

ON POINTWISE ESTIMATES FOR THE LITTLEWOOD-PALEY OPERATORS

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ABSTRACT. In a recent paper we proved pointwise estimates relating some classical maximal and singular integral operators. Here we show that inequalities essentially of the same type hold for the Littlewood-Paley operators.

1. INTRODUCTION

Let ω be a non-negative, locally integrable function on \mathbb{R}^n . Given a measurable set E , let $\omega(E) = \int_E \omega(x) dx$. The non-increasing rearrangement of a measurable function f on \mathbb{R}^n with respect to ω is defined by

$$f_\omega^*(t) = \sup_{\omega(E)=t} \inf_{x \in E} |f(x)| \quad (0 < t < \infty).$$

If $\omega \equiv 1$ we use the notation $f^*(t)$.

Let us consider the maximal function (cf. [13])

$$m_\lambda f(x) = \sup_{Q \ni x} (f \chi_Q)^*(\lambda |Q|) \quad (0 < \lambda < 1),$$

where the supremum is taken over all cubes Q containing x , χ_Q denotes the indicator function of Q and $|Q|$ denotes the Lebesgue measure of Q .

The function $m_\lambda f$, as well as the Hardy-Littlewood maximal function Mf , is a pointwise majorant of $|f|$. However, $m_\lambda f$ and f have the same integrability properties in the L^p scale for any $p > 0$ (this follows, for example, from inequality (3.1) below), unlike Mf for which it is true only for $p > 1$.

In [9], the following pointwise estimate was proved:

$$(1.1) \quad m_\lambda(T_1 f)(x) \leq c_{\lambda, n} T_2 f(x) + T_1 f(x),$$

where T_1, T_2 are certain maximal and singular integral operators. More precisely, (1.1) was proved in the cases when either T_1 or T_2 is the Hardy-Littlewood maximal function, while T_2 or T_1 is the Fefferman-Stein sharp maximal function and the Calderón-Zygmund maximal singular integral operator, respectively. Such estimates easily imply weighted rearrangement and L^p -inequalities with respect to a weight satisfying the A_∞ condition; BLO -inequalities can be derived as well (see Lemma 1 in [9] for further details). Thus they allow us to unify and simplify some basic techniques extensively applied to the above-mentioned operators.

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The aim of this paper is to show that estimates essentially of the same type as (1.1) hold for the Littlewood-Paley operators. The role of these operators in harmonic analysis is well-known (see, e.g., [12, Ch. 4]). Among numerous papers on this subject, let us mention only those of specific interest to us, namely those where “good- λ ” [1, 4, 5, 10, 11], rearrangement [7] and *BLO* [8] inequalities were proved.

Roughly speaking, our central result states that for certain Littlewood-Paley operators T_1 and T_2 we have the pointwise estimates of the form

$$(1.2) \quad m_\lambda(T_1 f)(x) \leq c_{\lambda,n} m_{\lambda/4}(T_2 f)(x) + T_1 f(x).$$

Below, in Lemma 3.5, we show that such estimates imply the results of no less importance than (1.1) implies.

Two main approaches to the study of the Littlewood-Paley operators are well-known. The first one is based on the vector-valued analysis, while the second is based on the theory of harmonic functions. We shall use the latter approach, since it is not quite clear how one can combine the rearrangement and vector-valued techniques.

The paper is organized as follows. Section 2 contains some preliminaries. In Section 3, we state the main results, Theorems 3.1–3.4, and show how to apply them. In Section 4, we state and prove two lemmas which form the core of the proofs of Theorems 3.1 and 3.2. In Section 5, we prove Theorems 3.1–3.4.

2. PRELIMINARIES

Throughout the paper, u will be a harmonic function on the upper half space $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$.

Define the truncated Lusin area integral and nontangential maximal function by

$$S_{\alpha,h}u(x) = \left(\int_{\Gamma_{\alpha,h}(x)} t^{1-n} |\nabla u(y,t)|^2 dy dt \right)^{1/2}$$

and

$$N_{\alpha,h}u(x) = \sup_{(y,t) \in \Gamma_{\alpha,h}(x)} |u(y,t)|,$$

respectively, where $\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial t} \right)$ and $\Gamma_{\alpha,h}$ is the truncated cone with vertex at $x \in \mathbb{R}^n$ and aperture α :

$$\Gamma_{\alpha,h}(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : |y-x| < \alpha t, 0 < t < h\}.$$

When $h = +\infty$ we drop the subscript h and write $S_\alpha, N_\alpha, \Gamma_\alpha$. Also we consider the Littlewood-Paley functions g and g_μ defined by

$$g(u)(x) = \left(\int_0^\infty t |\nabla u(x,t)|^2 dt \right)^{1/2}$$

and

$$g_\mu(u)(x) = \left(\int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{\mu n} t^{1-n} |\nabla u(y,t)|^2 dy dt \right)^{1/2} \quad (\mu > 1).$$

Next, we define the vertical and box maximal functions by

$$N^+u(x) = \sup_{t>0} |u(x,t)|$$

and

$$T_\mu u(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^\mu} \int_{T(Q)} t^{(\mu-1)n-1} |u(y,t)|^2 dy dt \right)^{1/2} \quad (\mu > 1),$$

respectively, where Q is a cube, $T(Q) = \{(x,t) : x \in Q, 0 < t < d_Q\}$, and d_Q denotes the diameter of Q .

Following Strömberg [13], define the local square function by

$$S_{\lambda,\alpha}^\# u(x) = \sup_{Q \ni x} \left((S_{\alpha,d_Q} u) \chi_Q \right)^* (\lambda |Q|) \quad (0 < \lambda < 1).$$

Consider the “local nontangential” maximal function defined by

$$N_{\lambda,\alpha}^\# u(x) = \sup_{Q \ni x} \inf_{c \in \mathbb{R}} \left((N_{\alpha,d_Q} (u - c)) \chi_Q \right)^* (\lambda |Q|) \quad (0 < \lambda < 1).$$

It is easy to see that $S_{\lambda,\alpha}^\# u(x) \leq m_\lambda S_\alpha u(x)$ and $N_{\lambda,\alpha}^\# u(x) \leq m_\lambda N_\alpha u(x)$.

Now, we recall some well-known facts concerning harmonic functions. First, by Green’s Theorem,

$$(2.1) \quad \int_G t |\nabla u(y,t)|^2 dy dt = \int_{\partial G} \left(tu \frac{\partial u}{\partial n} - \frac{u^2}{2} \frac{\partial t}{\partial n} \right) d\sigma,$$

where G is a domain in \mathbb{R}_+^{n+1} with piecewise smooth boundary ∂G . Further, by mean-value property we have for $\alpha < \beta$, $h < h'$ (see [12, p. 207]),

$$(2.2) \quad \sup_{(y,t) \in \Gamma_{\alpha,h}(x)} t |\nabla u(y,t)| \leq c N_{\beta,h'} u(x)$$

and

$$(2.3) \quad \sup_{(y,t) \in \Gamma_{\alpha,h}(x)} t |\nabla u(y,t)| \leq c S_{\beta,h'} u(x),$$

where $c = c(\alpha, \beta, h, h', n)$.

Recall that a weight ω satisfies Muckenhoupt’s condition A_∞ if there exist $c, \delta > 0$ such that for any Q and $E \subset Q$,

$$\omega(E) \leq c(|E|/|Q|)^\delta \omega(Q).$$

The space BLO [3] consists of all functions $f \in L^1_{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{BLO} = \sup_Q \frac{1}{|Q|} \int_Q \left(f(x) - \inf_Q f \right) dx < \infty.$$

3. MAIN RESULTS AND THEIR APPLICATIONS

Our main results are the following.

Theorem 3.1. *For any harmonic function u on \mathbb{R}_+^{n+1} and for all $x \in \mathbb{R}^n$,*

$$m_\lambda (S_\alpha^2 u)(x) \leq c m_{\lambda/4} (N_\beta^2 u)(x) + S_\alpha^2 u(x) \quad (0 < \lambda < 1, \alpha < \beta),$$

where c is a constant depending on λ, α, β and n .

Theorem 3.2. *For any harmonic function u on \mathbb{R}_+^{n+1} and for all $x \in \mathbb{R}^n$,*

$$m_\lambda (N_\alpha u)(x) \leq c m_{\lambda/4} (S_\beta u)(x) + N^+ u(x) \quad (0 < \lambda < 1, \alpha < \beta),$$

where c is a constant depending on λ, α, β and n .

Theorem 3.3. For any harmonic function u on \mathbb{R}_+^{n+1} and for all $x \in \mathbb{R}^n$,

$$m_\lambda(g^2 u)(x) \leq cm_{\lambda/4}(N_\alpha^2 u)(x) + g^2 u(x) \quad (0 < \lambda < 1, \alpha > 0),$$

where c is a constant depending on λ, α and n .

Theorem 3.4. For any harmonic function u on \mathbb{R}_+^{n+1} and for all $x \in \mathbb{R}^n$,

$$m_\lambda(g_\mu^2(u))(x) \leq cm_{\lambda/4}(T_\mu^2 u)(x) + g_\mu^2 u(x) \quad (0 < \lambda < 1, \mu > 1),$$

where c is a constant depending on λ, μ and n .

The following result clarifies the sense of (1.2), or more precisely of Theorems 3.1 – 3.4. As its counterpart for (1.1), Lemma 1 in [9], it serves as a bridge between a certain relation for two functions and corresponding relations for their rearrangements and norms.

Lemma 3.5. For any $\lambda, 0 < \lambda \leq 1/2$, let there exist a constant $c_\lambda > 0$, so that the non-negative functions f, g satisfy the inequality

$$m_\lambda f(x) \leq c_\lambda m_{\lambda/4} g(x) + f(x)$$

for all $x \in \mathbb{R}^n$, and let $\omega \in A_\infty$. Then we have:

(i) there exist constants $c_1, c_2 > 0$, so that for all $t > 0$,

$$f_\omega^*(t) \leq c_1 g_\omega^*(c_2 t) + f_\omega^*(2t);$$

(ii) if $f_\omega^*(+\infty) = 0$, then

$$\|f\|_{L^p_\omega} \leq c_p \|g\|_{L^p_\omega} \quad (0 < p < \infty);$$

(iii) if $g \in L^\infty$, then f has bounded lower oscillation, $f \in BLO$, and

$$\|f\|_{BLO} \leq c \|g\|_\infty.$$

Since the proof goes along the same lines as Lemma 1 in [9], we outline it briefly. Actually, all items of this lemma are an immediate combination of Lemma 1 in [9] and the next inequality

$$(3.1) \quad (m_\lambda f)_\omega^*(t) \leq f_\omega^*(ct),$$

where $c = c(\lambda, \omega)$. To prove (3.1), observe that from the definition of the rearrangement we have

$$\{x : m_\lambda f(x) > \alpha\} \subset \{x : M\chi_{\{y : |f(y)| > \alpha\}}(x) \geq \lambda\} \quad (\lambda, \alpha > 0).$$

On the other hand, it is well known (see, e.g., [2]) that a weight ω belongs to A_∞ iff there exist $k, r \geq 1$ so that

$$\omega\{x : Mf(x) > \lambda\} \leq \frac{k}{\lambda^r} \int_{\mathbb{R}^n} |f(x)|^r \omega(x) dx$$

for all measurable functions f on \mathbb{R}^n and all $\lambda > 0$. Using these inequalities, we obtain

$$\omega\{x : m_\lambda f(x) > \alpha\} \leq c\omega\{x : |f(x)| > \alpha\},$$

which proves (3.1).

Remark 3.6. It is easy to see that all assertions of Lemma 3.5 hold under the assumption that non-negative functions f, g satisfy the inequality

$$m_\lambda f(x) \leq c_\lambda m_{\varphi(\lambda)} g(x) + f(x),$$

where $\varphi(\lambda)$ is a measurable function satisfying the only condition $0 < \varphi(\lambda) < 1$.

Theorems 3.1–3.4 together with (ii) of Lemma 3.5 give classical L^p_ω -estimates obtained earlier with the help of “good- λ ” inequalities. Also, using Theorems 3.1, 3.3, 3.4 and (iii) of Lemma 3.5, we get a new approach to *BLO* inequalities (cf. [8]).

Next we note that Theorems 3.1, 3.3, 3.4 and (i) of Lemma 3.5 give a refinement of the rearrangement inequalities proved in [7].

Finally, we would like to point out that combining Theorem 3.2 and (i), (iii) of Lemma 3.5 gives two new results concerning the nontangential maximal function.

Corollary 3.7. *Let $\omega \in A_\infty$. For any harmonic function u on \mathbb{R}^{n+1}_+ we have:*

(i) *there exist constants $c', c'' > 0$ depending on α, β, ω , so that*

$$(N_\alpha u)_\omega^*(t) \leq c'(S_\beta u)_\omega^*(c''t) + (N_\alpha u)_\omega^*(2t) \quad (t > 0, \alpha < \beta);$$

(ii) *if $S_\beta u \in L^\infty$, then $N_\alpha u \in BLO$, and*

$$\|N_\alpha u\|_{BLO} \leq c\|S_\beta u\|_\infty \quad (\alpha < \beta).$$

4. TWO LEMMAS

We shall establish pointwise relations between $S_{\lambda, \alpha}^\# u$ and $N_{\lambda, \alpha}^\# u$. This is a key ingredient in the proving Theorems 3.1 and 3.2. Note that we partially use the arguments from [1, 5, 11].

Lemma 4.1. *For all $x \in \mathbb{R}^n$,*

$$(4.1) \quad S_{\lambda, \alpha}^\# u(x) \leq cN_{\lambda', \beta}^\# u(x) \quad (0 < \lambda' < \lambda < 1, \alpha < \beta),$$

where $c = c(\lambda, \lambda', \alpha, \beta, n)$.

Proof. Let $x \in Q$. For $\delta > 0$ let $h = (1 + \delta)d_Q$, and

$$A = \{z \in Q : N_{\beta, h}u(z) \leq ((N_{\beta, h}u)\chi_Q)^* ((\lambda - \delta)|Q|)\}.$$

It is clear that $|A| \geq (1 - \lambda + \delta)|Q|$. Let $E \subset Q$ be an arbitrary set with $|E| = \lambda|Q|$. Then $|E \cap A| \geq \delta|Q|$.

Set $D = \bigcup_{\xi \in E \cap A} \Gamma_{\alpha, d_Q}(\xi)$. Fubini’s theorem yields

$$(4.2) \quad \begin{aligned} \inf_{z \in E} S_{\alpha, d_Q}^2 u(z) &\leq \frac{1}{|E \cap A|} \int_{E \cap A} S_{\alpha, d_Q}^2 u(z) dz \\ &\leq \frac{1}{\delta|Q|} \int_D t^{1-n} |\nabla u(y, t)|^2 |\{\xi \in E \cap A : (y, t) \in \Gamma_{\alpha, d_Q}(\xi)\}| dy dt \\ &\leq \frac{c}{|Q|} \int_D t |\nabla u(y, t)|^2 dy dt. \end{aligned}$$

Now one can estimate the last integral in a standard way. We approximate D by a family of subdomains $D_\varepsilon \subset D$ with sufficiently smooth boundaries (see [12, p. 206]). It is easy to see that $\sigma(D_\varepsilon) \leq c|Q|$. Applying (2.1) and (2.2), we get

$$\begin{aligned} \int_{D_\varepsilon} t |\nabla u(y, t)|^2 dy dt &\leq \int_{\partial D_\varepsilon} (t|u(y, t)| |\nabla u(y, t)| + |u(y, t)|^2/2) d\sigma \\ &\leq c\sigma(\partial D_\varepsilon) \sup_{z \in A} N_{\beta, h}^2 u(z) \leq c|Q| ((N_{\beta, h}^2 u)\chi_Q)^* ((\lambda - \delta)|Q|). \end{aligned}$$

Letting $D_\varepsilon \rightarrow D$ as $\varepsilon \rightarrow 0+$ and using (4.2), we have

$$\inf_{z \in E} S_{\alpha, d_Q} u(z) \leq c((N_{\beta, h}u)\chi_Q)^* ((\lambda - \delta)|Q|).$$

Taking the supremum over all sets $E \subset Q$ with $|E| = \lambda|Q|$ yields

$$\begin{aligned} ((S_{\alpha,d_Q}u)\chi_Q)^*(\lambda|Q|) &\leq c((N_{\beta,h}u)\chi_Q)^*((\lambda - \delta)|Q|) \\ &\leq c((N_{\beta,h}u)\chi_{(1+\delta)Q})^*(\lambda'(1 + \delta)|Q|), \end{aligned}$$

where $\lambda' = (\lambda - \delta)/(1 + \delta)^n$. This completes the proof, since $S_{\alpha,d_Q}(u - c) = S_{\alpha,d_Q}(u)$. □

Now we introduce the following notation. Let $\alpha' = \max(1/\alpha, 1)$. If $x \in Q$, we set

$$\tilde{u}(y, t) = u(y, t) - u(x, \alpha'd_Q).$$

Lemma 4.2. *For any cube $Q \subset \mathbb{R}^n$ and all $x \in Q$,*

$$((N_{\alpha,d_Q}\tilde{u})\chi_Q)^*(\lambda|Q|) \leq cm_{\lambda/2}S_{\beta}u(x) \quad (0 < \lambda < 1, \alpha < \beta),$$

where $c = c(\lambda, \alpha, \beta, n)$.

Proof. Let $x \in Q$. For $\delta > 0$ let $h = (1 + \delta)\alpha'd_Q$, $S_{\beta}(Q) = ((S_{\beta,h}u)\chi_Q)^*((\lambda - \delta)|Q|)$ and

$$A = \{z \in Q : S_{\beta,h}u(z) \leq S_{\beta}(Q)\}.$$

Let $E \subset Q$ be an arbitrary set with $|E| = \lambda|Q|$. Then $|E \cap A| \geq \delta|Q|$.

For any $z \in E \cap A$ there exists a point $(y, t) \in \Gamma_{\alpha,d_Q}(z)$ so that $N_{\alpha,d_Q}\tilde{u}(z) \leq 2|\tilde{u}(y, t)|$. Let $\xi \in B(z, \alpha t)$, where $B(z, \alpha t)$ is the ball centered at z of radius αt . Since $(\xi, t), (y, t) \in \Gamma_{\alpha,d_Q}(z)$, all points of the segment with the end points (ξ, t) and (y, t) belong to $\Gamma_{\alpha,d_Q}(z)$. Using Mean Value Theorem, (2.3), and the fact that $\nabla\tilde{u} = \nabla u$, we obtain

$$|\tilde{u}(y, t)| \leq cS_{\beta}(Q) + \inf_{\xi \in B(z, \alpha t)} |\tilde{u}(\xi, t)|.$$

The balls $B(z, 2\alpha t)$ form the covering of the set $E \cap A$. Using Besicovitch's theorem [6, p. 5], choose a finite number of pairwise disjoint balls $B(z_i, 2\alpha t_i)$ ($i = 1, \dots, k$) so that $\sum_{i=1}^k |B(z_i, \alpha t_i)| \geq c|E \cap A| \geq c'|Q|$. Then

$$\begin{aligned} (4.3) \quad \inf_{z \in E} N_{\alpha,d_Q}\tilde{u}(z) &\leq \min_{1 \leq i \leq k} N_{\alpha,d_Q}\tilde{u}(z_i) \\ &\leq c(S_{\beta}(Q) + \min_{1 \leq i \leq k} \inf_{\xi \in B(z_i, \alpha t_i)} |\tilde{u}(\xi, t_i)|). \end{aligned}$$

Now set $D = \bigcup_{i=1}^k \Gamma_{\alpha,d_Q}(z_i) \setminus \Gamma_{\alpha,t_i}(z_i)$. Let us denote $\partial D^+ = \{(y, d_Q) \in \partial D\}$ and $\partial D^- = \partial D \setminus \partial D^+$. Since $B(z_i, 2\alpha t_i)$ are disjoint and $\xi \notin B(z, 2\alpha t)$ implies $\Gamma_{\alpha,t}(\xi) \cap \Gamma_{\alpha,t}(z) = \emptyset$, we see that $\bigcup_{i=1}^k \{(\xi, t_i) : \xi \in B(z_i, \alpha t_i)\} \subset \partial D^-$. Therefore,

$$\begin{aligned} (4.4) \quad \min_{1 \leq i \leq k} \inf_{\xi \in B(z_i, \alpha t_i)} |\tilde{u}(\xi, t_i)| &\leq \min_{1 \leq i \leq k} \left(\frac{1}{|B(z_i, \alpha t_i)|} \int_{B(z_i, \alpha t_i)} \tilde{u}^2(\xi, t_i) d\xi \right)^{1/2} \\ &\leq \left(\frac{1}{\sum_{i=1}^k |B(z_i, \alpha t_i)|} \sum_{i=1}^k \int_{B(z_i, \alpha t_i)} \tilde{u}^2(\xi, t_i) d\xi \right)^{1/2} \\ &\leq \left(\frac{c}{|Q|} \int_{\partial D^-} \tilde{u}^2 d\sigma \right)^{1/2}. \end{aligned}$$

It is easy to see that $|\partial t/\partial n| \leq 1, \sigma(\partial D) \leq c|Q|$ and $-\partial t/\partial n \geq c > 0$ for $(y, t) \in \partial D^-$. Thus, by (2.1) we obtain

$$(4.5) \quad \int_{\partial D^-} \tilde{u}^2 d\sigma \leq c \left(\int_{\partial D^+} \tilde{u}^2 d\sigma + \int_{\partial D} t|\tilde{u}||\nabla u| d\sigma + \int_D t|\nabla u(y, t)|^2 dy dt \right).$$

Let $(y, d_Q) \in \partial D$. There exists $z_i \in A$ so that $|y - z_i| < \alpha d_Q$. Besides, $|x - z_i| < d_Q \leq \alpha \cdot \alpha' d_Q$. Since (y, d_Q) and $(x, \alpha' d_Q)$ belong to $\Gamma_\alpha(z_i)$, using Mean Value Theorem and (2.3), we have

$$|\tilde{u}(y, d_Q)| \leq cS_{\beta,h}u(z_i) \leq cS_\beta(Q).$$

Hence,

$$(4.6) \quad \int_{\partial D^+} \tilde{u}^2 d\sigma \leq c|Q|S_\beta^2(Q).$$

Next, by (2.3) and Hölder's inequality,

$$(4.7) \quad \begin{aligned} \int_{\partial D} t|\tilde{u}||\nabla u|d\sigma &= \int_{\partial D^+} t|\tilde{u}||\nabla u|d\sigma + \int_{\partial D^-} t|\tilde{u}||\nabla u|d\sigma \\ &\leq c|Q|S_\beta^2(Q) + \left(\int_{\partial D^-} (t|\nabla u|)^2 d\sigma\right)^{1/2} \left(\int_{\partial D^-} \tilde{u}^2 d\sigma\right)^{1/2} \\ &\leq c|Q|S_\beta^2(Q) + c|Q|^{1/2}S_\beta(Q) \left(\int_{\partial D^-} \tilde{u}^2 d\sigma\right)^{1/2}. \end{aligned}$$

Let us estimate the last term in (4.5). We invoke the following geometrical observation (see [5]): for any point $\xi \in \mathbb{R}^n$ there exists $z \in A$ so that

$$\left(\bigcup_{\eta \in A} \Gamma_{\alpha,d_Q}(\eta)\right) \cap \Gamma_{(\beta-\alpha)/2,d_Q}(\xi) \subset \Gamma_{\beta,d_Q}(z).$$

It follows from this that for all $\xi \in \mathbb{R}^n$,

$$(4.8) \quad \int_{\Gamma_{\frac{\beta-\alpha}{2},d_Q}(\xi)} t^{1-n}|\nabla u(y,t)|^2 \chi_{\bigcup_{\eta \in A} \Gamma_{\alpha,d_Q}(\eta)}(y,t) dy dt \leq S_\beta^2(Q).$$

Note that $\bigcup_{\eta \in A} \Gamma_{\alpha,d_Q}(\eta) \subset c'Q \times d_Q \subset \bigcup_{\xi \in c'Q} \Gamma_{(\beta-\alpha)/2,d_Q}(\xi)$, with $c' = 2\alpha\sqrt{n} + 1$. Integrating (4.8) over $c'Q$ and using Fubini's theorem, we get

$$\begin{aligned} \int_D t|\nabla u(y,t)|^2 dy dt &\leq \int_{\bigcup_{\eta \in A} \Gamma_{\alpha,d_Q}(\eta)} t|\nabla u(y,t)|^2 dy dt \\ &\leq c \int_{\bigcup_{\eta \in A} \Gamma_{\alpha,d_Q}(\eta)} t^{1-n}|\nabla u(y,t)|^2 |\{x' \in c'Q : (y,t) \in \Gamma_{\frac{\beta-\alpha}{2},d_Q}(x')\}| dy dt \\ &\leq c|c'Q|S_\beta^2(Q) \leq c|Q|S_\beta^2(Q). \end{aligned}$$

From this and (4.5)–(4.7) we obtain

$$\int_{\partial D^-} \tilde{u}^2 d\sigma \leq c|Q|S_\beta^2(Q) + c|Q|^{1/2}S_\beta(Q) \left(\int_{\partial D^-} \tilde{u}^2 d\sigma\right)^{1/2}.$$

This yields $\int_{\partial D^-} \tilde{u}^2 d\sigma \leq c|Q|S_\beta^2(Q)$. Hence, from this and (4.3), (4.4) we get

$$(4.9) \quad \inf_{z \in E} N_{\alpha,d_Q} \tilde{u}(z) \leq c((S_{\beta,h}u)\chi_Q)^*((\lambda - \delta)|Q|).$$

Choosing $\delta = \lambda/2$ and taking the supremum over all $E \subset Q$ with $|E| = \lambda|Q|$ completes the proof of Lemma 4.2. \square

Remark 4.3. It is easy to see that (4.9) implies the inequality

$$N_{\lambda, \alpha}^{\#} u(x) \leq c S_{\lambda', \min(\alpha^n, 1), \beta}^{\#} u(x) \quad (0 < \lambda' < \lambda < 1, \alpha < \beta),$$

where $c = c(\lambda, \lambda', \alpha, \beta, n)$. Hence, for $\alpha \geq 1$ we have the inequality “converse” to (4.1).

5. PROOFS OF MAIN RESULTS

Here we always suppose that $x, z \in Q$.

Proof of Theorem 3.1. Let as above $\alpha' = \max(1/\alpha, 1)$. Since

$$K_{\alpha}(t) = \{y : |y - z| < \alpha t\} \setminus \{y : |y - x| < \alpha t\} \subset \{y : \alpha t - d_Q \leq |y - z| < \alpha t\},$$

for $t > \alpha' d_Q$ we get

$$|K_{\alpha}(t)| \leq c_n n \alpha^{n-1} d_Q t^{n-1}.$$

From this and (2.2) we obtain

$$\begin{aligned} & \int_{\alpha' d_Q}^{\infty} \int_{\{|y-z| < \alpha t\}} t^{1-n} |\nabla u(y, t)|^2 dy dt \\ & \leq \int_{\alpha' d_Q}^{\infty} \int_{K_{\alpha}(t)} t^{1-n} |\nabla u(y, t)|^2 dy dt \\ & \quad + \int_{\alpha' d_Q}^{\infty} \int_{\{|y-x| < \alpha t\}} t^{1-n} |\nabla u(y, t)|^2 dy dt \\ & \leq c d_Q N_{\beta}^2 u(z) \int_{\alpha' d_Q}^{\infty} \frac{dt}{t^2} + S_{\alpha}^2 u(x) \leq c N_{\beta}^2 u(z) + S_{\alpha}^2 u(x). \end{aligned}$$

If $\alpha < 1$, then using (2.2) again, we get

$$\int_{d_Q}^{\alpha' d_Q} \int_{\{|y-z| < \alpha t\}} t^{1-n} |\nabla u(y, t)|^2 dy dt \leq c N_{\beta}^2 u(z) \int_{d_Q}^{\alpha' d_Q} \frac{dt}{t} \leq c N_{\beta}^2 u(z).$$

Thus, two last estimates show that for any $\alpha > 0$ and for all $z \in Q$,

$$\begin{aligned} S_{\alpha}^2 u(z) &= S_{\alpha, d_Q}^2 u(z) + \int_{d_Q}^{\infty} \int_{\{|y-z| < \alpha t\}} t^{1-n} |\nabla u(y, t)|^2 dy dt \\ &\leq S_{\alpha, d_Q}^2 u(z) + c N_{\beta}^2 u(z) + S_{\alpha}^2 u(x). \end{aligned}$$

From this and (4.1) we have

$$\begin{aligned} ((S_{\alpha}^2 u) \chi_Q)^*(\lambda |Q|) &\leq ((S_{\alpha, d_Q}^2 u) \chi_Q)^*(\lambda/2 |Q|) + c((N_{\beta}^2 u) \chi_Q)^*(\lambda/2 |Q|) + S_{\alpha}^2 u(x) \\ &\leq c m_{\lambda/4} N_{\beta}^2 u(x) + c m_{\lambda/2} N_{\beta}^2 u(x) + S_{\alpha}^2 u(x) \\ &\leq c m_{\lambda/4} N_{\beta}^2 u(x) + S_{\alpha}^2 u(x) \end{aligned}$$

(with $\lambda' = \lambda/2$ in (4.1)). Taking the upper bound over all $Q \ni x$ completes the proof of the theorem. \square

Proof of Theorem 3.2. We use here the notation from Lemma 4.2. Let $(y, t) \in \Gamma_\alpha(z)$. If $t \geq d_Q$, then $(x, \alpha't) \in \Gamma_\alpha(z)$. Using Mean Value Theorem and (2.3), we obtain

$$|u(y, t)| \leq |u(y, t) - u(x, \alpha't)| + N^+u(x) \leq cS_\beta u(z) + N^+u(x).$$

Hence,

$$\begin{aligned} N_\alpha u(z) &= \max\left(\sup_{(y,t) \in \Gamma_{\alpha,d_Q}(z)} |u(y, t)|, \sup_{(y,t) \in \Gamma_\alpha \setminus \Gamma_{\alpha,d_Q}(z)} |u(y, t)|\right) \\ &\leq \max(N_{\alpha,d_Q} \tilde{u}(z) + N^+u(x), cS_\beta u(z) + N^+u(x)) \\ &\leq N_{\alpha,d_Q} \tilde{u}(z) + cS_\beta u(z) + N^+u(x). \end{aligned}$$

From this and Lemma 4.2 we get

$$\begin{aligned} ((N_\alpha u)\chi_Q)^*(\lambda|Q|) &\leq ((N_{\alpha,d_Q} \tilde{u})\chi_Q)^*(\lambda|Q|/2) + c((S_\beta u)\chi_Q)^*(\lambda|Q|/2) + N^+u(x) \\ &\leq cm_{\lambda/4} S_\beta u(x) + cm_{\lambda/2} S_\beta u(x) + N^+u(x) \\ &\leq cm_{\lambda/4} S_\beta u(x) + N^+u(x), \end{aligned}$$

which implies the desired result. □

Proof of Theorem 3.3. It suffices to prove the theorem for $0 < \alpha < 1$. Let $\eta < \alpha$.

If $t > d_Q/\eta$, then $(x, t) \in \Gamma_\eta(z)$. Using Mean Value Theorem and (2.2), we have

$$\begin{aligned} \left| |\nabla u(z, t)|^2 - |\nabla u(x, t)|^2 \right| &\leq \left(|\nabla u(z, t)| + |\nabla u(x, t)| \right) \left| |\nabla u(z, t)| - |\nabla u(x, t)| \right| \\ &\leq c \frac{|z-x|}{t} N_\alpha u(z) \left| \nabla |\nabla u(\xi, t)| \right| \leq c \frac{d_Q}{t^3} N_\alpha^2 u(z), \end{aligned}$$

where (ξ, t) is a point on the line with the end points (z, t) and (x, t) . From this and (2.2) we obtain

$$\begin{aligned} \int_{d_Q}^\infty t |\nabla u(z, t)|^2 dt &= \int_{d_Q}^{d_Q/\eta} t |\nabla u(z, t)|^2 dt + \int_{d_Q/\eta}^\infty t |\nabla u(z, t)|^2 dt \\ &\leq cN_\alpha^2 u(z) \int_{d_Q}^{d_Q/\eta} \frac{dt}{t} + cd_Q N_\alpha^2 u(z) \int_{d_Q}^\infty \frac{dt}{t^2} + \int_{d_Q}^\infty t |\nabla u(x, t)|^2 dt \\ (5.1) \quad &\leq cN_\alpha^2 u(z) + g^2 u(x). \end{aligned}$$

Next, we note that besides the well-known inequality $g(x) \leq cS(x)$ (see [12]), one can prove its “truncated” variant using exactly the same argument:

$$\int_0^{d_Q} t |\nabla u(z, t)|^2 dt \leq c_{\eta,n} S_{\eta,d_Q}^2 u(z).$$

Thus, applying Lemma 4.1 and (5.1), we get

$$\begin{aligned} ((g^2 u)\chi_Q)^*(\lambda|Q|) &\leq c \left(S_{\lambda/2,\eta}^\# u(x) \right)^2 + cm_{\lambda/2} N_\alpha^2 u(x) + g^2 u(x) \\ &\leq cm_{\lambda/4} (N_\alpha^2 u)(x) + g^2 u(x), \end{aligned}$$

as required. □

Proof of Theorem 3.4. Note, that in [10] the “good- λ ” inequality relating g_μ and T_μ have been proved. Using essentially the same arguments, one can prove the desired pointwise estimate. So, we shall omit some details in the proof.

We shall use the following inequalities [10], [12, p. 275]:

$$(5.2) \quad N_\alpha u(x) \leq cT_\mu u(x),$$

$$(5.3) \quad |\nabla u(y, t)|^2 \leq ct^{-2} \frac{1}{t^{n+1}} \int_{B_{t/2}(y, t)} u^2(\xi, \eta) d\xi d\eta,$$

where $B_{t/2}(y, t)$ is the ball centered at (y, t) of radius $t/2$.

We split the integral defining g_μ into two as follows:

$$\begin{aligned} g_\mu^2(u)(z) &= \int_{T(3Q)} \left(\frac{t}{t+|z-y|}\right)^{\mu n} t^{1-n} |\nabla u(y, t)|^2 dy dt \\ &+ \int_{\mathbb{R}_+^{n+1} \setminus T(3Q)} \left(\frac{t}{t+|z-y|}\right)^{\mu n} t^{1-n} |\nabla u(y, t)|^2 dy dt \\ &\equiv g_{\mu,1}u(z) + g_{\mu,2}u(z). \end{aligned}$$

Using (5.3) and the standard techniques for maximal functions, one can show that

$$g_{\mu,2}u(z) \leq cT_\mu^2 u(z) + g_\mu^2(x).$$

Now we set $A = \{z \in Q : T_\mu u(z) \leq ((T_\mu u)\chi_Q)^*(\lambda|Q|/2)\}$. Let $E \subset Q$, $|E| = \lambda|Q|$. Then $|E \cap A| \geq \lambda|Q|/2$. Denote $D = \bigcup_{z \in A} \Gamma_{1,3d_Q}(z)$. We have

$$\begin{aligned} \inf_{z \in E} g_{\mu,1}u(z) &\leq \frac{1}{|E \cap A|} \int_{E \cap A} \int_{T(3Q)} \left(\frac{t}{t+|z-y|}\right)^{\mu n} t^{1-n} |\nabla u(y, t)|^2 dy dt dz \\ &\leq \frac{c}{|Q|} \int_{T(3Q) \cap D} \int_A \left(\frac{t}{t+|z-y|}\right)^{\mu n} t^{1-n} |\nabla u(y, t)|^2 dz dy dt \\ (5.4) \quad &+ \frac{c}{|Q|} \int_{T(3Q) \setminus D} \int_A \left(\frac{t}{t+|z-y|}\right)^{\mu n} t^{1-n} |\nabla u(y, t)|^2 dz dy dt. \end{aligned}$$

Since $\int_{\mathbb{R}^n} \frac{t^{(\mu-1)n}}{(t+|z-y|)^{\mu n}} dz \leq c_{\mu,n}$, we see that

$$\int_{T(3Q) \cap D} \int_A \left(\frac{t}{t+|z-y|}\right)^{\mu n} t^{1-n} |\nabla u(y, t)|^2 dz dy dt \leq c \int_D t |\nabla u(y, t)|^2 dy dt.$$

Using (5.2) and the same arguments as in Lemma 4.1, we obtain

$$\begin{aligned} \int_D t |\nabla u(y, t)|^2 dy dt &\leq c|Q| \sup_{z \in A} N_\alpha^2 u(z) \\ &\leq c|Q| \sup_{z \in A} T_\mu^2 u(z) \leq c|Q| ((T_\mu^2 u)\chi_Q)^*(\lambda|Q|/2). \end{aligned}$$

The last integral in (5.4) is estimated exactly as in [10], by applying Whitney's decomposition to the set $3Q \setminus A$. Then, using (5.3), one can show that

$$\int_{T(3Q) \setminus D} \int_A \left(\frac{t}{t+|z-y|}\right)^{\mu n} t^{1-n} |\nabla u(y, t)|^2 dz dy dt \leq c|Q| ((T_\mu^2 u)\chi_Q)^*(\lambda|Q|/2).$$

Combining the obtained estimates gives

$$((g_{\mu,1}u)\chi_Q)^*(\lambda|Q|) \leq c ((T_\mu^2 u)\chi_Q)^*(\lambda|Q|/2).$$

Hence,

$$\begin{aligned} ((g_\mu^2 u)\chi_Q)^*(\lambda|Q|) &\leq c((T_\mu^2 u)\chi_Q)^*(\lambda|Q|/4) + c((T_\mu^2 u)\chi_Q)^*(\lambda|Q|/2) + g_\mu^2 u(x) \\ &\leq c((T_\mu^2 u)\chi_Q)^*(\lambda|Q|/4) + g_\mu^2 u(x), \end{aligned}$$

which proves the theorem. \square

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