immediately from the previous lemma.

Lemma 4. Let \( \tau(x) \) be a convex continuous function on \( R \). Then:
(i) the ratio
\[
\frac{\tau(x + h) - \tau(x)}{h} \quad (h \neq 0)
\]
is an increasing function of \( x \) and \( h \), bounded for bounded \( x \) and \( h \);
(ii) the equality
\[
\tau(x_0) = \min_x \tau(x)
\]
is equivalent to
\[
\tau'_-(x_0) \leq 0 \leq \tau'_+(x_0).
\]
It follows from known properties of convex functions [1, 3.18].

To prove Theorem 4, note that the function \( \mu(x) = E\phi(X + x) \) is also convex and continuous on \( R \) for any fixed \( X \in B_0 \). By Lemmas 3 and 4,
\[
\mu'_\pm(0) = \lim_{h \to \pm 0} \frac{E\phi(X + h) - \phi(X)}{h} = E\phi'_\pm(X).
\]
By Lemma 4, \( X \in B_\phi \) if and only if \( \mu'_-(0) \leq 0 \leq \mu'_+(0) \). Taking (14) into account, we obtain (13).

References