LINEAR PERTURBATIONS OF A NONOSCILLATORY SECOND ORDER DIFFERENTIAL EQUATION II

WILLIAM F. TRENCH

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Abstract. Let $y_1$ and $y_2$ be principal and nonprincipal solutions of the nonoscillatory differential equation $(r(t)y')' + f(t)y = 0$. In an earlier paper we showed that if $\int_\infty (f - g)y_1y_2 \, dt$ converges (perhaps conditionally), and a related improper integral converges absolutely and sufficiently rapidly, then the differential equation $(r(t)x')' + g(t)x = 0$ has solutions $x_1$ and $x_2$ that behave asymptotically like $y_1$ and $y_2$. Here we consider the case where $\int_\infty (f - g)y_2^2 \, dt$ converges (perhaps conditionally) without any additional assumption requiring absolute convergence.

1. Introduction

We consider the differential equation

$$
(r(t)x')' + g(t)x = 0
$$

as a perturbation of

$$
(r(t)y')' + f(t)y = 0,
$$

under the following standing assumption.

Assumption A. Let $r$ and $f$ be real-valued and continuous, with $r > 0$, on $[a, \infty)$. Suppose that (2) is nonoscillatory at infinity. Let $g$ be continuous and possibly complex-valued on $[a, \infty)$.

It is known [4, p. 355] that since (2) is nonoscillatory at infinity, it has solutions $y_1$ and $y_2$ which are positive on $[b, \infty)$ for some $b \geq a$ and satisfy the following conditions:

$$
r(y_1y_2' - y_1'y_2) = 1, \quad t \geq a,
$$

$$
\lim_{t \to \infty} \frac{y_2(t)}{y_1(t)} = \infty.
$$

Without loss of generality we let $b = a$. Henceforth $t \geq a$. It is convenient to define

$$
\rho = \frac{y_2}{y_1}.
$$
From (3) and (4),
\[ \rho' = \frac{1}{\rho y_1^2} > 0 \quad \text{and} \quad \lim_{t \to \infty} \rho(t) = \infty. \] (6)

We use the Landau symbols “\( o \)” and “\( O \)” in the standard way to denote behavior as \( t \to \infty \). In [6] we proved the following theorem.

**Theorem 1.** Suppose that \( \int_{\infty}^{\infty} (f - g) y_1 y_2 \ dt \) converges (perhaps conditionally) and
\[ \sup_{\tau \geq t} \left| \int_{\tau}^{\infty} (f - g) y_1 y_2 \ ds \right| \leq \phi(t), \] (7)
where \( \phi(t) \to 0 \) monotonically as \( t \to \infty \). Define
\[ G(t) = \int_{t}^{\infty} (f - g) y_1^2 \ ds, \] (8)
and suppose that
\[ \int_{\infty}^{\infty} |G| \phi \rho' \ dt < \infty \]
and
\[ \limsup_{t \to \infty} (\phi(t))^{-1} \int_{t}^{\infty} |G| \phi \rho' \ ds = A < 1/3. \] (9)

Then (1) has a solution \( x_1 \) such that
\[ x_1 = y_1 (1 + O(\phi)) \]
and
\[ (x_1/y_1)' = O(\phi \rho'/\rho), \]
and a solution \( x_2 \) such that
\[ x_2 = y_2 (1 + O(\phi_m)) \]
and
\[ (x_2/y_2)' = O(\phi_m \rho'/\rho), \]
where
\[ \phi_m = \max\{\phi, \hat{\phi}\} \]
with
\[ \hat{\phi}(t) = \frac{1}{\rho(t)} \int_{0}^{t} \rho' \phi \ ds. \]

This result was an improvement on a theorem of Hartman and Wintner [4, p. 379], and it was subsequently improved by Chen [1] and Šimša [5]. (For more on the Hartman-Wintner problem, see [2] and [3].) In this continuation of [6] we consider the case where \( \int_{\infty}^{\infty} (f - g) y_2^2 \ dt \) converges, perhaps conditionally. To motivate the present work, we first apply Theorem 1 under this assumption.

Let
\[ H(t) = \int_{t}^{\infty} (f - g) y_1 y_2 \ ds, \]
and recall from (7) that
\[ \sup_{\tau \geq t} \{|H(\tau)|\} \leq \phi(t). \] (10)
Let
\[ I(t) = \int_t^\infty (f - g)y^2 \, ds, \tag{11} \]
and suppose that
\[ \sup_{\tau \geq t} |I(\tau)| \leq \sigma(t), \tag{12} \]
where \( \sigma(t) \to 0 \) monotonically as \( t \to \infty \). From (8), (10), and (11),
\[ H(t) = -\int_t^\infty \frac{I'}{\rho} \, ds = \frac{I(t)}{\rho(t)} + \int_t^\infty I \left( \frac{1}{\rho} \right)' \, ds \tag{13} \]
and
\[ G(t) = -\int_t^\infty \frac{I'}{\rho^2} \, ds = \frac{I(t)}{\rho^2(t)} + \int_t^\infty I \left( \frac{1}{\rho^2} \right)' \, ds, \tag{14} \]
so
\[ |H(t)| \leq 2\sigma(t)/\rho(t) \quad \text{and} \quad |G(t)| \leq 2\sigma(t)/\rho^2(t). \]

It is straightforward to verify that (9) holds with \( \phi = \sigma/\rho \) and \( A = 0 \). Therefore Theorem 1 implies that (1) has solutions \( x_1 \) and \( x_2 \) such that
\[ x_1 = y_1(1 + O(\sigma/\rho)), \tag{15} \]
\[ (x_1/y_1)' = O(\sigma^2/\rho^2), \tag{16} \]
\[ x_2 = y_2(1 + O(\hat{\phi})), \tag{17} \]
and
\[ (x_2/y_2)' = O(\hat{\phi}^2/\rho), \tag{18} \]
with
\[ \hat{\phi}(t) = \frac{1}{\rho(t)} \int_t^\infty \frac{\sigma'}{\rho} \, ds. \]

At best, (17) and (18) imply that
\[ x_2 = y_2(1 + O(1/\rho)) \]
and
\[ (x_2/y_2)' = O(\rho'/\rho^2) \]
if \( \int_a^\infty \sigma'/\rho \, ds < \infty \), which may be false. Among other things, we will show that (17) and (18) can be replaced by
\[ x_2 = y_2(1 + O(\sigma/\rho)) \tag{19} \]
and
\[ (x_2/y_2)' = O(\sigma^2/\rho^2). \tag{20} \]

These two equations are improvements over (17) and (18), since \( \lim_{t \to \infty} \rho(t)\hat{\phi}(t)/\sigma(t) = \infty \) in any case. In fact, it can be seen from (15), (16), (19), and (20) that \((x_i/y_i) - 1, i = 1, 2\), approach zero at the same rate as \( t \to \infty \), as do \((x_i/y_i)'\), \( i = 1, 2 \). We also note that the results of these four equations can be written as
\[ x_i/y_i = 1 + O(\sigma y_1/y_2) \quad \text{and} \quad (x_i/y_i)' = O(\sigma/\rho y_2^2), \quad i = 1, 2. \]
2. MAINE RESULTS

Theorem 2. Suppose that \( \int_0^x (f - g) y^2 dt \) converges. Let \( I \) and \( \sigma \) be as in (11) and (12). Then (11) has a solution \( x_1 \) that satisfies (15) and (16), and a solution \( x_2 \) such that

\[
\frac{x_2 - y_2}{y_1} = O(\sigma)
\]
and

\[
\left( \frac{x_2 - y_2}{y_1} \right)' = O \left( \frac{\sigma \rho'}{\rho} \right).
\]

Proof. We have already proved the assertion concerning \( x_1 \). For the assertion concerning \( x_2 \), we use the contraction mapping theorem. If

\[
x_2(t) = y_2(t) + \int_t^\infty (y_2(s)y_1(t) - y_1(s)y_2(t))(f(s) - g(s))x_2(s) ds,
\]
then \( x_2 \) satisfies (11). Although this suggests a transformation to work with, it is better to use a transformation with the fixed point \( \zeta \), where

\[
\zeta = \frac{x_2 - y_2}{y_1}.
\]

Rewriting (23) in terms of \( \zeta \) yields

\[
\zeta(t) = \int_t^\infty (y_2(s) - y_1(s)\rho(t))(f(s) - g(s))y_2(s) ds
\]
plus

\[
\int_t^\infty (y_2(s) - y_1(s)\rho(t))(f(s) - g(s))y_1(s)\zeta(s) ds.
\]

We use the transformation \( Tz = Q + Lz \), where

\[
Q(t) = \int_t^\infty (y_2(s) - y_1(s)\rho(t))(f(s) - g(s))y_2(s) ds
\]
and

\[
(Lz)(t) = \int_t^\infty (y_2(s) - y_1(s)\rho(t))(f(s) - g(s))y_1(s)z(s) ds.
\]

From (11), (11), and (13),

\[
Q(t) = I(t) - \rho(t) H(t) = -\rho(t) \int_t^\infty I(1/\rho)' ds,
\]
so \( |Q(t)| \leq \sigma(t) \), from (12). Moreover,

\[
Q' = I' - \rho H' - H\rho' = -H\rho',
\]
so

\[
|Q'(t)| \leq 2\sigma(t)\rho'(t)/\rho(t),
\]
from (14). Therefore we let \( T \) act on the Banach space \( B \) of functions \( z \) on \([t_0, \infty)\) such that

\[
z = O(\sigma) \quad \text{and} \quad z' = O(\sigma \rho'/\rho),
\]
with norm

\[
\|z\| = \sup_{t \geq t_0} \left\{ \max \left\{ \frac{|z|}{\sigma}, \frac{|\rho z'|}{\sigma \rho'} \right\} \right\}.
\]
We will show that $T$ maps $B$ into $B$, and is a contraction if $t_0$ is sufficiently large. Since $Q \in B$, it suffices to show that $L$ is a contraction of $B$ if $t_0$ is sufficiently large. To this end, suppose $z \in B$ and $t_0 \leq t < T$, and consider the finite integral

$$w_T(t; z) = \int_t^T (y_2(s) - y_1(s)\rho(t))(f(s) - g(s))y_1(s)z(s) \, ds.$$

From (5) and (8),

$$w_T(t; z) = -\int_t^T (\rho(s) - \rho(t))z(s)G'(s) \, ds
= -(\rho(T) - \rho(t))z(T)G(T)
+ \int_t^T (\rho(s) - \rho(t))G(s)z'(s) \, ds
+ \int_t^T z(s)G(s)\rho'(s) \, ds.$$

From (14) and (24),

$$\left| (\rho(T) - \rho(t))z(T)G(T) \right| < 2\|z\|\sigma^2(T)/\rho(T) \to 0 \text{ as } T \to \infty,$$

$$\left| (\rho(s) - \rho(t))G(s)z'(s) \right| \leq 2\|z\|\sigma^2(s)\rho'(s)/\rho^2(s), \quad s \geq t,$$

and

$$\left| z(s)G(s)\rho'(s) \right| \leq 2\|z\|\sigma^2(s)\rho'(s)/\rho^2(s).$$

Therefore we can let $T \to \infty$ in (25) and conclude that

$$(Lz)(t) = -\int_t^\infty (\rho(s) - \rho(t))z(s)G'(s) \, ds$$

exists and satisfies the inequality

$$|(Lz)(t)| < 4\|z\| \int_t^\infty \frac{\sigma^2\rho'}{\rho^2} \, ds < 4\|z\| \frac{\sigma^2(t)}{\rho(t)}.$$

From (26),

$$(Lz)'(t) = \rho'(t) \int_t^\infty zG' \, ds = -\rho'(t) \left( z(t)G(t) + \int_t^\infty Gz' \, ds \right).$$

From (14) and (24), the last integral converges absolutely and

$$|(Lz)'(t)| \leq 2\|z\|\rho'(t) \left( \frac{\sigma^2(t)}{\rho^2(t)} + \int_t^\infty \frac{\sigma^2\rho'}{\rho^2} \, ds \right) < 4\|z\| \frac{\sigma^2(t)\rho'(t)}{\rho^2(t)}.$$n

From this and (20),

$$||(Lz)|| < 4\|z\|\sigma(t)/\rho(t).$$

Hence $L$ (and consequently $T$) is a contraction of $B$ if $\sigma(t_0)/\rho(t_0) < 1/4$. Therefore there is a unique $\zeta \in B$ such that $T\zeta = \zeta$, and the function $x_2$ defined by

$$x_2 = y_2 + y_1\zeta \quad (t \geq t_0)$$

is a solution of (11) that satisfies (21) and (22). We can extend the definition of $x_2$ back to $t = a$.

**Corollary 1.** Under the assumptions of Theorem 3, $x_2$ satisfies (14) and (20).
Proof. Since \( y_2/y_1 = \rho \), (21) implies that \( y_2 \) satisfies (19) and
\[
x_2/y_1 = \rho + O(\sigma).
\]
From (22),
\[
(x_2/y_1)' = \rho' (1 + O(\sigma/\rho)).
\]
Therefore
\[
\left( \frac{x_2}{y_2} \right)' = \left( \frac{x_2}{y_1} \right)' \frac{1}{\rho} - \frac{x_2 \rho'}{y_1 \rho^2} = \frac{\rho'}{\rho} (1 + O(\sigma/\rho)) - \frac{\rho'}{\rho^2} (\rho + O(\sigma)) = O \left( \frac{\sigma \rho'}{\rho^2} \right).
\]

It is natural to ask whether the convergence of \( \int_{t}^\infty (f - g)y_2^2 \, dt \) is necessary for the existence of a solution \( x_2 \) of (1) such that
\[
x_2 = y_2 (1 + o(1/\rho)) \quad \text{and} \quad (x_2/y_2)' = o(\rho'/\rho^2).
\]
Although we do not know the answer to this question, we offer the following related theorem.

**Theorem 3.** If (1) has a solution \( x_2 \) that satisfies (19) and (20) for some positive monotonic function \( \sigma \) such that \( \lim_{t \to \infty} \sigma(t) = 0 \), then
\[
\int_{t}^\infty (f - g)y_1y_2 \, dt = O(\sigma/\rho).
\]
Moreover, if
\[
\int_{t}^\infty \frac{\sigma \rho'}{\rho} \, dt < \infty,
\]
then \( \int_{t}^\infty (f - g)y_2^2 \, dt \) converges.

**Proof.** From (20), \( R(t) = \int_{t}^\infty (x_2/y_2)' \, ds \) converges absolutely and
\[
R = O(\sigma/\rho).
\]
If \( t > T \), define
\[
R_T(t) = \int_{t}^{T} \left( \frac{x_2}{y_2} \right)' \, ds.
\]
From (19) and (18),
\[
\left( \frac{x_2}{y_2} \right)' = y_2 x_2' - x_2 y_2' = u \frac{\rho'}{\rho^2},
\]
where
\[
u = r(y_2 x_2' - x_2 y_2').
\]
From (11) and (17),
\[
u' = (f - g)y_2 x_2.
\]
Therefore
\[
R_T(t) = \frac{u(t)}{\rho(t)} - \frac{u(T)}{\rho(T)} + \int_{t}^{T} (f - g)y_1 x_2 \, ds.
\]
From (20) and (31), \( u = o(\sigma) \), so we can let \( T \to \infty \) and invoke (30) to conclude that

\[
R(t) = \int_t^\infty (f - g)y_1x_2 \, ds = O(\sigma/\rho).
\]

Now let

\[
s_T(t) = \int_t^T (f - g)y_1y_2 \, ds = -\int_t^T \frac{y_2}{x_2} \hat{R}' \, ds
\]

\[
= \frac{y_2(t)}{x_2(t)} \hat{R}(t) - \frac{y_2(T)}{x_2(T)} \hat{R}(T) + \int_t^T \hat{R} \left( \frac{y_2}{x_2} \right)' \, ds.
\]

But

\[
\left( \frac{y_2}{x_2} \right)' = -\frac{y_2}{x_2} \left( \frac{x_2}{y_2} \right)' = O \left( \frac{\sigma \rho'}{\rho^2} \right)
\]

from (19) and (20). From this and (32), we can let \( T \to \infty \) in (33) to conclude that

\[
S(t) = \int_t^\infty (f - g)y_1y_2 = O(\sigma/\rho).
\]

This verifies (28). If (29) holds and \( T > a \), then

\[
\int_a^T (f - g)y_2^2 \, dt = -\int_a^T \rho S' \, dt = \rho(a)S(a) - \rho(T)S(T) + \int_a^T S' \rho' \, dt.
\]

Since (34) implies that \( \lim_{T \to \infty} \rho(T)S(T) = 0 \) and (29) and (34) together imply that \( \int_a^\infty S' \rho' \, dt \) converges, (35) implies that \( \int_a^\infty (f - g)y_2^2 \, dt \) converges.

3. Examples

Examples illustrating our results can be constructed by letting

\[
g(t) = f(t) + \frac{u(t)S(t)}{y_2^2(t)}, \quad t \geq a,
\]

where \( u \) and \( S \) are continuously differentiable and \( S \) has a bounded antiderivative \( C \) on \([a, \infty)\), while \( \lim_{t \to \infty} u(t) = 0 \) and \( \int_a^\infty |u'(t)| \, dt < \infty \). Then

\[
\int_a^\infty (f(s) - g(s))y_2^2(s) \, ds = -\int_a^\infty u(s)S(s) \, ds = -u(s)C(s) \bigg|_t^\infty \ + \int_t^\infty u'(s)C(s) \, ds
\]

converges, and the convergence may be conditional. Here we may take

\[
\sigma(t) = M \sup_{\tau \geq t} \left( |u(\tau)| + \int_\tau^\infty |u'(s)| \, ds \right),
\]

where \( M \) is an upper bound for \( C \) on \([a, \infty)\).

For a specific example, consider the equation

\[
x'' + \frac{\sin t}{t^2(\log t)^\alpha} x = 0, \quad t \geq a > 0 \quad (\alpha > 0),
\]

as a perturbation of \( y'' = 0 \). Our results imply that (36) has solutions \( x_1 \) and \( x_2 \) such that

\[
x_1(t) = 1 + O \left( t^{-1}(\log t)^{-\alpha} \right), \quad x_1'(t) = O \left( t^{-2}(\log t)^{-\alpha} \right)
\]

and

\[
x_2(t) = t + O((\log t)^{-\alpha}), \quad x_2'(t) = 1 + O(t^{-1}(\log t)^{-\alpha}).
\]
References


