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# THE FIRST COHOMOLOGY GROUP OF THE GENERALIZED MORAVA STABILIZER ALGEBRA

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ABSTRACT. There exists a $p$-local spectrum $T(m)$ with $BP_*(T(m)) = BP_*[t_1, \ldots, t_m]$. Its Adams-Novikov $E_2$-term is isomorphic to

$$\mathrm{Ext}_{\Gamma(m+1)}(BP_*, BP_*),$$

where

$$\Gamma(m+1) = BP_*(BP)/(t_1, \ldots, t_m) = BP_*[t_{m+1}, t_{m+2}, \ldots].$$

In this paper we determine the groups

$$\mathrm{Ext}^1_{\Gamma(m+1)}(BP_*, v_n^{-1}BP_*/I_n)$$

for all $m, n > 0$. Its rank ranges from $n+1$ to $n^2$ depending on the value of $m$.

## 1. INTRODUCTION AND MAIN THEOREM

The object of this paper is to compute the first cohomology ($H^1$) of certain subgroups $S_{n,m}$ of the pro-$p$-group $S_n$ known as the Morava stabilizer group. $S_n$ can be described as a group of automorphisms of a certain formal group law $F_n$ of height $n$ in characteristic $p$, and as a group of units in the maximal order $E_n$ of a certain $p$-adic division algebra $D_n$. $E_n$ is also the endomorphism ring of $F_n$. The group $S_n$ has a well known role in the chromatic approach to stable homotopy theory and the Adams–Novikov spectral sequence introduced in [MRW77]. We refer the reader to [Rav86, Chapters 5 and 6] for a detailed description.

The subgroups in question can be described in three equivalent ways:

(i) in terms of the the formal group law $F_n$ over the field $F_{p^n}$ defined in [Rav86, A2.2.10],

(ii) in terms of the maximal order $E_n$ described in [Rav86, A2.2.16], and

(iii) in terms of topological constructions related to the Adams–Novikov spectral sequence described in [Rav86, §6.2].

For (i) $F_n \in F_{p^n}[[x, y]]$ is a certain power series in two variables. An automorphism of it is an invertible (as a function) power series $f(x)$ in one variable satisfying the condition

$$f(F_n(x, y)) = F(f(x), f(y)).$$

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It is known that if two such automorphisms agree modulo $(x^i)$, then they also agree modulo $(x^{p^m})$ where $p^m$ is the smallest power of $p$ not less than $i$. (There is a similar statement about formal group laws themselves known as the Lazard Comparison Lemma [Rav86, A2.1.12].) $S_n$ is the group of automorphisms that are congruent to $x$ modulo $(x^p)$, and $S_{n,m} \subset S_n$ is the subgroup of automorphisms congruent to $x$ modulo $(x^{p^{m+1}})$. In particular $S_{n,0} = S_n$.

For (ii) recall that $E_n$ is the algebra obtained from the Witt ring $W(F_{p^n})$ by adjoining an indeterminate $S$ subject to the relations $S^n = p$ and $Sw = w^\sigma S$ for $w \in W(F_{p^n})$ where $\sigma$ denotes the Frobenius automorphism of $W(F_{p^n})$. Then $S_{n,m}$ is the group of units in $E_n$ congruent to 1 modulo $(S^{m+1})$.

For (iii) we need to recall the role of $S_n$ in stable homotopy theory. We refer the reader not familiar with the Adams–Novikov spectral sequence to [Rav86, Chapter 4]. In the chromatic spectral sequence (see [Rav86, Chapter 5]) one is interested in computing the group

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, v_n^{-1}BP_*/I_n). \tag{1.1}$$

Here $BP$ denotes the Brown-Peterson spectrum for a fixed prime $p$. Its homotopy is

$$BP_* := \pi_*(BP) = \mathbf{Z}_{(p)}[v_1, v_2, \dots]$$

and its self-homology is

$$BP_*(BP) := \pi_*(BP \wedge BP) = BP_*[t_1, t_2, \dots]$$

where $|v_i| = |t_i| = 2p^i - 2$. $I_n$ denotes the ideal $(p, v_1, \dots, v_{n-1})$.

A change-of-rings-isomorphism (see [MR77] or [Rav86, 6.1.1]) equates the Ext group of (1.1) with

$$\mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*),$$

where $\Sigma(n)$ is the Morava stabilizer algebra

$$\Sigma(n) = K(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(n)_*.$$

As an algebra,

$$\Sigma(n) = K(n)_*[t_1, t_2, \dots]/(v_n t_i^{p^n} - v_n^{p^i} t_i),$$

where $t_i$ is the image of the generator of the same name in $BP_*(BP)$. $\Sigma(n)$ is closely related to the dual of the group ring $F_{p^n}[S_n]$; we refer the reader to [Rav86, §6.2] for the precise statement. As in [Rav86, §6.5] we let

$$\Sigma(n, m+1) = \Sigma(n)/(t_1, \dots, t_m);$$

we call this the *generalized Morava stabilizer algebra*. It bears a similar relation to the dual of the group ring $F_{p^n}[S_{n,m}]$. The object of this paper is to determine its first cohomology group,

$$\mathrm{Ext}^1_{\Sigma(n,m+1)}(K(n)_*, K(n)_*)$$

(which we will abbreviate by $\mathrm{Ext}^1_{\Sigma(n,m+1)}$), for all values of $m \geq 0$ and $n > 0$ and for all primes $p$. This amounts to identifying the primitive elements in $\Sigma(n, m+1)$. The case $m = 0$ was described in [Rav86, 6.3.12].

There is a deeper reason to consider these particular subgroups of $S_n$. In [Rav86, §6.5], the second author has introduced the spectrum $T(m)$ which has $BP_*$-homology

$$BP_*(T(m)) = BP_*[t_1, \cdots\cdots, t_m],$$

and is homotopy equivalent to $BP$ below dimension $2p^{m+1}-3$.

Then the Adams-Novikov $E_2$-term converging to the homotopy groups of $T(m)$

$$E_2^{*,*}(T(m)) = \mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(T(m)))$$

is isomorphic by [Rav86, 7.1.3] to

$$\mathrm{Ext}_{\Gamma(m+1)}(BP_*, BP_*),$$

where

$$\Gamma(m+1) = BP_*(BP)/\,(t_1, \dots, t_m) = BP_*[t_{m+1}, t_{m+2}, \dots].$$

In particular $\Gamma(1) = BP_*(BP)$ by definition. When using the chromatic spectral sequence to compute Ext over $\Gamma(m+1)$, the group $S_{n,m}$ has a role analogous to that of $S_n$ in the classical case. The groups appearing in the $E_1$-term of this version of the chromatic spectral sequence are known as *generalized chromatic Ext groups*. Recently they have been the subject of several papers: [KS01], [IK00], [Ich], [Shic], [Shia], [Shi95], [KS93], [MS93b], [MS93a], [NY], [INR], and [NRb].

The spectra $T(m)$ and their Ext groups figure in the method of infinite descent, the technique for explicitly computing the Adams-Novikov $E_2$-term that was used in [Rav86, Chapter 7] and described further in [Rav02] and [NRa]. An approach to the limiting behavior of these groups as $m$ approaches infinity is described in [Rav00].

The ring $E_n$ has an embedding in the ring of $n \times n$ matrices over the Witt ring $W(\mathbf{F}_{p^n})$ described in [Rav86, 6.2.6]. This means that $S_n$ and each of its subgroups supports a homomorphism induced by the determinant to the group of units in $W(\mathbf{F}_{p^n})$, and it is known that its image is contained in the $p$-adic units $\mathbf{Z}_p^\times$. The structure of this group is

$$\mathbf{Z}_p^\times \cong \begin{cases} \mathbf{Z}/(p-1) \oplus \mathbf{Z}_p & \text{for } p \text{ odd}, \\ \mathbf{Z}/(2) \oplus \mathbf{Z}_2 & \text{for } p = 2. \end{cases}$$

From this is it possible to construct primitives $T_n \in \Sigma(n)$ for all primes $p$ and $U_n \in \Sigma(n)$ for $p = 2$ [Rav86, 6.3.12] satisfying

$$T_n \equiv \sum_{0 \le j < n} t_n^{p^j} \bmod (t_1, \dots, t_{n-1})$$

and

$$U_n - T_n \equiv \sum_{0 \le j < n} t_{2n}^{2^j} \bmod (t_1, \dots, t_{n-1}).$$

The corresponding elements in $\mathrm{Ext}^1_{\Sigma(n)}$, and their images in $\mathrm{Ext}^1_{\Sigma(n,m+1)}$, are denoted by $\zeta_n$ and $\rho_n$, respectively.

The results of [Rav86, §6.3] are stated in terms of $S(n) = \Sigma(n) \otimes_{K(n)_*} \mathbf{F}_p$ and $S(n, m+1) = \Sigma(n, m+1) \otimes_{K(n)_*} \mathbf{F}_p$. Passing from $\Sigma(n)$ to $S(n)$ amounts to dropping the grading and setting $v_n$ equal to 1. Formulas are given for $T_n$ and (for $p = 2$) $U_n$ in $S(n)$. It is straightforward to lift them to homogeneous elements in $\Sigma(n)$.

We can now state our main result.

**Theorem 1.2.** *For $p$ odd the rank of $\mathrm{Ext}^1_{\Sigma(n,m+1)}$ (as a vector space over $K(n)_*$) is*

$$\begin{cases} (m+1)n+1 & \text{for } m < \frac{n-2}{2}, \\ (m+1)n+n/2 & \text{for } n \text{ even and } m = \frac{n-2}{2}, \\ (m+1)n & \text{for } \frac{n-1}{2} \le m \le n-1, \\ n^2 & \text{for } m \ge n-1. \end{cases}$$

*Let $h_{m+i,j} \in \mathrm{Ext}^1$ be the element corresponding to $t_{m+i}^{p^j}$ when it is primitive. Then a basis for $\mathrm{Ext}^1$ is given by*

$$\begin{cases} \{\zeta_n\} \cup \{h_{m+i,j} \colon 1 \le i \le m+1, j \in \mathbf{Z}/(n)\} & \text{for } m < \frac{n-2}{2}, \\ \{\zeta_{n,j} \colon j \in \mathbf{Z}/(n/2)\} \\ \quad \cup \{h_{m+i,j} \colon 1 \le i \le m+1, j \in \mathbf{Z}/(n)\} & \text{for } n \text{ even and } m = \frac{n-2}{2}, \\ \{h_{m+i,j} \colon 1 \le i \le m+1, j \in \mathbf{Z}/(n)\} & \text{for } \frac{n-1}{2} \le m \le n-1, \\ \{h_{m+i,j} \colon 1 \le i \le n, j \in \mathbf{Z}/(n)\} & \text{for } m \ge n, \end{cases}$$

*where $\zeta_n$ is as above and*

$$\zeta_{n,j} = v_n^{-p^j}(t_n + v_n^{1-p^{n/2}} t_n^{p^{n/2}} - t_{n/2}^{1+p^{n/2}})^{p^j}.$$

*For $p = 2$ the rank is*

$$\begin{cases} (m+1)n+2 & \text{for } m < \frac{n-2}{2}, \\ (m+1)n+n/2+1 & \text{for } n \text{ even and } m = \frac{n-2}{2}, \\ (m+1)n+1 & \text{for } \frac{n-1}{2} \le m \le n-1, \\ n^2 & \text{for } m \ge n. \end{cases}$$

*The basis is as in the odd primary case but with $\rho_n$ added when $m < n$.*

Note that for $m = 0$ this result gives the same answer as [Rav86, 6.3.12]. Also [Rav86, 6.5.6] implies that $\mathrm{Ext}^1$ has rank $n^2$ with the basis indicated above when $m > \frac{pn}{2p-2} - 1$ and $m \ge n-1$; it says that in this case the full Ext group is the exterior algebra on those generators. [There is a missing hypothesis in [Rav86, 6.5.6] and [Rav86, 6.3.7]; see the online errata for details.]

**Corollary 1.3.** *For $n \le 3$ the rank of $\mathrm{Ext}^1_{\Sigma(n,m+1)}$ is as indicated in the following table:*

| $n=1$ | | | | $n=2$ | | | | $n=3$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p=2$ | | $p$ odd | | $p=2$ | | $p$ odd | | $p=2$ | | $p$ odd | |
| $m$ | rank | $m$ | rank | $m$ | rank | $m$ | rank | $m$ | rank | $m$ | rank |
| 0 | 2 | $\ge 0$ | 1 | 0 | 4 | 0 | 3 | 0 | 5 | 0 | 4 |
| $\ge 1$ | 1 | | | 1 | 5 | $\ge 1$ | 4 | 1 | 7 | 1 | 6 |
| | | | | $\ge 2$ | 4 | | | 2 | 10 | $\ge 2$ | 9 |
| | | | | | | | | $\ge 3$ | 9 | | |

## 2. The proof

We need to show that the indicated basis elements are primitive and that there are no other primitives. The primitivity of $\zeta_n$ and (for $p = 2$) $\rho_n$ was established in [Rav86, 6.3.12].

For the rest we need to study the coproduct in $\Sigma(n, m+1)$. A formula for the coproduct in $BP_*(BP)$ was given in [Rav86, 4.3.13]. In $BP_*(BP)/I_n$ for $i \leq 2n$ we have [Rav86, 4.3.15]

$$\Delta(t_i) = \sum_{0\leq j\leq i} t_j \otimes t_{i-j}^{p^j} + \sum_{0\leq j\leq i-n-1} v_{n+j} b_{i-n-j,n+j-1},$$

where $b_{i,j}$ satisfies

$$b_{i,j} \equiv -\frac{1}{p} \sum_{0<k<p^{j+1}} \binom{p^{j+1}}{k} t_i^k \otimes t_i^{p^{j+1}-k} \qquad \mathrm{mod}\ (t_1, \ldots, t_{i-1}).$$

It is defined precisely in [Rav86, 4.3.14]. Similar methods yield the following formula for the coproduct in $\Gamma(m+1)/I_n$ for $i \leq 2n$:

$$\begin{aligned}\Delta(t_{m+i}) \quad &= \quad t_{m+i} \otimes 1 + 1 \otimes t_{m+i} + \sum_{m<k<i} t_k \otimes t_{m+i-k}^{p^k} \\ &\qquad + \sum_{0\leq k\leq i-n-1} v_{n+k} b_{m+i-n-k,n+k-1}.\end{aligned}$$

In $\Sigma(n, m+1)$ this simplifies to

$$\begin{aligned}\Delta(t_{m+i}) \quad &= \quad t_{m+i} \otimes 1 + 1 \otimes t_{m+i} + \sum_{m<k<i} t_k \otimes t_{m+i-k}^{p^k} \\ &\qquad + v_n b_{m+i-n,n-1},\end{aligned} \tag{2.1}$$

where the last term vanishes when $i \leq n$. This formula implies that $t_{m+i}$ is primitive for $i \leq \min(m+1, n)$.

When $n$ is even and $m = \frac{n-2}{2}$ we have

$$\Delta(t_n) = t_n \otimes 1 + 1 \otimes t_n + t_{n/2} \otimes t_{n/2}^{p^{n/2}},$$

$$\begin{aligned}\Delta(v_n^{1-p^{n/2}} t_n^{p^{n/2}}) &= v_n^{1-p^{n/2}} \left(t_n \otimes 1 + 1 \otimes t_n + t_{n/2} \otimes t_{n/2}^{p^{n/2}}\right)^{p^{n/2}} \\ &= v_n^{1-p^{n/2}} \left(t_n^{p^{n/2}} \otimes 1 + 1 \otimes t_n^{p^{n/2}} + t_{n/2}^{p^{n/2}} \otimes t_{n/2}^{p^n}\right) \\ &= v_n^{1-p^{n/2}} \left(t_n^{p^{n/2}} \otimes 1 + 1 \otimes t_n^{p^{n/2}} + v_n^{p^{n/2}-1} t_{n/2}^{p^{n/2}} \otimes t_{n/2}\right) \\ &= v_n^{1-p^{n/2}} \left(t_n^{p^{n/2}} \otimes 1 + 1 \otimes t_n^{p^{n/2}}\right) + t_{n/2}^{p^{n/2}} \otimes t_{n/2}\end{aligned}$$

and

$$\begin{aligned}\Delta(t_{n/2}^{1+p^{n/2}}) &= \left(t_{n/2} \otimes 1 + 1 \otimes t_{n/2}\right)^{1+p^{n/2}} \\ &= t_{n/2}^{1+p^{n/2}} \otimes 1 + t_{n/2}^{p^{n/2}} \otimes t_{n/2} + t_{n/2} \otimes t_{n/2}^{p^{n/2}} + 1 \otimes t_{n/2}^{1+p^{n/2}},\end{aligned}$$

so $\zeta_{n,j}$ is primitive.

*This means that each basis element specified in Theorem 1.2 is indeed primitive.*

To show that there are no other primitives in $\Sigma(n, m+1)$ we need the methods of [Rav86, §6.3]. As noted above, results there are stated in terms of $S(n) =$

$\Sigma(n) \otimes_{K(n)_*} \mathbf{F}_p$ and $S(n, m+1) = \Sigma(n, m+1) \otimes_{K(n)_*} \mathbf{F}_p$. An increasing filtration on $S(n)$ is described in [Rav86, 6.3.1]. The weight of $t_i^{p^j}$ for each $j$ is the integer $d_{n,i}$ defined recursively by

$$d_{n,i} = \begin{cases} 0 & \text{if } i \le 0, \\ \max(i, pd_{n,i-n}) & \text{if } i > 0. \end{cases}$$

The bigraded object $E^0S(n)$ is described in [Rav86, 6.3.2]. It is considerably simpler than the coproduct in the unfiltered object. It contains elements $t_{m+i,j}$ (with $j \in \mathbf{Z}/(n)$) which are the projections of $t_{m+i}^{p^j}$. The coproduct on these elements is given by

$$\Delta(t_{m+i,j}) = \begin{cases} t_{m+i,j} \otimes 1 + 1 \otimes t_{m+i,j} \\ \qquad + \sum_{m<k<i} t_{k,j} \otimes t_{m+i-k,j+k} & \text{if } i < c-m, \\ \\ t_{m+i,j} \otimes 1 + 1 \otimes t_{m+i,j} \\ \qquad + \sum_{m<k<i} t_{k,j} \otimes t_{m+i-k,j+k} \\ \qquad + \overline{b}_{m+i-n,n-1+j} & \text{if } i = c-m, \\ \\ t_{m+i,j} \otimes 1 + 1 \otimes t_{m+i,j} \\ \qquad + \overline{b}_{m+i-n,n-1+j} & \text{if } i > c-m, \end{cases} \tag{2.2}$$

where $c = pn/(p-1)$ and $\overline{b}_{m+i-n,n-1+j}$ is the projection of $b_{m+i-n,n-1+j}$, which is 0 for $i \le n$.

Note that $t_{m+i,j}$ is primitive for $i \le m+1$ as expected, but it is also primitive for $c - m < i \le n$, which can occur when $m > n/(p-1)$.

To proceed further we use the fact that the dual of $E^0S(n, m+1)$ is a primitively generated Hopf algebra and therefore isomorphic to the universal enveloping algebra of its restricted Lie algebra of primitives, by a theorem of Milnor-Moore [MM65]. The cohomology of the unrestricted Lie algebra $L(n, m+1)$ (this notation differs from that of [Rav86, §6.3]) is that of the Koszul complex

$$C(n, m+1) = E(h_{m+i,j} \colon i > 0, j \in \mathbf{Z}/(n)), \tag{2.3}$$

where each $h_{m+i,j}$ has cohomological degree 1, with

$$d(h_{m+i,j}) = \begin{cases} \sum_{m<k<i} h_{k,j} h_{m+i-k,j+k} & \text{if } i \le c-m, \\ 0 & \text{if } i > c-m. \end{cases}$$

**Lemma 2.4.** *Let $C(n, m+1)$ be the complex of (2.3). Then $H^1(L(n, m+1)) = H^1(C(n, m+1))$ is spanned by*

$$\{h_{m+i,j} \colon 1 \le i \le m+1\} \cup \{h_{m+i,j} \colon i > c-m\} \cup \left\{ \sum_j h_{n,j}, \sum_j h_{2n,j} \right\},$$

*(where $c = pn/(p-1)$) unless $n = 2m+2$, in which case we must adjoin the set*

$$\left\{ h_{n,j} + h_{n,j+n/2} \colon j \in \mathbf{Z}/(n/2) \right\}.$$

Note that $h_{n,j}$ is either trivial or in the first subset unless $n \geq 2m+2$ and that $h_{n,j}$ is either trivial or in the second subset unless $p = 2$. Note also that the first and second subsets overlap when $m \geq c/2$.

*Proof.* The primitivity of the elements in the first and second subsets is obvious. For $\sum_j h_{n,j}$ we have

$$\begin{aligned} d\left(\sum_j h_{n,j}\right) &= \sum_j \sum_{m<k<n-m} h_{k,j}h_{n-k,j+k} \\ &= \sum_{m<k<n/2} \sum_j h_{k,j}h_{n-k,j+k} \\ &\qquad + \begin{cases} \sum_j h_{n/2,j}h_{n/2,j+n/2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd} \end{cases} \\ &\qquad + \sum_{n/2<k<n-m} \sum_j h_{k,j}h_{n-k,j+k} \\ &= \sum_{m<k<n/2} \sum_j h_{k,j}h_{n-k,j+k} + h_{n-k,j+k}h_{k,j} \\ &\qquad + \begin{cases} \sum_{0\leq j<n/2} h_{n/2,j}h_{n/2,j+n/2} \\ \qquad + \sum_{n/2\leq j<n} h_{n/2,j}h_{n/2,j+n/2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd} \end{cases} \\ &= \begin{cases} \sum_{0\leq j<n/2} h_{n/2,j}h_{n/2,j+n/2} + h_{n/2,j+n/2}h_{n/2,j} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd} \end{cases} \\ &= 0. \end{aligned}$$

Similar calculations show that for $p = 2$, $\sum_j h_{2n,j}$ is a cocycle, and that for $n = 2m+2$, $h_{n,j} + h_{n,j+n/2}$ is one.

It remains to show that there are no other cocycles in the subspace spanned by

$$\{h_{m+i,j} \colon m+1 < i \leq c-m\},$$

which is nonempty only when

$$m < \frac{pn-p+1}{2(p-1)}.$$

It suffices to consider elements which are homogeneous with respect to the filtration grading, i.e., to restrict our attention to one value of $i$ at a time. Thus we need to show that the subspace spanned by

$$\left\{\sum_{m<k<i} h_{k,j}h_{m+i-k,j+k} \colon j \in \mathbf{Z}/(n)\right\} \tag{2.5}$$

has dimension

$$\begin{cases} n/2 & \text{if } m+i = n \text{ and } n = 2m+2, \\ n-1 & \text{if } m+i = n \text{ and } n > 2m+2, \\ n-1 & \text{if } m+i = 2n, \\ n & \text{otherwise.} \end{cases} \tag{2.6}$$

When $n = 2m + 2$ and $m + i = n$, the set of (2.5) is

$$\begin{aligned}
&\left\{h_{n/2,j}h_{n/2,j+n/2} \colon j \in \mathbf{Z}/(n)\right\} \\
&\qquad = \left\{h_{n/2,j}h_{n/2,j+n/2} \colon 0 \leq j < n/2\right\} \\
&\qquad\qquad \cup \left\{h_{n/2,j}h_{n/2,j+n/2} \colon n/2 \leq j < n\right\} \\
&\qquad = \left\{h_{n/2,j}h_{n/2,j+n/2} \colon 0 \leq j < n/2\right\} \\
&\qquad\qquad \cup \left\{-h_{n/2,j+n/2}h_{n/2,j} \colon n/2 \leq j < n\right\} \\
&\qquad = \left\{h_{n/2,j}h_{n/2,j+n/2} \colon 0 \leq j < n/2\right\} \\
&\qquad\qquad \cup \left\{-h_{n/2,j}h_{n/2,j+n/2} \colon 0 \leq j < n/2\right\},
\end{aligned}$$

so the subspace it spans has dimension $n/2$.

Now suppose that $m + i = n$, $n > 2m + 2$, and $n$ is odd. It suffices to consider the middle two terms in the sum. Let $\ell = (n-1)/2$. Then we have

$$d(h_{n,j}) = h_{\ell,j}h_{\ell+1,j+\ell} + h_{\ell+1,j}h_{\ell,j+\ell+1} + \dots .$$

We can cancel the second term by adding $d(h_{n,j+\ell+1})$, i.e.,

$$\begin{aligned}
&d(h_{n,j} + h_{n,j+\ell+1}) \\
&\qquad = h_{\ell,j}h_{\ell+1,j+\ell} + h_{\ell+1,j}h_{\ell,j+\ell+1} \\
&\qquad\qquad + h_{\ell,j+\ell+1}h_{\ell+1,j+\ell+\ell+1} + h_{\ell+1,j+\ell+1}h_{\ell,j+\ell+1+\ell+1} + \dots \\
&\qquad = h_{\ell,j}h_{\ell+1,j+\ell} + h_{\ell+1,j}h_{\ell,j+\ell+1} \\
&\qquad\qquad + h_{\ell,j+\ell+1}h_{\ell+1,j} + h_{\ell+1,j+\ell+1}h_{\ell,j+1} + \dots \\
&\qquad = h_{\ell,j}h_{\ell+1,j+\ell} + h_{\ell+1,j+\ell+1}h_{\ell,j+1} + \dots .
\end{aligned}$$

Similarly we can cancel the second term here by adding $d(h_{n,j+1})$. Since $(n+1)/2$ and $n$ are relatively prime, we will need to sum up the $h_{n,j}$ over all $j$ to get a cocycle. It follows that this subspace has dimensions $n-1$ as claimed.

For $m + i = n$ and $n$ even, let $\ell = n/2$. Then it suffices to consider the middle three terms of the sum, i.e.,

$$d(h_{n,j}) = h_{\ell-1,j}h_{\ell+1,j+\ell-1} + h_{\ell,j}h_{\ell,j+\ell} + h_{\ell+1,j}h_{\ell-1,j+\ell+1} + \dots .$$

We can cancel the middle term by adding $d(h_{n,j+\ell})$, so we get

$$\begin{aligned}
&d(h_{n,j} + h_{n,j+\ell}) \\
&\qquad = h_{\ell-1,j}h_{\ell+1,j+\ell-1} + h_{\ell,j}h_{\ell,j+\ell} + h_{\ell+1,j}h_{\ell-1,j+\ell+1} \\
&\qquad\qquad + h_{\ell-1,j+\ell}h_{\ell+1,j-1} + h_{\ell,j+\ell}h_{\ell,j} + h_{\ell+1,j+\ell}h_{\ell-1,j+1} + \dots \\
&\qquad = h_{\ell-1,j}h_{\ell+1,j+\ell-1} + h_{\ell-1,j+\ell}h_{\ell+1,j-1} \\
&\qquad\qquad + h_{\ell+1,j}h_{\ell-1,j+\ell+1} + h_{\ell+1,j+\ell}h_{\ell-1,j+1} + \dots .
\end{aligned}$$

Now we can cancel the third and fourth terms by adding $d(h_{n,j+1}+h_{n,j+\ell+1})$, and we have

$$\begin{aligned} d(h_{n,j} &+ h_{n,j+\ell} + h_{n,j+1} + h_{n,j+\ell+1}) \\ &= h_{\ell-1,j}h_{\ell+1,j+\ell-1} + h_{\ell-1,j+\ell}h_{\ell+1,j-1} \\ &\quad +h_{\ell+1,j}h_{\ell-1,j+\ell+1} + h_{\ell+1,j+\ell}h_{\ell-1,j+1} \\ &\quad +h_{\ell-1,j+1}h_{\ell+1,j+\ell} + h_{\ell-1,j+\ell+1}h_{\ell+1,j} \\ &\quad +h_{\ell+1,j+1}h_{\ell-1,j+\ell+2} + h_{\ell+1,j+\ell+1}h_{\ell-1,j+2} + \dots \\ &= h_{\ell-1,j}h_{\ell+1,j+\ell-1} + h_{\ell-1,j+\ell}h_{\ell+1,j-1} \\ &\quad +h_{\ell+1,j+1}h_{\ell-1,j+\ell+2} + h_{\ell+1,j+\ell+1}h_{\ell-1,j+2} + \dots . \end{aligned}$$

Again in order to get complete cancellation we need to sum over all $j$, so the subspace has dimension $n-1$ as claimed.

We can make a similar argument for $m+i=2n$ when $p=2$, namely

$$\begin{aligned} d(h_{2n,j}) &= h_{n-1,j}h_{n+1,j-1} + h_{n,j}h_{n,j} + h_{n+1,j}h_{n-1,j+1} + \dots \\ &= h_{n-1,j}h_{n+1,j-1} + h_{n+1,j}h_{n-1,j+1} + \dots , \end{aligned}$$

so

$$\begin{aligned} d(h_{2n,j} &+ h_{2n,j+1}) \\ &= h_{n-1,j}h_{n+1,j-1} + h_{n+1,j}h_{n-1,j+1} \\ &\quad h_{n-1,j+1}h_{n+1,j} + h_{n+1,j+1}h_{n-1,j+2} + \dots \\ &= h_{n-1,j}h_{n+1,j-1} + h_{n+1,j+1}h_{n-1,j+2} + \dots , \end{aligned}$$

and so on.

Finally we need to consider the cases of (2.6) where $m+i$ is not divisible by $n$. For this we can show that the expressions

$$\sum_{m<k<i} h_{k,j}h_{m+i-k,j+k}$$

are linearly independent. Suppose the term

$$\pm h_{k,x}h_{m+i-k,y}$$

appears in the sums for some value of $j$. Then modulo $n$ either $j=x$ and $y\equiv k+x$, so $x\equiv y-k$, or $j=y$ and $x\equiv m+i+y-k$. These conditions on $x$ are mutually exclusive since $m+i$ is not divisible by $n$. This means that each monomial of this form can appear in the sum for at most one value of $j$, so the sums for various $j$ are linearly independent. □

Now $\mathrm{Ext}^1_{S(n,m+1)}$ is a subspace of $H^1(L(n,m+1))$. To finish the proof of the theorem we need to show that the elements $h_{m+i,j}$ with $i>\max(c-m,m+1)$ do not survive passage to $\mathrm{Ext}^1_{E^0S(n,m+1)}$ or from it to $\mathrm{Ext}^1_{S(n,m+1)}$. We need to look at the first and second spectral sequences constructed for this purpose by May in [May66] and described (for $m=0$) in [Rav86, 6.3.4]. It follows from (2.2) that in the first May spectral sequence

$$d_r(h_{m+i,j}) = b_{m+i-n,j-1} \neq 0 \qquad \text{for } i>n$$

for some $r$.

This eliminates all of the unwanted primitives except the ones with

$$\max(c-m, m+1) < i \leq n.$$

For this we can use (2.1), which implies that in the second May spectral sequence,

$$d_r(h_{m+i,j}) = \sum_{m<k<i} h_{k,j} h_{m+i-k,j+k}$$

where

$$\begin{aligned} r &= \min(d_{n,m+i} - d_{n,k} - d_{n,m+i-k} \colon m < k < i) \\ &= p(m+i-n) - (m+i) \\ &\qquad \text{since } k \text{ and } m-i-k \text{ do not exceed } n \text{ and } m+i < 2n \\ &= (p-1)(m+i) - pn. \end{aligned}$$

Note that

$$n < c < m+i \leq m+n < 2n$$

so $m+i$ is not divisible by $n$. Thus we can argue as in the last paragraph of the proof of Lemma 2.4 that the sums $\sum_{m<k<i} h_{k,j} h_{m+i-k,j+k}$ are linearly independent. It follows that no linear combination of the unwanted $h_{m+i,j}$ can survive to $\mathrm{Ext}^1_{S(n,m+1)}$, so $\mathrm{Ext}^1_{\Sigma(n,m+1)}$ is as claimed.

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