LINEAR PERTURBATIONS OF A NONOSCILLATORY SECOND ORDER DIFFERENTIAL EQUATION II

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Abstract. Let \( y_1 \) and \( y_2 \) be principal and nonprincipal solutions of the nonoscillatory differential equation \( (r(t)y')' + f(t)y = 0 \). In an earlier paper we showed that if \( \int_\infty^\infty (f - g)y_1y_2 \, dt \) converges (perhaps conditionally), and a related improper integral converges absolutely and sufficiently rapidly, then the differential equation \( (r(t)x')' + g(t)x = 0 \) has solutions \( x_1 \) and \( x_2 \) that behave asymptotically like \( y_1 \) and \( y_2 \). Here we consider the case where \( \int_\infty^\infty (f - g)y_2^2 \, dt \) converges (perhaps conditionally) without any additional assumption requiring absolute convergence.

1. Introduction

We consider the differential equation
\[
(r(t)x')' + g(t)x = 0
\]
as a perturbation of
\[
(r(t)y')' + f(t)y = 0,
\]
under the following standing assumption.

Assumption A. Let \( r \) and \( f \) be real-valued and continuous, with \( r > 0 \), on \([a, \infty)\). Suppose that (2) is nonoscillatory at infinity. Let \( g \) be continuous and possibly complex-valued on \([a, \infty)\).

It is known \[4, p. 355\] that since (2) is nonoscillatory at infinity, it has solutions \( y_1 \) and \( y_2 \) which are positive on \([b, \infty)\) for some \( b \geq a \) and satisfy the following conditions:
\[
(3) \quad r(y_1y_2' - y_1'y_2) = 1, \quad t \geq a,
\]
\[
(4) \quad \lim_{t \to \infty} \frac{y_2(t)}{y_1(t)} = \infty.
\]

Without loss of generality we let \( b = a \). Henceforth \( t \geq a \). It is convenient to define
\[
(5) \quad \rho = y_2/y_1.
\]

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From (3) and (4),
\[ \rho' = \frac{1}{\rho y_1^2} > 0 \quad \text{and} \quad \lim_{t \to \infty} \rho(t) = \infty. \]

We use the Landau symbols “\( o \)" and “\( O \)" in the standard way to denote behavior as \( t \to \infty \). In [6] we proved the following theorem.

**Theorem 1.** Suppose that \( \int_{-\infty}^{\infty} (f - g) y_1 y_2 \, dt \) converges (perhaps conditionally) and
\[ \sup_{\tau \geq t} \left| \int_{\tau}^{\infty} (f - g) y_1 y_2 \, ds \right| \leq \phi(t), \]
where \( \phi(t) \to 0 \) monotonically as \( t \to \infty \). Define
\[ G(t) = \int_{t}^{\infty} (f - g) y_1 y_2 \, ds, \]
and suppose that
\[ \int_{-\infty}^{\infty} |G| \rho' \, dt < \infty \]
and
\[ \limsup_{t \to \infty} (\phi(t))^{-1} \int_{t}^{\infty} |G| \rho' \, ds = A < 1/3. \]
Then (7) has a solution \( x_1 \) such that
\[ x_1 = y_1 (1 + O(\phi)) \]
and
\[ (x_1 / y_1)' = O(\phi' / \rho), \]
and a solution \( x_2 \) such that
\[ x_2 = y_2 (1 + O(\phi_m)) \]
and
\[ (x_2 / y_2)' = O(\phi_m \rho' / \rho), \]
where
\[ \phi_m = \max\{\phi, \hat{\phi}\} \]
with
\[ \hat{\phi}(t) = \frac{1}{\rho(t)} \int_{a}^{t} \rho' \phi \, ds. \]

This result was an improvement on a theorem of Hartman and Wintner [4, p. 379], and it was subsequently improved by Chen [1] and Šimša [5]. (For more on the Hartman-Wintner problem, see [2] and [3].) In this continuation of [6] we consider the case where \( \int_{-\infty}^{\infty} (f - g) y_2^2 \, dt \) converges, perhaps conditionally. To motivate the present work, we first apply Theorem 1 under this assumption.

Let
\[ H(t) = \int_{t}^{\infty} (f - g) y_1 y_2 \, ds, \]
and recall from (7) that
\[ \sup_{\tau \geq t} |H(\tau)| \leq \phi(t). \]
Let
\begin{equation}
I(t) = \int_t^\infty (f - g)y_2^2 \, ds,
\end{equation}
and suppose that
\begin{equation}
\sup_{\tau \geq \xi} |I(\tau)| \leq \sigma(t),
\end{equation}
where \( \sigma(t) \to 0 \) monotonically as \( t \to \infty \). From (8), (10), and (11),
\begin{equation}
H(t) = -\int_t^\infty \frac{I'}{\rho} \, ds = \frac{I(t)}{\rho(t)} + \int_t^\infty I \left( \frac{1}{\rho} \right)' \, ds
\end{equation}
and
\begin{equation}
G(t) = -\int_t^\infty \frac{I'}{\rho^2} \, ds = \frac{I(t)}{\rho^2(t)} + \int_t^\infty I \left( \frac{1}{\rho^2} \right)' \, ds,
\end{equation}
so
\begin{equation}
|H(t)| \leq 2\sigma(t)/\rho(t) \quad \text{and} \quad |G(t)| \leq 2\sigma(t)/\rho^2(t).
\end{equation}
It is straightforward to verify that (12) holds with \( \phi = \sigma/\rho \) and \( A = 0 \). Therefore Theorem 1 implies that (1) has solutions \( x_1 \) and \( x_2 \) such that
\begin{equation}
x_1 = y_1(1 + O(\sigma/\rho)),
\end{equation}
\begin{equation}
(x_1/y_1)' = O(\sigma\rho'/\rho^2),
\end{equation}
\begin{equation}
x_2 = y_2(1 + O(\hat{\phi})),$n and
\begin{equation}
(x_2/y_2)' = O(\hat{\phi}\rho'/\rho),
\end{equation}
with
\begin{equation}
\hat{\phi}(t) = \frac{1}{\rho(t)} \int_t^\xi \frac{\sigma'}{\rho} \, ds.
\end{equation}
At best, (17) and (18) imply that
\begin{equation}
x_2 = y_2(1 + O(1/\rho))
\end{equation}
and
\begin{equation}
(x_2/y_2)' = O(\rho'/\rho^2)
\end{equation}
if \( \int_t^\infty \sigma' / \rho \, ds < \infty \), which may be false. Among other things, we will show that (17) and (18) can be replaced by
\begin{equation}
x_2 = y_2(1 + O(\sigma/\rho))
\end{equation}
and
\begin{equation}
(x_2/y_2)' = O(\sigma\rho'/\rho^2).
\end{equation}
These two equations are improvements over (17) and (18), since \( \lim_{t \to \infty} \rho(t)\hat{\phi}(t)/\sigma(t) = \infty \) in any case. In fact, it can be seen from (15), (16), (19), and (20) that \( (x_i/y_i) - 1, \) \( i = 1, 2 \), approach zero at the same rate as \( t \to \infty \), as do \( (x_1/y_1)' \), \( i = 1, 2 \). We also note that the results of these four equations can be written as
\begin{equation}
x_i/y_i = 1 + O(\sigma y_1/y_2) \quad \text{and} \quad (x_i/y_i)' = O(\sigma/ry_2^2), \quad i = 1, 2.
\end{equation}
2. Main results

Theorem 2. Suppose that $\int_{-\infty}^{\infty} (f-g)y_2^2\,dt$ converges. Let $I$ and $\sigma$ be as in (11) and (12). Then (11) has a solution $x_1$ that satisfies (15) and (16), and a solution $x_2$ such that

\[(21) \quad \frac{x_2 - y_2}{y_1} = O(\sigma)\]

and

\[(22) \quad \left(\frac{x_2 - y_2}{y_1}\right)' = O\left(\frac{\sigma\rho'}{\rho}\right).\]

Proof. We have already proved the assertion concerning $x_1$. For the assertion concerning $x_2$, we use the contraction mapping theorem. If

\[(23) \quad x_2(t) = y_2(t) + \int_t^\infty (y_2(s)y_1(t) - y_1(s)y_2(t))(f(s) - g(s))x_2(s)\,ds,\]

then $x_2$ satisfies (11). Although this suggests a transformation to work with, it is better to use a transformation with the fixed point $\zeta$, where

\[\zeta = (x_2 - y_2)/y_1.\]

Rewriting (23) in terms of $\zeta$ yields

\[\zeta(t) = \int_t^\infty (y_2(s) - y_1(s)\rho(t))(f(s) - g(s))y_2(s)\,ds\]

\[+ \int_t^\infty (y_2(s) - y_1(s)\rho(t))(f(s) - g(s))y_1(s)\zeta(s)\,ds.\]

We use the transformation $Tz = Q + Lz$, where

\[Q(t) = \int_t^\infty (y_2(s) - y_1(s)\rho(t))(f(s) - g(s))y_2(s)\,ds\]

and

\[(Lz)(t) = \int_t^\infty (y_2(s) - y_1(s)\rho(t))(f(s) - g(s))y_1(s)z(s)\,ds.\]

From (11), (11), and (13),

\[Q(t) = I(t) - \rho(t)H(t) = -\rho(t)\int_t^\infty I(1/\rho)'\,ds,\]

so $|Q(t)| \leq \sigma(t)$, from (12). Moreover,

\[Q' = I' - \rho H' - H\rho' = -H\rho',\]

so

\[|Q'(t)| \leq 2\sigma(t)\rho'(t)/\rho(t),\]

from (14). Therefore we let $T$ act on the Banach space $B$ of functions $z$ on $[t_0, \infty)$ such that

\[z = O(\sigma)\quad \text{and} \quad z' = O(\sigma\rho'/\rho),\]

with norm

\[(24) \quad ||z|| = \sup_{t \geq t_0} \left\{ \max \left\{ \frac{|z|}{\sigma}, \frac{\rho|z'|}{\sigma\rho'} \right\} \right\}.\]
We will show that $T$ maps $B$ into $B$, and is a contraction if $t_0$ is sufficiently large. Since $Q \in B$, it suffices to show that $L$ is a contraction of $B$ if $t_0$ is sufficiently large. To this end, suppose $z \in B$ and $t_0 \leq t < T$, and consider the finite integral

$$w_T(t; z) = \int_t^T (y_2(s) - y_1(s)\rho(t))(f(s) - g(s))y_1(s)z(s) \, ds.$$ 

From (5) and (8),

$$w_T(t; z) = -\int_t^T (\rho(s) - \rho(t))z(s)G'(s) \, ds = -z(T)z(T)G(T) + \int_t^T (\rho(s) - \rho(t))G(s)z'(s) \, ds + \int_t^T z(s)G(s)\rho'(s) \, ds.$$ 

From (26) and (24),

$$|z(s)G(s)\rho'(s)| \leq 2\|z\|\sigma^2(s)\rho'(s)/\rho^2(s).$$

Therefore we can let $T \to \infty$ in (25) and conclude that

$$\mathcal{L}z(t) = -\int_t^\infty (\rho(s) - \rho(t))z(s)G'(s) \, ds$$

exists and satisfies the inequality

$$|\mathcal{L}z(t)| \leq 4\|z\| \int_t^\infty \frac{\sigma^2\rho'}{\rho^2} \, ds < 4\|z\| \frac{\sigma^2(t)}{\rho(t)}.$$ 

From (26),

$$\mathcal{L}z'(t) = \rho'(t) \int_t^\infty zG' \, ds = -\rho'(t) \left( z(t)G(t) + \int_t^\infty Gz' \, ds \right).$$

From (14) and (24), the last integral converges absolutely and

$$|\mathcal{L}z'(t)| \leq 2\|z\|\rho'(t) \left( \frac{\sigma^2(t)}{\rho^2(t)} + \int_t^\infty \frac{\sigma^2\rho'}{\rho^3} \, ds \right) < 4\|z\| \frac{\sigma^2(t)\rho'(t)}{\rho^2(t)}.$$ 

From this and (26),

$$\|\mathcal{L}z\| \leq 4\|z\|\sigma(t)/\rho(t).$$

Hence $\mathcal{L}$ (and consequently $T$) is a contraction of $B$ if $\sigma(t_0)/\rho(t_0) < 1/4$. Therefore there is a (unique) $\zeta \in B$ such that $T\zeta = \zeta$, and the function $x_2$ defined by $x_2 = y_2 + y_1\zeta$ ($t \geq t_0$) is a solution of (1) that satisfies (21) and (22). We can extend the definition of $x_2$ back to $t = a$.

Corollary 1. Under the assumptions of Theorem 2, $x_2$ satisfies (14) and (20).
Proof. Since \( \frac{y_2}{y_1} = \rho \), (21) implies that \( y_2 \) satisfies (19) and
\[
x_2/y_1 = \rho + O(\sigma).
\]
From (22),
\[
(x_2/y_1)' = \rho' \left(1 + O\left(\frac{\sigma}{\rho}\right)\right).
\]
Therefore
\[
\left(\frac{x_2}{y_2}\right)' = \left(\frac{x_2}{y_1\rho}\right)' = \left(\frac{x_2}{y_1}\right)' \frac{1}{\rho} - \frac{x_2}{y_1} \frac{\rho'}{\rho^2}
= \frac{\rho'}{\rho} \left(1 + O(\sigma/\rho)\right) - \frac{\rho'}{\rho^2} (\rho + O(\sigma)) = O\left(\frac{\sigma \rho'}{\rho^2}\right).
\]

It is natural to ask whether the convergence of \( \int_\infty^\infty (f - g)y_2^2 dt \) is necessary for the existence of a solution \( x_2 \) of (1) such that \( x_2 = y_2(1 + o(1/\rho)) \) and \( (x_2/y_2)' = o(\rho'/\rho^2) \).

Although we do not know the answer to this question, we offer the following related theorem.

**Theorem 3.** If (1) has a solution \( x_2 \) that satisfies (19) and (20) for some positive monotonic function \( \sigma \) such that \( \lim_{t \to \infty} \sigma(t) = 0 \), then
\[
\int_\infty^\infty (f - g)y_1y_2 dt = O(\sigma/\rho).
\]
Moreover, if
\[
\int_\infty^\infty \frac{\sigma \rho'}{\rho} dt < \infty,
\]
then \( \int_\infty^\infty (f - g)y_2^2 dt \) converges.

**Proof.** From (20), \( R(t) = \int_t^\infty (x_2/y_2)' dt \) converges absolutely and
\[
R = O(\sigma/\rho).
\]
If \( t > T \), define
\[
R_T(t) = \int_t^T \left(\frac{x_2}{y_2}\right)' ds.
\]
From (10) and (6),
\[
\left(\frac{x_2}{y_2}\right)' = \frac{y_2x_2' - x_2y_2'}{y_2^2} = u \frac{\rho'}{\rho^2},
\]
where
\[
u = r(y_2x_2' - x_2y_2').
\]
From (11) and (7),
\[
u' = (f - g)y_2x_2.
\]
Therefore
\[
R_T(t) = \frac{u(t)}{\rho(t)} - \frac{u(T)}{\rho(T)} + \int_t^T (f - g)y_1x_2 ds.
\]
From (20) and (31), \( u = o(\sigma) \), so we can let \( T \to \infty \) and invoke (30) to conclude that
\[
\hat{R}(t) = \int_t^\infty (f - g)y_1x_2 \, ds = O(\sigma/\rho).
\]

Now let
\[
S_T(t) = \int_t^T (f - g)y_1y_2 \, ds = -\int_t^T \frac{y_2}{x_2} \hat{R}' \, ds
\]
\[
= \frac{y_2(t)}{x_2(t)} \hat{R}(t) - \frac{y_2(T)}{x_2(T)} \hat{R}(T) + \int_t^T \left( \frac{y_2}{x_2} \right)' \, ds.
\]

But
\[
\left( \frac{y_2}{x_2} \right)' = \frac{y_2}{x_2} \left( \frac{x_2'}{y_2} \right) = O \left( \frac{\sigma \rho'}{\rho^2} \right)
\]
from (19) and (20). From this and (32), we can let \( T \to \infty \) in (33) to conclude that
\[
S(t) = \int_t^\infty (f - g)y_1y_2 = O(\sigma/\rho).
\]
This verifies (28). If (29) holds and \( T > a \), then
\[
\int_a^T (f - g)y_2^2 \, dt = -\int_a^T \rho \sigma' \, dt = \rho(a)S(a) - \rho(T)S(T) + \int_a^T S \rho' \, dt.
\]
Since (34) implies that \( \lim_{T \to \infty} \rho(T)S(T) = 0 \) and (29) and (34) together imply that \( \int^\infty S \rho' \, dt \) converges, (35) implies that \( \int^\infty (f - g)y_2^2 \, dt \) converges.

3. Examples

Examples illustrating our results can be constructed by letting
\[
g(t) = f(t) + \frac{u(t)S(t)}{y_2^2(t)}, \quad t \geq a,
\]
where \( u \) and \( S \) are continuously differentiable and \( S \) has a bounded antiderivative \( C \) on \([a, \infty)\), while \( \lim_{t \to \infty} u(t) = 0 \) and \( \int^\infty |u'(t)| \, dt < \infty \). Then
\[
\int_t^\infty (f(s) - g(s))y_2^2(s) \, ds = -\int_t^\infty u(s)S(s) \, ds = -u(s)C(s) \bigg|_t^\infty + \int_t^\infty u'(s)C(s) \, ds
\]
converges, and the convergence may be conditional. Here we may take
\[
\sigma(t) = M \sup_{t \geq t} \left( |u(\tau)| + \int_{\tau}^\infty |u'(s)| \, ds \right),
\]
where \( M \) is an upper bound for \( C \) on \([a, \infty)\).

For a specific example, consider the equation
\[
x'' + \frac{\sin t}{t^2(\log t)^\alpha} x = 0, \quad t \geq a > 0 \quad (\alpha > 0),
\]
as a perturbation of \( y'' = 0 \). Our results imply that (36) has solutions \( x_1 \) and \( x_2 \) such that
\[
x_1(t) = 1 + O \left( t^{-1}(\log t)^{-\alpha} \right), \quad x_1'(t) = O \left( t^{-2}(\log t)^{-\alpha} \right)
\]
and
\[
x_2(t) = t + O(\log t^{-\alpha}), \quad x_2'(t) = 1 + O(t^{-1}(\log t)^{-\alpha}).
\]
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