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PROPAGATION OF NORMALITY ALONG REGULAR ANALYTIC JORDAN ARCS IN ANALYTIC FUNCTIONS WITH VALUES IN A COMPLEX UNITAL BANACH ALGEBRA WITH CONTINUOUS INVOLUTION

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ABSTRACT. Globevnik and Vidav have studied the propagation of normality from an open subset V of a region $\mathcal{D}$ of the complex plane for analytic functions with values in the space $\mathcal{L}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. We obtain a propagation of normality in the more general setting of a converging sequence located on a regular analytic Jordan arc in the complex plane for analytic functions with values in a complex unital Banach algebra with continuous involution. We show that in this more general setting, the propagation of normality does not imply functional commutativity anymore as it does in the case studied by Globevnik and Vidav. An immediate consequence of the Propagation of Normality Theorem is that the further generalization given by Wolf of Jamison's generalization of Rellich's theorem is equivalent to Jamison's result. We obtain a propagation property within Banach subspaces for analytic Banach space-valued functions. The propagation of normality differs from the propagation within Banach subspaces since the set of all normal elements does not form a Banach subspace.

Notation. We denote by $\mathcal{D}$ a region of the complex plane, by $\mathcal{O}$ an open set of the complex plane and by $\mathcal{B}$ be a complex unital Banach algebra with continuous involution. Let $\mathcal{X}$ be a complex normed vector space. We denote by $\mathrm{H}(\mathcal{O},\mathcal{X})$ the space of all analytic $\mathcal{X}$-valued functions from $\mathcal{D}$.

In the context of the space of bounded linear operators $\mathcal{L}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, Globevnik and Vidav [2] have shown that if $f \in \mathrm{H}(\mathcal{D},\mathcal{L}(\mathcal{H}))$ is normal in an open subset $V \subseteq \mathcal{D}$, then f and f^* form a *functionally commutative* pair (f, f^*) in $\mathcal{D}$ (i.e. the commutator $[f(s), f^*(t)]$ is zero for every $s, t \in \mathcal{D}$). Hence by Fuglede's theorem, f is *functionally commutative* (i.e. $[f(s), f(t)]$ is zero for every

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$s, t \in \mathcal{D}$). In particular, $f(s)$ is normal for every $s \in \mathcal{D}$. In other words, the normality of $f(s)$ *propagates* from the open subset V into the whole region $\mathcal{D}$. We note that the first part of this result remains valid in a complex unital Banach algebra with continuous involution and the second part remains valid in a C^*-algebra.

Theorem 1 (Globevnik-Vidav). *Let $f \in \mathrm{H}(\mathcal{D};\mathcal{B})$. If there exists an open subset $V \subseteq \mathcal{D}$ such that $f(s)$ is normal for every $s \in \mathcal{D}$, then (f, f^*) is a functionally commutative pair. In particular, f is normal in $\mathcal{D}$. Furthermore, if $\mathcal{B}$ is a C^*-algebra, then f is functionally commutative.*

We will now study the propagation of normality on regular analytic Jordan arcs for analytic functions with values in a complex unital Banach algebra with continuous involution. We will see in an example that in this more general setting the propagation of normality does not imply functional commutativity. We note that in general the function $f^* : s \mapsto f(s)^*$ is *not* analytic. However, we will see that the principle of analytic continuation for vector-valued functions can still be applied in a restricted way to f^* to obtain the Propagation of Normality Theorem.

Definition. A curve $\gamma : [a, b] \mapsto \mathbb{C}$ is said to be an *analytic arc* if for each point s_{o} in $[a, b]$ there exists a power series

$$\gamma(s) = \sum_{n=0}^{\infty} a_n (s - s_{\mathrm{o}})^n$$

that converges for $s \in B(s_{\mathrm{o}}, r_{\mathrm{o}})$ for a certain $r_{\mathrm{o}} > 0$.

An analytic arc γ is said to be *regular* if $\gamma'(s) \neq 0$ for every $s \in [a, b]$.

Theorem 2 (Propagation of Normality). *Let $f \in \mathrm{H}(\mathcal{O}, \mathcal{B})$. Let Γ be a regular analytic Jordan arc contained in $\mathcal{O}$ and (s_n) a converging sequence contained in Γ such that $s_n \neq s_m$ for $n \neq m$. If $f(s_n)$ is normal for every s_n, then $f(s)$ is normal for every $s \in \Gamma$.*

Proof. Since Γ is an analytic Jordan arc, it has an analytic continuation in a simply connected open neighborhood $\mathcal{D}$ of $[a, b]$ in $\mathbb{C}$, symmetrical with respect to the real axis. As Γ is regular, shrinking $\mathcal{D}$ if necessary we may suppose that the continuation has a non-vanishing derivative and is therefore injective. Let us denote this continuation by j. The region $\mathcal{D}$ can be chosen small enough that $j(\mathcal{D}) \subset \mathcal{O}$. We define

$$F(t) = f(j(t)), \qquad t \in \mathcal{D}.$$

So $F \in \mathrm{H}(\mathcal{D}, \mathcal{B})$. We set

$$t_n = j^{-1}(s_n),$$

where (t_n) is a converging sequence of real numbers and $F(t_n) = f(s_n)$. So, $F(t_n)$ is normal for every t_n. Let t_{o} denote the limit of (t_n) (it is clear that $t_{\mathrm{o}} = j^{-1}(s_{\mathrm{o}})$ where s_{o} is the limit of s_n). We have

$$F(t) = \sum_{n=0}^{\infty} a_n (t - t_{\mathrm{o}})^n, \qquad t \in B(t_{\mathrm{o}}, r_{\mathrm{o}}), \text{ for a certain } r_{\mathrm{o}} > 0.$$

Since the involution is continuous, we have

$$(F(t))^* = \sum_{n=0}^{\infty} a_n^* \left(\overline{t - t_{\mathrm{o}}}\right)^n, \qquad t \in B(t_{\mathrm{o}}, r_{\mathrm{o}}).$$

Let g be defined by

$$g(t) := \sum_{n=0}^{\infty} a_n^*(t-t_o)^n\,, \qquad t \in B(t_o, r_o)\,.$$

We have $g \in \mathrm{H}(B(t_o,r_o),\mathcal{B})$ and

$$F(t)^* = g(t)\,, \qquad \text{for every } t \in (t_o - r_o, t_o + r_o)\,.$$

Hence, the normality of $F(t_n)$ can now be written in terms of the commutation of *two analytic $\mathcal{B}$-valued functions*:

$$[F(t_n), g(t_n)] = 0\,, \qquad t_n \in B(t_o, r_o)\,.$$

The principal of analytic continuation implies that

$$[F(t), g(t)] = 0\,, \qquad t \in B(t_o, r_o)\,.$$

In particular, F is normal on $(t_o - r_o, t_o + r_o)$. Since F is continuous, $F(t_o - r_o)$ and $F(t_o + r_o)$ are also normal.

Consequently, by continuing this process, we see that $F(t)$ is normal for every $t \in \mathbb{R} \cap \mathcal{D}$. In particular, this implies that $f(s)$ is normal for every $s \in \Gamma$. □

Jamison [3, p. 109] has shown that Rellich's theorem on the analyticity of eigenvalues of an analytic matrix-valued function that is self-adjoint on the real axis can be extended to an analytic family of operators in $\mathcal{L}(\mathcal{H})$ that is normal on the real axis.

Theorem (Jamison, [3])**.** *Let $f \in \mathrm{H}(B(0,d),\mathcal{L}(\mathcal{H}))$, $d > 0$, be such that $f(s)$ is normal for s real. If $f(0)$ has an isolated eigenvalue λ_o of finite multiplicity m, then there exists $r, \delta > 0$ such that for each $s \in B(0,r)$, the set $\mathrm{Sp}\, f(s) \cap B(\lambda_o, \delta)$ is composed of m eigenvalues (counting the multiplicity) of $f(s)$; thus*

$$\mathrm{Sp}\, f(s) \cap B(\lambda_o, \delta) = \{\boldsymbol{\lambda}_o^{(1)}(s), \ldots, \boldsymbol{\lambda}_o^{(n)}(s)\}\,, \quad s \in B(0,r)\,,$$

where n is the number of distinct eigenvalues contained in $\mathrm{Sp}\, f(s) \cap B(\lambda_o, \delta)$ for every $s \in B'(0,r)$, $n \le m$, and $\boldsymbol{\lambda}_o^{(1)}, \ldots, \boldsymbol{\lambda}_o^{(n)}$ are distinct analytic functions in $B(0,r)$ and $\boldsymbol{\lambda}_o^{(1)}(0) = \cdots = \boldsymbol{\lambda}_o^{(n)}(0)$.

Wolf [6, Theorem 6.4] has given a generalization of Jamison's theorem by replacing the condition of normality of $f(s)$ for s on the real line by the following: *$f(s)$ is normal for every $s \in E$ where E is a subset of the real line having 0 as an accumulation point.* An immediate consequence of the Propagation of Normality Theorem is that Wolf's hypothesis that $f(s)$ is normal for every $s \in E$ implies that $f(s)$ is normal for every s real, i.e. Wolf's theorem is equivalent to Jamison's theorem.

We note that up to this day, Jamison's theorem and its generalization given by Wolf had each received separate attention. In 1959, Butler, using a completely different approach from the one used here, gave a simple proof of the theorem as stated by Wolf. The simplicity of Butler's paper led Reed and Simon [4, p. 60] to comment concerning the theorem of Wolf: "*Considerable light has been shed on this theorem by J. Butler*". The Propagation of Normality Theorem provides a new and descriptive solution to the difficulties of Wolf's theorem.

The Propagation of Normality Theorem and the Globevnik-Vidav theorem lead us to formulate the following problem: When Γ is a regular analytic Jordan curve and $f \in \mathrm{H}(\mathcal{D}, \mathcal{B})$ is normal on a subset $E \subseteq \Gamma$ with a limit point, does the normality propagate on all of $\mathcal{D}$ when $\mathcal{D}$ is a simply connected region of the complex plane? We will construct an example that shows that *the answer to this question is negative in all non-commutative unital complex Banach algebras $\mathcal{B}$ with continuous involution*.

Example. Let $f \in \mathrm{H}(\mathbb{C}; \mathcal{B})$ where $\mathcal{B}$ is non-commutative. We want f to be normal on the unit circle without being normal on the unit disc. Thus, suppose that $f(s)$ is normal for every $s \in \partial U$. We have

$$f\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} A_n r^n e^{i\theta n} \ .$$

Since the involution is continuous, we have

$$f^*\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} A_n^* r^n e^{-i\theta n} \ .$$

By hypothesis, the function f is normal on ∂U, thus

$$\begin{aligned} e^{i\theta(p-1)} \left[f(e^{i\theta}), f^*(e^{i\theta})\right] &= 0 \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left[A_k, A_{j-k}^*\right] e^{i\theta(2k-j+p-1)}\,, \qquad p = 0, 1, 2, \dots . \end{aligned}$$

Thus, by integrating on the unit circle, we have

$$\sum_{j=0}^{\infty} \sum_{k=0}^{j} \left[A_k, A_{j-k}^*\right] \int_{-\pi}^{\pi} e^{i\theta(2k-j+p)}\, d\theta = 0\,, \qquad p = 0, 1, 2, \dots . \tag{1}$$

The only terms contributing in (1) are those for which $2k - j + p = 0$, therefore we get

$$\sum_{k=0}^{\infty} \left[A_k, A_{p+k}^*\right] = 0\,, \qquad p = 0, 1, 2, \dots . \tag{2}$$

Now consider the particular case that f is a polynomial of degree 2 with values in $\mathcal{B}$, i.e. $A_k = 0$ for every $k \geq 3$. From (2), we get the following system of equations:

$$\begin{aligned} [A_0, A_0^*] + [A_1, A_1^*] + [A_2, A_2^*] &= 0, \\ [A_0, A_1^*] + [A_1, A_2^*] &= 0, \\ [A_0, A_2^*] &= 0\ . \end{aligned} \tag{3}$$

In order to simplify the system of equations (3), we set $A_0 = I$. Thus we get

$$\begin{aligned} [A_1, A_1^*] + [A_2, A_2^*] &= 0, \\ [A_1, A_2^*] &= 0\ . \end{aligned} \tag{4}$$

Since the algebra $\mathcal{B}$ is non-commutative, it follows from the decomposition of $\mathcal{B}$ into the direct sum $\mathcal{B} = \mathrm{Sym}(\mathcal{B}) \oplus i\,\mathrm{Sym}(\mathcal{B})$ that there exist two self-adjoint elements $x_o, y_o \in \mathrm{Sym}(\mathcal{B})$ that do not commute. Therefore $A_1 := x_o + iy_o$ is an element of $\mathcal{B}$ that is not normal. Hence, we get a solution of the system (4) by posing $A_2 := A_1^*$. Therefore, the polynomial

$$f(z) = I + A_1 z + A_1^* z^2$$

is normal on the unit circle without being normal on the unit disc.

Similarly,[1] if we choose $A_2 = 0$ in (3) (instead of $A_0 = I$) we get the following system which is analogous to (4):

$$[A_0, A_0^*] + [A_1, A_1^*] = 0,$$
$$[A_0, A_1^*] = 0\ .$$

We can set $A_1 = A_0^*$, where $A_0 := x + iy$ is not a normal element. The polynomial

$$p(z) = A_0 + A_0^* z$$

is normal on the unit circle and is not normal anywhere else in the plane since $[p(z), p(z)^*] = (1 - |z|^2)[A_o, A_o^*]$. Furthermore, p is not functionally commutative, indeed $[p(z_1), p(z_2)] = 0$ if and only if $z_1 = z_2$ since $[p(z_1), p(z_2)] = (z_1 - z_2)[A_o^*, A_o]$.

The next theorem is a direct consequence of the principle of analytic continuation for analytic vector-valued functions. It describes a propagation property within Banach subspaces.

Theorem 3. *Let $f \in \mathrm{H}(\mathcal{D}, \mathcal{X})$, where $\mathcal{X}$ is a complex Banach space. Let G be a subset of $\mathcal{D}$ with a limit point in $\mathcal{D}$. Then for each $s \in \mathcal{D}$ and $j = 0, 1, 2, \ldots$, $\mathrm{D}^j f(\mathcal{D})$ is included in the Banach subspace generated by $f(G)$.*

Proof. Let us first show that $f(\mathcal{D})$ is included in the Banach subspace generated by $f(G)$. Denote by $S(f(G))$ the Banach subspace generated by $f(G)$. Let $\dot{f}(z)$ denote the class of $f(z)$ in the Banach space $\mathcal{X}/S(f(G))$. We have $\dot{f} \in \mathrm{H}(\mathcal{D}, \mathcal{X}/S(f(G)))$ and $\dot{f}|_G = 0$, therefore the principle of analytic continuation implies that $\dot{f}$ is identically equal to zero. Hence $f(\mathcal{D}) \subset S(f(G))$.

Let us now show that $\mathrm{D}f(\mathcal{D})$ is included in $S(f(G))$. Let s_o be an arbitrary element of $\mathcal{D}$. Since the set $\mathcal{D}$ is open, there exists $r > 0$ such that $s_o + B(0, r)$ is contained in $\mathcal{D}$. Let (h_n) be a sequence in $B(0, r)$ converging to 0 and such that $h_n \neq 0$. Then, we have

$$\mathrm{D}f(s_o) = \lim_{n \to \infty} \frac{f(s_o + h_n) - f(s_o)}{h_n}\ .$$

But we have already shown that $f(s_o)$ and $f(s_o + h_n)$ are elements of $S(f(G))$, so $\frac{f(s_o+h_n)-f(s_o)}{h_n}$ is an element of $S(f(G))$. Thus, $\mathrm{D}f(s_o)$ is the limit of a sequence of elements of $S(f(G))$. Hence, $\mathrm{D}f(s_o)$ is an element of $S(f(G))$. But s_o is an arbitrary point in $\mathcal{D}$, so $\mathrm{D}f(s)$ is an element of $S(f(G))$ for each $s \in \mathcal{D}$. By induction on the order j of the derivative, that implies that $\mathrm{D}^j f(s)$ is an element of $S(f(G))$ for each $s \in \mathcal{D}$ and $j = 0, 1, 2, \ldots$. □

Corollary. *Let $A \in \mathrm{H}(\mathcal{D}; \mathcal{L}(\mathcal{X}, \mathcal{Y}))$ where $\mathcal{X}$ and $\mathcal{Y}$ are complex Banach spaces and $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the space of bounded linear operators between $\mathcal{X}$ and $\mathcal{Y}$. Suppose that there exists a sequence (s_n) converging in $\mathcal{D}$ such that $A(s_n)$ is a compact operator. Then $\mathrm{D}^j A(s)$ is a compact operator for each $s \in \mathcal{D}$ and $j = 0, 1, 2, \ldots$.*

Proof. Let $\mathcal{LC}(\mathcal{X}, \mathcal{Y})$ denote the set of compact operators in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. $\mathcal{LC}(\mathcal{X}, \mathcal{Y})$ is a closed vector subspace of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ [5, theorem 4.18]. So, the closed vector subspace generated by the set of $f(s_n)$ is contained in $\mathcal{LC}(\mathcal{X}, \mathcal{Y})$. Therefore, Theorem 3 implies that $\mathrm{D}^j A(\mathcal{D})$ is contained in $\mathcal{LC}(\mathcal{X}, \mathcal{Y})$. □

[1] The choice of $A_2 = 0$ in (3) was suggested by M.D. Choi, University of Toronto.

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