

INTERPOLATION INEQUALITIES IN BESOV SPACES

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Dedicated to Professor Takaaki Nishida on the occasion of his sixtieth birthday

ABSTRACT. In this paper we present an interpolation inequality in the homogeneous Besov spaces on \mathbb{R}^n , which reduces to a number of well-known inequalities in special cases.

1. INTRODUCTION

There are several types of interpolation inequalities in the Sobolev and Besov spaces on \mathbb{R}^n ; see for instance [1]–[15] and references therein. We prove the following theorem (see below for notation).

Theorem 1. *Let $\lambda, \mu, p, q, r, \theta$ satisfy $\lambda, \mu \in \mathbb{R}$, $1 \leq p, q \leq r \leq \infty$, $0 < \theta < 1$,*

$$(1.1) \quad \lambda > \frac{n}{p} - \frac{n}{r},$$

$$(1.2) \quad \mu < \frac{n}{q} - \frac{n}{r},$$

$$(1.3) \quad \theta \left(\lambda - \frac{n}{p} + \frac{n}{r} \right) + (1 - \theta) \left(\mu - \frac{n}{q} + \frac{n}{r} \right) = 0.$$

Then there exists a constant $C > 0$ such that

$$(1.4) \quad \|f; \dot{B}_{r,1}^0\| \leq C \|f; \dot{B}_{p,\infty}^\lambda\|^\theta \|f; \dot{B}_{q,\infty}^\mu\|^{1-\theta}$$

for all $f \in \dot{B}_{p,\infty}^\lambda \cap \dot{B}_{q,\infty}^\mu$.

By the embeddings $\dot{B}_{r,1}^0 \hookrightarrow L^r$ and $\dot{H}_r^\rho \hookrightarrow \dot{B}_{r,\infty}^\rho$ with $1 \leq r \leq \infty$, $\rho \in \mathbb{R}$, we have the following corollary.

Corollary 2. *Let $\lambda, \mu, p, q, r, \theta$ be as above. Then there exists a constant $C > 0$ such that*

$$(1.5) \quad \|f; L^r\| \leq C \|f; \dot{H}_p^\lambda\|^\theta \|f; \dot{H}_q^\mu\|^{1-\theta}$$

for all $f \in \dot{H}_p^\lambda \cap \dot{H}_q^\mu$.

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The results above generalize various previously known interpolation inequalities. In [9], Miyakawa proved (1.5) in the special cases: (a) $r = \infty$, $\lambda = \mu$; (b) $r = \infty$, $p = q$, $\mu = 0$. In [5], Escobedo and Vega proved (1.5) in the special case: (c) $r = \infty$, $p, q > 1$, $0 < \lambda$, $\mu < n$. Corollary 2 ensures that (1.5) holds when $\mu \leq 0$. In [11], (1.5) is proved in the special case: (d) $r = \infty$, $p = q = 2$.

There are some available interpolation inequalities not covered by the results above. Complex interpolation yields (1.5) with $1 < p, q, r < \infty$, $1/r = (1 - \theta)/p + \theta/q$, $(1 - \theta)\lambda + \theta\mu = 0$ (see also [7]), and therefore (1.1) and (1.2) amount to additional restrictions. The known complex interpolation formulas for Besov spaces do not cover (1.4), however. A possible dependence of the third index on the interpolation inequalities is a novelty of (1.4) (see also [8, 12]).

We know a few more results which seem to be related to (1.4) and (1.5). In [6], Gérard, Meyer, and Oru proved

$$(1.6) \quad \|f; L^r\| \leq C \|f; \dot{H}_p^\lambda\|^{p/r} \|f; \dot{B}_{\infty, \infty}^{-\alpha}\|^{1-p/r},$$

where $1 < p < r < \infty$, $\alpha(r/p - 1) = \lambda > 0$, in particular, $p = 2$, $r = 6$, $\lambda = 1$, $\alpha = 1/2$. In [4], Cohen, Dahmen, Daubechies, and De Vore proved

$$(1.7) \quad \|f; L^2\| \leq C \|f; BV\|^{1/2} \|f; B_{\infty, \infty}^{-1}\|^{1/2},$$

where BV denotes the space of functions vanishing at infinity in the weak sense and satisfying the estimate

$$\sup_{y \in \mathbb{R}^n} |y|^{-1} \int |f(x + y) - f(x)| dx \leq C.$$

We prove the theorem in the next section. The proof depends on the standard technique from the Littlewood–Paley theory (see [1, 2, 3, 6, 8, 11, 15] for instance) and therefore the theorem holds for the homogeneous Besov spaces on the Heisenberg group \mathbb{H}^n with necessary modifications (see [1]).

We finally introduce the notation. For any r with $1 \leq r \leq \infty$, $L^r = L^r(\mathbb{R}^n)$ denotes the Lebesgue space on \mathbb{R}^n . For any $\rho \in \mathbb{R}$ and any r with $1 \leq r \leq \infty$, \dot{H}_r^ρ denotes the homogeneous Sobolev space defined as the space of classes of distributions f modulo polynomials such that $(-\Delta)^{\rho/2} f \in L^r$, where Δ is the Laplacian in \mathbb{R}^n . For any $\rho \in \mathbb{R}$ and any r, m with $1 \leq r, m \leq \infty$, $\dot{B}_{r, m}^\rho$ denotes the homogeneous Besov space defined as the space of classes of distributions f modulo polynomials such that $\{2^{\rho j} \|\varphi_j * f; L^r\|\} \in l^m(\mathbb{Z})$, where $*$ denotes the convolution in \mathbb{R}^n and the Fourier transformed functions $\{\hat{\varphi}_j\} \subset C_0^\infty$ satisfy $\sum_{j \in \mathbb{Z}} \hat{\varphi}_j(\xi) = 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, $0 \leq \hat{\varphi}_j \leq 1$, $\text{supp } \hat{\varphi}_j \subset \{\xi; 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, $\hat{\varphi}_j(\xi) = \hat{\varphi}_0(2^{-j}\xi)$. We refer to [2, 3, 7, 15] for general information on homogeneous Besov and Triebel–Lizorkin spaces.

2. PROOF OF THE THEOREM

We may assume that $\|f; \dot{B}_{p, \infty}^\lambda\| \neq 0$ and $\|f; \dot{B}_{q, \infty}^\mu\| \neq 0$. From the support properties of $\hat{\varphi}_j$ it follows that

$$(2.1) \quad \|f; \dot{B}_{r, 1}^0\| = \sum_{j \in \mathbb{Z}} \|\varphi_j * f; L^r\| \leq \sum_{j \in \mathbb{Z}} \sum_{k=j-1}^{j+1} \|\varphi_k * \varphi_j * f; L^r\|.$$

By the Young inequality, we have

$$(2.2) \quad \begin{aligned} \|\varphi_k * \varphi_j * f; L^r\| &\leq \|\varphi_k; L^m\| \|\varphi_j * f; L^s\| \\ &= 2^{nk(1-1/m)} \|\varphi_0; L^m\| \|\varphi_j * f; L^s\|, \end{aligned}$$

where $1 + 1/r = 1/m + 1/s$. We apply (2.2) with $s = p, q$ to (2.1) to obtain

$$\begin{aligned} \|f; \dot{B}_{r,1}^0\| &\leq C \sum_{j \geq l} 2^{(n/p-n/r-\lambda)j} \cdot 2^{\lambda j} \|\varphi_j * f; L^p\| \\ &\quad + C \sum_{j < l} 2^{(n/q-n/r-\mu)j} \cdot 2^{\mu j} \|\varphi_j * f; L^q\| \\ &\leq C \sum_{j \geq l} 2^{(n/p-n/r-\lambda)j} \|f; \dot{B}_{p,\infty}^\lambda\| + C \sum_{j < l} 2^{(n/q-n/r-\mu)j} \|f; \dot{B}_{q,\infty}^\mu\| \\ &\leq C(2^{(n/p-n/r-\lambda)l} \|f; \dot{B}_{p,\infty}^\lambda\| + 2^{(n/q-n/r-\mu)l} \|f; \dot{B}_{q,\infty}^\mu\|) \\ &= C(2^{(n/p-n/r-\lambda)l} a^{1-\theta} + 2^{(n/q-n/r-\mu)l} a^{-\theta}) \|f; \dot{B}_{p,\infty}^\lambda\|^\theta \|f; \dot{B}_{q,\infty}^\mu\|^{1-\theta}, \end{aligned}$$

where $a = \|f; \dot{B}_{p,\infty}^\lambda\| / \|f; \dot{B}_{q,\infty}^\mu\|$.

Let $\sigma = (\lambda - n/p + n/r) - (\mu - n/q + n/r) > 0$ and let l be the largest integer that is less than or equal to $\sigma^{-1} \log_2 a$. Then,

$$2^l \leq a^{1/\sigma} \leq 2 \cdot 2^l, \quad \theta = -(\mu - n/q + n/r)/\sigma, \quad 1 - \theta = (\lambda - n/p + n/r)/\sigma,$$

and therefore

$$\begin{aligned} 2^{(n/p-n/r-\lambda)l} a^{1-\theta} &\leq (2a^{-1/\sigma})^{\lambda-n/p+n/r} a^{1-\theta} = 2^{\lambda-n/p+n/r}, \\ 2^{(n/q-n/r-\mu)l} a^{-\theta} &\leq a^{(n/q-n/r-\mu)/\sigma} a^{-\theta} = 1. \end{aligned}$$

This proves the theorem.

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REFERENCES

- [1] H. BAHOURI, P. GÉRARD AND C. -J. XU, *Espace de Besov et estimations de Strichartz généralisées sur le group de Heisenberg*, J. d'Anal. Math., **82** (2000), 93–118. MR **2001k:58054**
- [2] J. BERGH AND J. LÖFSTRÖM, *Interpolation spaces*, Springer, Berlin/Heiderberg/New York (1976). MR **58:2349**
- [3] P. BRENNER, V. THOMÉE AND L. WAHLBIN, *Besov spaces and applications to difference methods for initial value problems*, Lecture Notes in Math., **434**, Springer-Verlag, Berlin/Heidelberg/New York (1975). MR **57:1106**
- [4] A. COHEN, W. DAHMEN, I. DAUBECHIES AND R. DE VORE *Harmonic analysis of the space BV*, Preprint, RWTH Aachen, Institut für Geometrie und Praktische Mathematik, Nr. **195** (2000) (see <http://www.igpm.rwth-aachen.de>).
- [5] M. ESCOBEDO AND L. VEGA, *A semilinear Dirac equation in $H^s(\mathbb{R}^3)$ for $s > 1$* , SIAM J. Math. Anal., **28** (1997), 338–362. MR **97k:35239**
- [6] P. GÉRARD, Y. MEYER AND F. ORU, *Inégalités de Sobolev précisées, Seminaire de EDP, Ecole Polytechnique*, 1996–1997. CMP 98:04
- [7] J. GINIBRE AND G. VELO, *The global Cauchy problem for the nonlinear Klein–Gordon equation*, Math. Z., **189** (1985), 487–505. MR **86f:35149**
- [8] H. KOZONO, T. OGAWA AND Y. TANIUCHI, *Critical Sobolev inequalities and application to the semilinear evolution equations* (preprint).
- [9] T. MIYAKAWA, *On Morrey spaces of measures: Basic properties and potential estimates*, Hiroshima Math. J., **20** (1990), 213–222. MR **91k:46021**

- [10] Y. MEYER, *Oscillating patterns in image processing and in some nonlinear evolution equations*, The Fifteenth Dean J.B. Lewis Memorial Lectures (2001).
- [11] M. NAKAMURA AND T. OZAWA, *Small solutions to nonlinear Schrödinger equations in the Sobolev spaces*, J. d'Anal. Math., **81** (2000), 305–329. MR **2001j**:35256
- [12] T. OGAWA AND Y. TANIUCHI, *Critical Sobolev inequality and its application to nonlinear evolution equations in the fluid mechanics*, RIMS Kokyuroku, **1162** (2000), 99–106. CMP 2001:05
- [13] T. OZAWA, *On critical cases of Sobolev's inequalities*, J. Funct. Anal., **127** (1995), 259–269. MR **96c**:46039
- [14] T. OZAWA, *Characterization of Trudinger's inequality*, J. Inequal. & Appl., **1** (1997), 369–374. MR **2001e**:46061
- [15] H. TRIEBEL, *Theory of Function Spaces*, Birkhäuser, Basel (1983). MR **86j**:46026

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