DIRECTIONAL CONVEXITY OF LEVEL LINES
FOR FUNCTIONS CONVEX IN A GIVEN DIRECTION

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Abstract. Let $K(\varphi)$ be the class of functions $f(z) = z + a_2 z^2 + \ldots$ which are holomorphic and convex in direction $e^{i\varphi}$ in the unit disk $D$, i.e. the domain $f(D)$ is such that the intersection of $f(D)$ and any straight line \{ \( w : w = w_0 + te^{i\varphi}, t \in \mathbb{R} \) \} is a connected or empty set. In this note we determine the radius $r_{\psi,\varphi}$ of the biggest disk $|z| \leq r_{\psi,\varphi}$ with the property that each function $f \in K(\psi)$ maps this disk onto the convex domain in the direction $e^{i\varphi}$.

1.

The class of holomorphic functions in the unit disk $D = \{ z : |z| < 1 \}$ which are convex in some direction plays an important role in the extremal problems of univalent holomorphic and harmonic functions (e.g. Goodman [2], Clunie and Sheil-Small [1], Ruscheweyh and Salinas [7]).

The class of holomorphic functions

\[ f(z) = z + a_2 z^2 + \ldots, \quad z \in D, \]

which map $D$ onto a domain $f(D)$ such that the intersection of $f(D)$ with any straight line \{ $w : w = w_0 + te^{i\varphi}, t \in \mathbb{R}$ \} is a connected or empty set for all $w_0 \in \mathbb{C}$, is called the class of convex functions in direction $e^{i\varphi}$, $\varphi \in \mathbb{R}$, and is denoted by $K(\varphi)$. In particular, $K(\pi/2) = K(-\pi/2) = CIA$ is known as the class of convex functions in the direction of the imaginary axis.

Many properties of this class have been studied and different normalizations were considered (Goodman [2] and Hengartner and Schober [4], [5]). In the problem under consideration we can restrict ourselves to the classical normalization (1). Of course, the class $K(\varphi)$ consists only of univalent functions [2, p. 196].

It was observed by Hengartner and Schober [5] that $f(z) \in CIA$ does not imply $f(rz)/r \in CIA$ for all $r \in (0, 1)$, contrary to the other classes of univalent functions like convex, starlike and close-to-convex. It was shown by Prokhorov [6] and by a completely different method by Ruscheweyh and Salinas [7] that only the circles \{ $z : |z| = r \leq \sqrt{2} - 1$ \} are mapped by $f(z) \in CIA$ onto the domains convex in
the direction of the imaginary axis, and this result is sharp (see also Goodman and Saff [3]).

During the Conference on Computational Methods and Function Theory, Aveiro, 2001, Suffridge posed the problem of finding the radius of the biggest disk \(|z| \leq r_{\psi, \varphi}\) with the property that each function \(f \in K(\psi)\) maps this disk onto the convex domain in the direction \(\varphi\). In this note we find the exact value of \(r_{\psi, \varphi}\).

2.

In order to state our Theorem we observe that if \(f(z) \in K(\psi)\), then \(h(z) = e^{i\delta} f(e^{-i\delta} z) \in K(\psi + \delta)\) and therefore we can assume without loss of generality that \(\psi = \pi/2\) and \(\varphi \in [-\pi/2, \pi/2]\).

Denote \(r_{\varphi} := r_{\pi/2, \varphi}\). We have

**Theorem.** If \(f(z) \in CIA\), then \(f(r_{\varphi} z)/r_{\varphi} \in K(\varphi)\) for all \(\varphi \in [-\pi/2, \pi/2]\) where

\[
r_{\varphi} = 2 \cos \varphi - \sqrt{4 \cos^2 \varphi - 1}.
\]

**Proof.** Let \(f(z) \in CIA\). We have to prove that \(f(r_{\varphi} z)/r_{\varphi} \in K(\varphi)\). Observe that if \(\alpha = \pi/2 - \varphi\), then the function

\[
F_\alpha(z) = e^{i\alpha} f(e^{-i\alpha} z)
\]

possesses the property that \(F_\alpha(r_{\varphi} z)/r_{\varphi} \in CIA\).

From the geometric properties of the convexity in the direction of the imaginary axis it follows that the function

\[
u(\theta) := \frac{\partial}{\partial \theta} \Re [F_\alpha(r_{\varphi} e^{i\theta})] = -\Im [r_{\varphi} e^{i\theta} \frac{d}{d\theta} F_\alpha'(r_{\varphi} e^{i\theta})]
\]

has to be non-negative and non-positive respectively for \(\theta\) corresponding to two complementary arcs of the unit circle \(z = e^{i\theta}\) \([2\text{ p. 195}]\). This is equivalent to the fact that the function

\[
v(\theta) := \arg[r_{\varphi} e^{i\theta} F_\alpha'(r_{\varphi} e^{i\theta})]
\]

attains its extremal values on some intervals \(\Delta_1 = [\theta_1, \theta_2]\) and \(\Delta_2 = [\theta_2, \theta_1 + 2\pi]\), \(\theta_1 < \theta_2 < \theta_1 + 2\pi\), only at the end points.

Suppose \(r_{\varphi}\) is the critical radius for \(f(z)\), therefore there exists a critical point \(\theta_0\) such that

\[u'(\theta_0) = u''(\theta_0) = 0,\]

which can be expressed under the denotation \(z_0 = r_{\varphi} e^{i\theta_0}\) as

\[
\Im [z_0 F_\alpha'(z_0)] = 0
\]

and

\[
\Re \left[1 + \frac{z_0 F_\alpha''(z_0)}{F_\alpha'(z_0)}\right] = 0.
\]

According to (6) \(e^{i\theta_0} F_\alpha'(z_0)\) is real. Hence the function

\[
g(z) = \frac{e^{-i\theta_0} F_\alpha(z)}{F_\alpha'(z_0)(1 - r_{\varphi}^2)} = z + b_2 z^2 + \ldots
\]

belongs to \(K(\pi - \varphi)\) together with \(F_\alpha(z)\).
Substituting (8) into (6) and (7) we obtain

\[ \Re[1 + r_f^2 + 2r_f b_2] = 0. \]  

and

\[ \Re[\bar{z}e^{-2i\varphi} + 2z \cos \varphi] = 0. \]  

The function

\[ G_{-\alpha}(z) = e^{-i\alpha}g(e^{i\alpha}z) = z + c_2 z^2 + \ldots \]

belongs to CIA and \( c_2 = e^{i\alpha}b_2 \). Equations (9) and (10) are now equivalent to

\[ \Im[e^{-i\alpha}G'_{-\alpha}(-e^{-i\alpha}r_f)] = 0 \]

and

\[ \Re[1 + r_f^2 + 2e^{-i\alpha}c_2] = 0. \]

To estimate \( \Re[e^{-i\alpha}c_2] \) for \( G_{-\alpha}(z) \in CIA \) consider the integral representation

\[ G_{-\alpha}(z) = \int_0^z \frac{p(t)dt}{(1 - \tau_1 t)(1 - \tau_2 t)}, \]

where \( p(z), p(0) = 1 \), is holomorphic in \( D \) and satisfies \( \Re[e^{-i\gamma}p(z)] > 0, z \in D \), for a certain \( \gamma \in (-\pi/2, \pi/2) \). Points \( z_1 \) and \( z_2 \), \( |z_1| = |z_2| = 1 \), are such that the function

\[ w(z) = \frac{z}{(1 - \tau_1 z)(1 - \tau_2 z)} \]

maps \( D \) onto the complex plane minus a continuum on the straight line which passes through the origin and has the slope \((\pi/2 - \gamma)\). This means that \( z_1 = e^{i\theta_1} \) and \( z_2 = e^{i\theta_2} \) where \( (\theta_1 + \theta_2)/2 = -(\pi/2 + \gamma) \). From (14) we obtain \((\alpha = \frac{\pi}{2} - \varphi)\):

\[ \Re[e^{-i\alpha}c_2] = \frac{1}{2} \Re[e^{-i\alpha}[p'(0) + \bar{z}_1 + \bar{z}_2]] \geq - \frac{|p'(0)| - \cos(\theta_1 + \alpha) - \cos(\theta_2 + \alpha)}{2} \]

\[ \geq - \cos \gamma + |\cos(\frac{\theta_1 + \theta_2}{2} + \alpha)| = - \cos \gamma - |\cos(\varphi + \gamma)| \geq -2 \cos \frac{\varphi}{2}. \]

Inequality (15) and equation (13) imply the estimate

\[ r_\varphi \geq 2 \cos \frac{\varphi}{2} - \sqrt{4 \cos^2 \frac{\varphi}{2} - 1}. \]

Notice that all the inequalities in (15) are sharp. To show that estimate (16) cannot be improved we should determine a function \( G_{-\alpha}(z) \in CIA \) for which the inequalities (15) become equalities. Going back to \( g(z) \), \( F_\alpha(z) \) and \( f(z) \in CIA \) by (11), (8) and (3) we find the extremal function of the above problem.

We will give the explicit examples for \( \varphi \in [0, \pi/2] \). Let \( \beta = (\pi - \varphi)/2 = (\alpha + \pi/2)/2 \) and

\[ G_{-\alpha}(z) = \int_0^z \frac{1 + e^{3\beta t}}{(1 + e^{3\beta t})^3} dt. \]

This is a special case of the formula (14) with \( p(z) = (1 + e^{3\beta z})/(1 + e^{i\beta}z) \) and \( z_1 = z_2 = -e^{-i\beta} \). Therefore \( G_{-\alpha}(z) \in CIA \).
From (11) and (17) we have
\[ c_2 = \frac{1}{2}(e^{i\beta} - 3e^{i\beta}) \]
and therefore
\[ \Re[e^{-i\alpha}c_2] = \frac{1}{2}((\cos(3\beta - \alpha)) - 3\cos(\beta - \alpha)) = -2\cos\frac{\varphi}{2}. \]

Equations (2), (13) and (18) imply that if \( r = r_\varphi + \epsilon \) for \( \epsilon > 0 \) small enough, then
\[ \Re[1 + r^2 + 2e^{-i\alpha}rc_2] < 0. \]

Equation (13) does not depend on (12), and the critical value \( \theta_0 \) can be found directly from (12).

Inequality (19) together with (12) mean that there are at least four different points \( \theta \in [0, 2\pi] \) where the function
\[ u_r(\theta) := \frac{\partial}{\partial \theta} \Re[F_\alpha(re^{i\theta})] = -\Im[F_\alpha'(re^{i\theta})] \]
changes its sign. Therefore the function \( F_\alpha(rz)/r \) does not belong to CIA. This completes the proof of the Theorem.

Remark. The radius \( r_\varphi \) increases on \([0, \pi/2]\) from \( r_0 = 2 - \sqrt{3} \) which is the radius of convexity in the class \( S \) of holomorphic and univalent functions \( f(z) \) of form (1) to \( r_{\pi/2} = \sqrt{2} - 1 \). One can observe that the classes \( K(\varphi) \) are different for different \( \varphi \in [0, 2\pi] \). Moreover, the class \( C \) of convex univalent functions \( f(z) \) of form (1) is the proper subclass of \( K(\varphi) \) for any \( \varphi \).

The next example illustrates the Remark.

Example. Let \( B = B_1 \cup B_2 \) be the union of two rectangles \( B_1 = \{ z = x + iy : -2 < x < 2, -1 < y < 1 \} \) and \( B_2 = \{ z = x + iy : -1 < x < 1, -2 < y < 2 \} \). The function \( f(z) = d_1z + d_2z^2 + \ldots, d_1 > 0 \), mapping \( D \) onto \( B \) can be represented by the Schwarz-Christoffel integral. Then \( f(z)/d_1 \) belongs to CIA and \( K(0) \) but does not belong to any \( K(\varphi) \) for \( \varphi \in (0, \pi/2) \cup (\pi/2, \pi) \).

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REFERENCES


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