

## GEVREY VECTORS OF MULTI-QUASI-ELLIPTIC SYSTEMS

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ABSTRACT. We show that the multi-quasi-ellipticity is a necessary and sufficient condition for the property of elliptic iterates to hold for multi-quasi-homogenous differential operators.

### 1. INTRODUCTION

Let  $P_j(x, D) = \sum_{\alpha} a_{j\alpha}(x) D^{\alpha}$ ,  $j = 1, \dots, N$ , henceforth denoted  $(P_j)_{j=1}^N$ , be linear differential operators with  $C^{\infty}$  coefficients in an open subset  $\Omega$  of  $\mathbb{R}^n$ .

The aim of this work is to prove the property of elliptic iterates for multi-quasi-elliptic systems of differential operators in generalized Gevrey spaces  $G^{\mathcal{F},s}(\Omega)$ , where  $\mathcal{F}$  denotes Newton's polyhedron of the system  $(P_j)_{j=1}^N$ . The property of elliptic iterates for the system  $(P_j)_{j=1}^N$  in the generalized Gevrey classes  $G^{\mathcal{F},s}(\Omega)$  means the following inclusion:

$$G^s(\Omega, (P_j)_{j=1}^N) \subset G^{\mathcal{F},s}(\Omega).$$

**Definition 1.** Newton's polyhedron of the system  $(P_j)_{j=1}^N$  at the point  $x_0 \in \Omega$ , denoted  $\mathcal{F}(x_0)$ , is the convex hull of the set  $\{\alpha \in \mathbb{N}^n, \exists j \in \{1, \dots, N\}; a_{j\alpha}(x_0) \neq 0\}$ . A Newton's polyhedron  $\mathcal{F}$  is said to be regular if there exists a finite set  $Q(\mathcal{F}) \subset (\mathbb{R}_+^*)^n$  such that

$$\mathcal{F} = \bigcap_{q \in Q(\mathcal{F})} \{\alpha \in \mathbb{R}_+^n, \langle \alpha, q \rangle \leq 1\}.$$

Set

$$\begin{aligned} k(\alpha, \mathcal{F}) &= \inf \{t > 0, t^{-1}\alpha \in \mathcal{F}\}, \quad \alpha \in \mathbb{R}_+^n, \\ \mu(\mathcal{F}) &= \max_{1 \leq j \leq n} \mu_j(\mathcal{F}), \\ \mu_j(\mathcal{F}) &= \max_{q \in Q(\mathcal{F})} q_j^{-1}, \quad j = 1, \dots, n, \\ \theta(\mathcal{F}) &= \left( \frac{\mu(\mathcal{F})}{\mu_1(\mathcal{F})}, \dots, \frac{\mu(\mathcal{F})}{\mu_n(\mathcal{F})} \right). \end{aligned}$$

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**Definition 2.** Let  $\mathcal{F}$  be a regular Newton’s polyhedron and  $s \in \mathbb{R}_+$ . We define the generalized Gevrey space  $G^{\mathcal{F},s}(\Omega)$  by the space of  $u \in C^\infty(\Omega)$  such that  $\forall H$  compact of  $\Omega, \exists C > 0, \forall \alpha \in \mathbb{N}^n,$

$$(1.1) \quad \sup_H |D^\alpha u| \leq C^{|\alpha|+1} [\Gamma(\mu(\mathcal{F})k(\alpha, \mathcal{F}) + 1)]^s,$$

where  $\Gamma$  is the gamma function.

*Remark 1.* One can take  $\sup_H |D^\alpha u|$  or  $\|D^\alpha u\|_{L^2(H)}$  in the definition, according to Sobolev imbedding theorems.

**Definition 3.** The system  $(P_j)_{j=1}^N$  is said to be multi-quasi-elliptic in  $\Omega$  if

- 1) The  $\mathcal{F}(x)$  do not depend on  $x \in \Omega$ , i.e.  $\forall x, \mathcal{F}(x) = \mathcal{F}$ .
- 2)  $\mathcal{F}$  is regular.
- 3)  $\forall x \in \Omega, \exists C > 0, \exists R \geq 0, \forall \xi \in \mathbb{R}^n, |\xi| \geq R,$

$$\sum_{j=1}^N |P_j(x, \xi)| \geq C \sum_{\alpha \in \mathbb{Z}_+^n \cap \mathcal{F}} |\xi^\alpha|.$$

**Definition 4.** Let  $(P_j)_{j=1}^N$  be a system of linear differential operators satisfying conditions 1) and 2) of Definition 3 and  $s \in \mathbb{R}_+$ , the space of Gevrey vectors of the system  $(P_j)_{j=1}^N$ , denoted  $G^s(\Omega, (P_j)_{j=1}^N)$ , is the space of  $u \in C^\infty(\Omega)$  such that  $\forall H$  compact of  $\Omega, \exists C > 0, \forall l \in \mathbb{N}, 1 \leq i_l \leq N,$

$$(1.2) \quad \|P_{i_1} \dots P_{i_l} u\|_{L^2(H)} \leq C^{l+1} (l!)^{s\mu(\mathcal{F})}.$$

The aim of this work is to show the following theorem.

**Theorem 1.** Let  $\Omega$  be an open subset of  $\mathbb{R}^n, \sigma > s \geq 1$  and  $(P_j)_{j=1}^N$  be a system of linear differential operators with  $G^{\theta(\mathcal{F}),\sigma}(\Omega)$  coefficients. Then

$$(P_j)_{j=1}^N \text{ is multi-quasi-elliptic in } \Omega \iff G^s(\Omega, (P_j)_{j=1}^N) \subset G^{\mathcal{F},s}(\Omega).$$

Some consequences of this theorem are given in section 4. For differential operators with constant coefficients we have shown in [3] a more general result.

## 2. SUFFICIENT CONDITION

The proof of the sufficient condition follows essentially the work of Zanghirati [6], so we refer for details to this paper.

Instead of  $Q(\mathcal{F}), k(\mathcal{F}, \alpha), \mu(\mathcal{F}), \theta(\mathcal{F})$  we write, respectively,  $Q, k(\alpha), \mu, \theta$ . Denote  $\mathcal{K} = \{k = k(\alpha) : \alpha \in \mathbb{N}^n\}$ . If  $\omega$  is an open subset of  $\mathbb{R}^n, u \in C^\infty(\omega)$  and  $k \in \mathcal{K}$ , define  $|u|_{k,\omega} = \sum_{k(\alpha)=k} \|D^\alpha u\|_{L^2(\omega)}$ . When  $u \in C_0^\infty(\mathbb{R}^n)$  we write  $|u|_k$ .

Let  $(P_j)_{j=1}^N$  be a system of linear differential operators with coefficients defined in an open neighborhood  $\Omega$  of the origin satisfying the following conditions:

- (i) The system  $(P_j)_{j=1}^N$  is multi-quasi-elliptic in  $\Omega$ .
- (ii) The coefficients  $a_{j\alpha} \in G^{\theta,s}(\Omega), \forall \alpha \in \mathcal{F}, \forall j \in \{1, \dots, N\}$ .

For  $\rho > 0$ , we denote  $B_\rho = \{x \in \mathbb{R}^n, \sum_{j=1}^n x_j^{2\mu_j/\mu} < \rho^2\}$ . We define for  $h \in \mathbb{N}$ ,

$$P_j^h(x, D) = \underbrace{P_j(x, D) \circ \dots \circ P_j(x, D)}_{h \text{ times}}, j = 1, \dots, N.$$

From the multi-quasi-ellipticity of the system  $(P_j)_{j=1}^N$  and following the proof of Lemma 3.4 of [6], we obtain

**Lemma 1.** *There exist  $\rho_0 > 0$  and  $C_1 > 0, \forall \varepsilon \in ]0, \frac{1}{v(n)}[$  ( $v(n)$  denote the number of elements of  $\mathcal{K} \cap [0, n]$ ),  $\exists C_2(\varepsilon) > 0, \forall \delta \in ]0, 1[, \forall \rho > 0, B_{\rho+\delta} \subset B_{\rho_0}, \forall u \in C^\infty(B_{\rho_0}), \forall p \geq n,$*

$$(2.1)$$

$$|u|_{p+1, B_\rho} \leq C_1 \left( \sum_{j=1}^N |P_j^n(x, D)u|_{p-n+1, B_{\rho+\delta}} + \varepsilon |u|_{p+1, B_{\rho+\delta}} + (\varepsilon\delta)^{-n\mu} |u|_{p-n+1, B_{\rho+\delta}} + \sum_{h=0}^p \left( \frac{(p+1)!}{h!} \right)^{s\mu} C_2(\varepsilon)^{p+1-h} |u|_{h, B_{\rho+\delta}} \right),$$

and for  $p \leq n$ , we have

$$(2.2)$$

$$|u|_{p+1, B_\rho} \leq C_1 \left( \sum_{j=1}^N |P_j^n(x, D)u|_{p-n+1, B_{\rho+\delta}} + \varepsilon |u|_{p+1, B_{\rho+\delta}} + (\varepsilon\delta)^{-(p+1)\mu} |u|_{0, B_{\rho+\delta}} \right).$$

Let  $\lambda > 0$  and  $R > 0$ . For  $p \in \mathbb{N}$ , we set

$$\sigma_p(u, \lambda) = (p!)^{-s\mu} \lambda^{-p} \sup_{R/2 \leq \rho < R} (R - \rho)^{p\mu} |u|_{p, B_\rho}.$$

**Lemma 2.** *Let  $\rho_0$  be as in the Lemma 1 and let  $0 < R < 1$  such that  $\overline{B_R} \subset B_{\rho_0}$ . Then there exists  $\lambda_0 > 0$  ( $\lambda_0$  depends only on  $R$  and  $(P_j)_{j=1}^N$ ),  $\forall u \in C^\infty(B_{\rho_0}), \forall \lambda \geq \lambda_0, \forall p \geq n,$*

$$(2.3)$$

$$\sigma_{p+1}(u, \lambda) \leq [(p - n + 2) \dots (p + 1)]^{-s\mu} \sum_{j=1}^N \sigma_{p-n+1}(P_j^n u, \lambda) + \sum_{h=0}^p \sigma_h(u, \lambda),$$

and for  $p \leq n - 1$ ,

$$(2.4) \quad \sigma_{p+1}(u, \lambda) \leq (p + 1)!^{-s\mu} \sum_{j=1}^N \sigma_0(P_j^{p+1} u, \lambda) + \sigma_0(u, \lambda).$$

*Proof.* Let  $p \geq n$ , multiply both sides of (2.1) by  $(p + 1)!^{-s\mu} \lambda^{-p-1} (R - \rho)^{p\mu}$ , put  $\delta = \frac{R-\rho}{p-n+2}$  and then taking the sup over  $\rho \in [R/2, R[$ , we obtain

$$\sigma_{p+1}(u, \lambda) \leq C_1 (I_1 + \varepsilon I_2 + \varepsilon^{-n\mu} I_3 + I_4),$$

where  $I_1, I_2, I_3$  and  $I_4$  are such that

$$\begin{aligned} I_1 &\leq \sum_{j=1}^N \left( \frac{(p-n+1)!}{(p+1)!} \right)^{s\mu} \frac{e^\mu}{\lambda^n} \sigma_{p-n+1}(P_j^n u, \lambda), \\ I_2 &\leq (2^n e)^\mu \sigma_{p+1}(u, \lambda), \\ I_3 &\leq \frac{e^\mu}{\lambda^n} \sigma_{p-n+1}(u, \lambda), \\ I_4 &\leq \frac{e^\mu C_2(\varepsilon)}{\lambda} \sum_{h=0}^p \left( \frac{C_2(\varepsilon)}{\lambda} \right)^{p-h} \sigma_h(u, \lambda). \end{aligned}$$

By a suitable choice of  $\varepsilon$ , we find

$$\begin{aligned} \sigma_{p+1}(u, \lambda) &\leq \left(\frac{(p-n+1)!}{(p+1)!}\right)^{s\mu} \frac{\tilde{C}_1}{\lambda^n} \sum_{j=1}^N \sigma_{p-n+1}(P_j^n u, \lambda) + \frac{\tilde{C}_2}{\lambda^n} \sigma_{p-n+1}(u, \lambda) \\ &\quad + \frac{\tilde{C}_3}{\lambda} \sum_{h=0}^p \left(\frac{\tilde{C}_4}{\lambda}\right)^{p-h} \sigma_h(u, \lambda). \end{aligned}$$

It suffices to take  $\lambda_0 = \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3 + \tilde{C}_4$  to get (2.3). For the inequality (2.4) we multiply both sides of inequality (2.2) by  $\frac{(R-\rho)^{(p+1)\mu}}{(p+1)!^{s\mu} \lambda^{p+1}}$ , take  $\delta = \frac{R-\rho}{2}$  and then we follow the same procedure for obtaining (2.3).  $\square$

**Lemma 3.** *Let  $\rho_0, R$  and  $\lambda_0$  be as in Lemma 2. Then for any  $u \in C^\infty(B_{\rho_0})$ ,  $\forall \lambda \geq \lambda_0$ ,  $\forall p \in \mathbb{N}$ , we have*

(2.5)

$$\sigma_{p+1}(u, \lambda) \leq 2^{p+1} \sigma_0(u, \lambda) + \sum_{l=1}^{p+1} 2^{p+1-l} C_{p+1}^l \frac{1}{(l!)^{s\mu}} \sum_{1 \leq i_1, \dots, i_l \leq N} \sigma_0(P_{i_1} \dots P_{i_l} u, \lambda).$$

*Proof.* It is obtained by recurrence over  $p$ .  $\square$

Our first result is the following theorem, which generalizes the results of [6], [7] and [8] to systems.

**Theorem 2.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ ,  $s \geq 1$  and  $(P_j(x, D))_{j=1}^N$  be a system of linear differential operators with  $G^{\theta, s}(\Omega)$  coefficients. Then*

$$(P_j)_{j=1}^N \text{ is multi-quasi-elliptic in } \Omega \Rightarrow G^s(\Omega, (P_j)_{j=1}^N) \subset G^{\mathcal{F}, s}(\Omega).$$

*Proof.* It is sufficient to check (1.1) in a neighborhood of every point  $x$  of  $\Omega$ . Let us assume  $x$  is the origin. Then there exist  $\rho_0, \lambda_0$  and  $R$  such that the precedent lemmas hold. Let  $u \in G^s(\Omega, (P_j)_{j=1}^N)$ . Then there is  $C_1 > 0$  such that

$$\sigma_0(P_{i_1} \dots P_{i_l} u, \lambda_0) \leq C_1^{l+1} (l!)^{s\mu}, \forall l \in \mathbb{N},$$

hence from (2.5), we obtain

$$\sigma_{p+1}(u, \lambda_0) \leq C_1 (2 + NC_1)^{p+1}, \forall p \in \mathbb{N},$$

which gives

$$(2.6) \quad |u|_{p+1, B_{R/2}} \leq (p+1)!^{s\mu} C_2^{(p+1)\mu+1}, \forall p \in \mathbb{N}.$$

Following the same steps as in [6] we obtain

$$(2.7) \quad |u|_{k, B_{R/2}} \leq \tilde{C}^{k\mu+1} (\Gamma(k+1))^{s\mu}.$$

Consequently from (2.7) it is easy, as in [6], to obtain the estimate (1.1).  $\square$

### 3. NECESSARY CONDITION

In this section we prove the converse of Theorem 2. For this aim we need a characterization of the multi-quasi-ellipticity of the system  $(P_j(x, D))_{j=1}^N$ , known in the case of a scalar operator; see [4].

**Proposition 1.** *A system  $(P_j)_{j=1}^N$ , satisfying 1) and 2) of Definition 2, is multi-quasi-elliptic in  $\Omega$  if and only if for any  $x \in \Omega, \forall q \in Q$ ,*

$$\sum_{j=1}^N |P_{jq}(x, \xi)| \neq 0, \quad \forall \xi \in \mathbb{R}^n, \xi_1 \dots \xi_n \neq 0,$$

where  $P_{jq}$  is the  $q$ -quasi-homogenous part of  $P_j$ , i.e.

$$P_{jq}(x, \xi) = \sum_{\langle \alpha, q \rangle = 1} a_{j\alpha}(x) \xi^\alpha.$$

**Theorem 3.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  and  $P_j(x, D), j = 1, \dots, N$ , be differential operators with  $G^{\theta, \sigma}(\Omega)$  coefficients. If  $s > \sigma \geq 1$ , then*

$$G^s(\Omega, (P_j)_{j=1}^N) \subset G^{\mathcal{F}, s}(\Omega) \Rightarrow (P_j)_{j=1}^N \text{ is multi-quasi-elliptic in } \Omega.$$

*Proof.* Assume that the system  $(P_j)_{j=1}^N$  is not multi-quasi-elliptic. Then there exist  $x_0 \in \Omega, q \in Q$  and  $\xi_0 \in S^{n-1}, \xi_{0,1} \dots \xi_{0,n} \neq 0$ , such that

$$(3.1) \quad P_{jq}(x_0, \xi_0) = 0, \quad \forall j = 1, \dots, N.$$

We construct a function  $u \in G^s(\Omega, (P_j)_{j=1}^N)$  such that  $u \notin G^{\mathcal{F}, s}(\Omega)$ , which contradicts the hypothesis. Put  $\eta = \frac{1-\varepsilon/\mu}{\mu s}$ , and choose  $\varepsilon$  satisfying

$$0 < \varepsilon \leq \frac{\mu(s - \sigma)}{2\mu s - \sigma} < \frac{1}{2} \text{ and } \varepsilon < \min_{\langle \beta, q \rangle < 1} \mu(1 - \langle \beta, q \rangle).$$

Let  $\delta > 0$  such that the ball  $B_0 = B(x_0, 2\delta)$  is relatively compact in  $\Omega$  and  $\varphi \in G^{q, \sigma\mu}(\mathbb{R}^n)$  with compact support in  $B(0, 2\delta)$  and  $\varphi(x) \equiv 1$  in  $B(0, \delta)$ . The desired function is defined by

$$u(x) = \int_1^{+\infty} \varphi[r^{\varepsilon q}(x - x_0)] e^{-r^\eta} e^{i\langle x - x_0, r^q \xi_0 \rangle} dr,$$

where  $r^q x = (r^{q_1} x_1, r^{q_2} x_2, \dots, r^{q_n} x_n)$ .

Following [5] and [8] it is easy to show that  $u \notin G^{\mathcal{F}, s}(U)$  for any neighborhood  $U$  of  $x_0$ .

Let us verify that  $u \in G^s(\Omega, (P_j)_{j=1}^N)$ . Since the coefficients of the operators  $P_j$  are in  $G^{\theta, \sigma}(\Omega) \subset G^{q, \sigma\mu}(\Omega)$ , then  $\exists M > 0, \forall \alpha \in \mathbb{Z}_+^n, \forall \beta \in \mathbb{Z}_+^n, \forall x \in B_0, \forall r \geq 1, \forall j = 1, \dots, N$ , such that

$$(3.2) \quad \left| \left( D_x^\beta P_j^{(\alpha)} \right) (x, r^q \xi_0) \right| \leq M^{|\beta|+1} [\Gamma(\langle \beta, q \rangle + 1)]^{\sigma\mu} r^{1-\langle \alpha, q \rangle}.$$

On the other hand in view of (3.1) it is easy to obtain  $\forall \delta > 0, \exists C_1 > 0, \forall r \geq 1, \forall x \in \Omega, |x - x_0| < 2\delta r^{-\varepsilon/\mu}, \forall j = 1, \dots, N$ ,

$$(3.3) \quad |P_j(x, r^q \xi_0)| \leq C_1 r^{1-\varepsilon/\mu}.$$

Now we need a convenient form of  $P_{i_k} \dots P_{i_1} u$ , for any integer  $k \geq 1$ . The generalized Leibniz formula  $P_j(x, D)(uv) = \sum_{\alpha} \frac{1}{\alpha!} P_j^{(\alpha)} u D^\alpha v$  gives

$$P_{i_k} \dots P_{i_0} u(x) = \int_1^{+\infty} A_{i_k \dots i_0}(x, r) e^{-r^\eta} e^{i\langle x - x_0, r^q \xi_0 \rangle} dr,$$

where  $1 \leq i_l \leq N$ , for any integer  $l \leq k$ ,  $P_{i_0}$  designs the identity operator, and

$$(3.4) \quad \begin{cases} A_{i_0}(x, r) = \varphi[r^{\varepsilon q}(x - x_0)], \\ A_{i_{k+1}, i_k \dots i_0}(x, r) = \sum_{\langle \alpha, q \rangle \leq 1} \frac{1}{\alpha!} P_{i_{k+1}}^{(\alpha)}(x, r^q \xi_0) D_x^\alpha A_{i_k \dots i_0}(x, r). \end{cases}$$

To complete the proof we need the following

**Lemma 4.**  $\exists L > 0, \exists L_0 > 0, \exists C_0 > 0, \forall k \in \mathbb{Z}_+, \forall \gamma \in \mathbb{Z}_+^n, \forall x \in B_0, \forall r \geq 1,$

$$(3.5) \quad |D_x^\gamma A_{i_k \dots i_0}(x, r)| \leq C_0 (L_0 r^\varepsilon)^{\langle \gamma, q \rangle} L^k \left( r^{(1-\varepsilon/\mu)k} [\Gamma(\langle \gamma, q \rangle + 1)]^{\sigma\mu} + [\Gamma(\langle \gamma, q \rangle + k + 1)]^{\sigma\mu} r^{k\varepsilon(2-1/\mu)} \right).$$

*Proof.* It is obtained by recurrence over  $k$ . In fact for  $k = 0$ , the estimate (3.5) means  $\varphi \in G_0^{q, \sigma\mu}(\mathbb{R}^n)$ . So suppose that the estimate (3.5) holds up to the order  $k$  and let us check it at the order  $k + 1$ . Set  $\lambda = r^{1-\varepsilon/\mu}$  and  $\tau = r^{\varepsilon(2-1/\mu)}$ . Then the estimate (3.5) is written as

$$|D_x^\gamma A_{i_k \dots i_0}(x, r)| \leq C_0 (L_0 r^\varepsilon)^{\langle \gamma, q \rangle} L^k S(k, \gamma),$$

where

$$S(k, \beta) = \lambda^k [\Gamma(\langle \beta, q \rangle + 1)]^{\sigma\mu} + [\Gamma(\langle \beta, q \rangle + k + 1)]^{\sigma\mu} \tau^k.$$

Let  $\omega = \min_{1 \leq j \leq n} q_j$ . Then we have

$$(3.6) \quad \lambda^{1-\langle \alpha, q \rangle} \tau^{\langle \alpha, q \rangle} S(k, \beta + \alpha) \leq 2^{\frac{\sigma\mu}{\omega} + 1} S(k + 1, \beta), \quad \langle \alpha, q \rangle \leq 1.$$

From (3.4), we have

$$|D_x^\gamma A_{i_{k+1} \dots i_0}(x, r)| \leq I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= |P_{i_{k+1}}(x, r^q \xi_0)| |D_x^\gamma A_{i_k \dots i_0}(x, r)|, \\ I_2 &= \sum_{\beta < \gamma} \binom{\gamma}{\beta} |D_x^{\gamma-\beta} P_{i_{k+1}}(x, r^q \xi_0)| |D_x^\beta A_{i_k \dots i_0}(x, r)|, \\ I_3 &= \sum_{0 < \langle \alpha, q \rangle \leq 1} \sum_{\beta \leq \gamma} \frac{1}{\alpha!} \binom{\gamma}{\beta} |D_x^{\gamma-\beta} P_{i_{k+1}}^{(\alpha)}(x, r^q \xi_0)| |D_x^{\alpha+\beta} A_{i_k \dots i_0}(x, r)|. \end{aligned}$$

Since  $A_{i_k \dots i_0}$  are functions of compact supports in  $B(x_0, 2\delta r^{-\varepsilon/\mu})$ , and according to (3.3) and (3.6), we have

$$(3.7) \quad I_1 \leq 2^{\frac{\sigma\mu}{\omega} + 1} C_1 C_0 (L_0 r^\varepsilon)^{\langle \gamma, q \rangle} S(k + 1, \gamma) L^k.$$

The estimates (3.2) and (3.6) give

$$\begin{aligned} I_2 &\leq \sum_{\beta < \gamma} \binom{\gamma}{\beta} [\Gamma(\langle \gamma - \beta, q \rangle + 1)]^{\sigma\mu} M^{|\gamma-\beta|+1} r^\varepsilon C_0 (L_0 r^\varepsilon)^{\langle \beta, q \rangle} \\ &\quad \cdot 2^{\frac{\sigma\mu}{\omega} + 1} S(k + 1, \beta) L^k. \end{aligned}$$

On the other hand, using properties of the gamma function, we have

$$(3.8) \quad \binom{\gamma}{\beta} [\Gamma(\langle \gamma - \beta, q \rangle + 1)]^{\sigma\mu} S(k, \beta) \leq C_2^{\sigma\mu \langle \gamma - \beta, q \rangle} S(k, \gamma).$$

Thus we obtain

$$I_2 \leq \frac{nMC_2^{\sigma\mu}}{L_0 r^\varepsilon} \sum_{\beta \geq 0} \left( \frac{MC_2^{\sigma\mu}}{L_0 r^\varepsilon} \right)^{\langle \beta, q \rangle} r^\varepsilon 2^{\frac{\sigma\mu}{\omega} + 1} MC_0 (L_0 r^\varepsilon)^{\langle \gamma, q \rangle} S(k+1, \gamma) L^k.$$

Set  $C_3 = \sum_{\alpha \geq 0} \left(\frac{1}{2}\right)^{\langle \alpha, q \rangle}$ , take  $L_0 \geq 2MC_2^{\sigma\mu}$  and  $r \geq 1$ , and then

$$(3.9) \quad I_2 \leq \frac{nMC_2^{\sigma\mu}}{L_0} C_3 2^{\frac{\sigma\mu}{\omega} + 1} MC_0 (L_0 r^\varepsilon)^{\langle \gamma, q \rangle} S(k+1, \gamma) L^k.$$

Finally in view of (3.2)

$$I_3 \leq \sum_{0 < \langle \alpha, q \rangle \leq 1} \sum_{\beta \leq \gamma} \binom{\gamma}{\beta} [\Gamma(\langle \gamma - \beta, q \rangle + 1)]^{\sigma\mu} M^{|\gamma - \beta| + 1} r^{1 - \langle \alpha, q \rangle} \\ \times C_0 (L_0 r^\varepsilon)^{|\beta + \alpha|} S(k, \beta + \alpha) L^k.$$

For any  $\alpha \in \mathbb{Z}_+^n$ ,  $0 < \langle \alpha, q \rangle \leq 1$ , we have  $r^{1 - \langle \alpha, q \rangle + \varepsilon \langle \alpha, q \rangle} \leq \lambda^{1 - \langle \alpha, q \rangle} \tau^{\langle \alpha, q \rangle}$ , which gives, with (3.6) and (3.8),

$$I_3 \leq \sum_{0 < \langle \alpha, q \rangle \leq 1} \sum_{\beta \leq \gamma} \left( \frac{MC_2^{\sigma\mu}}{L_0 r^\varepsilon} \right)^{|\gamma|} 2^{\frac{\sigma\mu}{\omega} + 1} MC_0 L_0^{|\alpha|} (L_0 r^\varepsilon)^{|\gamma|} S(k+1, \gamma) L^k.$$

Put  $C_4 = \sum_{0 < \langle \alpha, q \rangle \leq 1} L_0^{\langle \alpha, q \rangle}$ . Then we obtain

$$(3.10) \quad I_3 \leq 2^{\frac{\sigma\mu}{\omega} + 1} MC_4 C_3 C_0 (L_0 r^\varepsilon)^{\langle \gamma, q \rangle} S(k+1, \gamma) L^k.$$

If we choose

$$L \geq 2^{\frac{\sigma\mu}{\omega} + 1} \left( C_1 + \frac{nM^2 C_2^{\sigma\mu}}{L_0} C_3 + MC_3 C_4 \right),$$

we get, from (3.7), (3.9) and (3.10),

$$I_1 + I_2 + I_3 \leq C_0 (L_0 r^\varepsilon)^{\langle \gamma, q \rangle} S(k+1, \gamma) L^{k+1},$$

which means that (3.5) holds at the order  $k+1$ . □

*End of the Proof of Theorem 3.* Applying the last lemma for  $\gamma = 0$ , we find

$$(3.11) \quad |A_{i_k \dots i_0}(x, r)| \leq C'_0 L^k \left( r^{(1-\varepsilon/\mu)k} + (k!)^{\sigma\mu} r^{k\varepsilon(2-1/\mu)} \right).$$

Thus we obtain

$$|A_{i_k \dots i_0}(x, r)| \leq C'_0 L^k (2s\mu)^{k\mu s} (k!)^{s\mu} \left[ \exp\left(\frac{r^\eta}{2}\right) + \exp\left(\frac{r^{\eta'}}{2}\right) \right],$$

where  $\eta' = \frac{\varepsilon(2-1/\mu)}{\mu(s-\sigma)} \leq \eta = \frac{1-\varepsilon/\mu}{\mu s}$ , since  $\varepsilon \leq \frac{\mu(s-\sigma)}{2\mu s - \sigma}$ .

Therefore

$$|P_{i_k \dots i_0} u(x)| \leq 2C'_0 L^k (k!)^{s\mu} \int_1^{+\infty} \exp\left(-\frac{r^\eta}{2}\right) dr \\ \leq C^{k+1} (k!)^{s\mu},$$

which means that  $u \in G^s \left( \Omega, (P_j)_{j=1}^N \right)$ . □

## 4. CONSEQUENCES

A first consequence of Theorem 2 is a result on Gevrey-hypoellipticity for multi-quasi-elliptic systems.

**Corollary 1.** *Under the assumptions of Theorem 2, the following propositions are equivalent:*

- (i)  $u \in \mathcal{D}'(\Omega)$ ,  $P_j u \in G^{\mathcal{F},s}(\Omega)$ ,  $\forall j = 1, \dots, N$ .
- (ii)  $u \in G^{\mathcal{F},s}(\Omega)$ .

The theorems of this work unify the results of Bolley-Camus [1] and Métivier [5] in the homogenous case, the results of Zanghirati [7] and [8] in the scalar quasi-homogenous case and generalize them to quasi-homogenous systems.

**Corollary 2.** *Let  $\Omega$  be an open subset of  $\mathbb{R}^n$  and  $\sigma > s \geq 1$ , and let  $(P_j)_{j=1}^N$  be a system of linear differential operators with coefficients in  $G^{q,\sigma}(\Omega)$ . Then*

$$(P_j)_{j=1}^N \text{ is } q\text{-quasi-elliptic in } \Omega \iff G^s\left(\Omega, (P_j)_{j=1}^N\right) \subset G^{q,s}(\Omega).$$

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