GEVREY VECTORS OF MULTI-QUASI-ELLIPITC SYSTEMS

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Abstract. We show that the multi-quasi-ellipticity is a necessary and sufficient condition for the property of elliptic iterates to hold for multi-quasi-homogenous differential operators.

1. Introduction

Let \( P_j(x, D) = \sum_{\alpha} a_{j\alpha}(x) D^\alpha, j = 1, \ldots, N, \) henceforth denoted \((P_j)^N\), be linear differential operators with \( C^\infty \) coefficients in an open subset \( \Omega \) of \( \mathbb{R}^n \).

The aim of this work is to prove the property of elliptic iterates for multi-quasi-elliptic systems of differential operators in generalized Gevrey spaces \( G^{F,s}(\Omega) \), where \( F \) denotes Newton’s polyhedron of the system \((P_j)^N\). The property of elliptic iterates for the system \((P_j)^N\) in the generalized Gevrey classes \( G^{F,s}(\Omega) \) means the following inclusion:

\[
G^s(\Omega, (P_j)^N) \subset G^{F,s}(\Omega).
\]

Definition 1. Newton’s polyhedron of the system \((P_j)^N\) at the point \( x_0 \in \Omega \), denoted \( \mathcal{F}(x_0) \), is the convex hull of the set \( \{ \alpha \in \mathbb{N}^n, \exists j \in \{1, \ldots, N\}; a_{j\alpha}(x_0) \neq 0 \} \).

A Newton’s polyhedron \( \mathcal{F} \) is said to be regular if there exists a finite set \( Q(\mathcal{F}) \subset (\mathbb{R}^*_+)^n \) such that

\[
\mathcal{F} = \bigcap_{q \in Q(\mathcal{F})} \{ \alpha \in \mathbb{R}^n, \langle \alpha, q \rangle \leq 1 \}.
\]

Set

\[
\begin{align*}
k(\alpha, \mathcal{F}) &= \inf \{ t > 0, t^{-1} \alpha \in \mathcal{F} \}, \alpha \in \mathbb{R}^n, \\
\mu(\mathcal{F}) &= \max_{1 \leq j \leq n} \mu_j(\mathcal{F}), \\
\mu_j(\mathcal{F}) &= \max_{q \in Q(\mathcal{F})} q_j^{-1}, j = 1, \ldots, n, \\
\theta(\mathcal{F}) &= \left( \frac{\mu(\mathcal{F})}{\mu_1(\mathcal{F})}, \ldots, \frac{\mu(n)(\mathcal{F})}{\mu_n(\mathcal{F})} \right).
\end{align*}
\]

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Definition 2. Let $\mathcal{F}$ be a regular Newton’s polyhedron and $s \in \mathbb{R}_+$. We define the generalized Gevrey space $G^{\mathcal{F},s}(\Omega)$ by the space of $u \in C^\infty(\Omega)$ such that
\begin{equation}
\sup_H |D^\alpha u| \leq C^{\alpha+1} \left[ \Gamma(\mu(\mathcal{F}) k(\alpha, \mathcal{F}) + 1) \right] s,
\end{equation}
where $\Gamma$ is the gamma function.

Remark 1. One can take $\sup_H |D^\alpha u|$ or $\|D^\alpha u\|_{L^2(\Omega)}$ in the definition, according to Sobolev imbedding theorems.

Definition 3. The system $(P_j)_{j=1}^N$ is said to be multi-quasi-elliptic in $\Omega$ if
1) The $\mathcal{F}(x)$ do not depend on $x \in \Omega$, i.e. $\forall x, \mathcal{F}(x) = \mathcal{F}$.
2) $\mathcal{F}$ is regular.
3) $\forall x \in \Omega, \exists C > 0, \exists R \geq 0, \forall \xi \in \mathbb{R}^n, |\xi| \geq R, \sum_{j=1}^N |P_j(x, \xi)| \leq C \sum_{\alpha \in \mathbb{Z}_+^N \cap \mathcal{F}} |\xi\alpha|.$

Definition 4. Let $(P_j)_{j=1}^N$ be a system of linear differential operators satisfying conditions 1) and 2) of Definition 3 and $s \in \mathbb{R}_+$, the space of Gevrey vectors of the system $(P_j)_{j=1}^N$, denoted $G^s(\Omega, (P_j)_{j=1}^N)$, is the space of $u \in C^\infty(\Omega)$ such that
\begin{equation}
\|P_1 \ldots P_n u\|_{L^2(\Omega)} \leq C^l + 1 (l!)^{s\mu(\mathcal{F})}.
\end{equation}
The aim of this work is to show the following theorem.

Theorem 1. Let $\Omega$ be an open subset of $\mathbb{R}^n$, $\sigma > s \geq 1$ and $(P_j)_{j=1}^N$ be a system of linear differential operators with $G^{\mu(\mathcal{F}), \sigma}(\Omega)$ coefficients. Then
$(P_j)_{j=1}^N$ is multi-quasi-elliptic in $\Omega$ $\iff$ $G^s(\Omega, (P_j)_{j=1}^N) \subset G^{\mathcal{F}, s}(\Omega)$.

Some consequences of this theorem are given in section 4. For differential operators with constant coefficients we have shown in [9] a more general result.

2. Sufficient condition

The proof of the sufficient condition follows essentially the work of Zanghirati [9], so we refer for details to this paper.

Instead of $Q(\mathcal{F}), k(\mathcal{F}, \alpha), \mu(\mathcal{F}), \theta(\mathcal{F})$ we write, respectively, $Q, k(\alpha), \mu, \theta$. Denote $\mathcal{K} = \{ k = k(\alpha) : \alpha \in \mathbb{N}_+^n \}$. If $\omega$ is an open subset of $\mathbb{R}^n$, $u \in C^\infty(\omega)$ and $k \in \mathcal{K}$, define $|u|_{k, \omega} = \sum_{k(\alpha) = k} \|D^\alpha u\|_{L^2(\omega)}$. When $u \in C^\infty(\mathbb{R}^n)$ we write $|u|_k$.

Let $(P_j)_{j=1}^N$ be a system of linear differential operators with coefficients defined in an open neighborhood $\Omega$ of the origin satisfying the following conditions:
(i) The system $(P_j)_{j=1}^N$ is multi-quasi-elliptic in $\Omega$.
(ii) The coefficients $a_{j\alpha} \in G^{\mu, s}(\Omega), \forall \alpha \in \mathcal{F}, \forall j \in \{1, \ldots, N\}$.

For $\rho > 0$, we denote $B_\rho = \{ x \in \mathbb{R}^n, \sum_{j=1}^n \rho^{2j} < \rho^2 \}$. We define for $h \in \mathbb{N}$,
$P_j^h(x, D) = P_j(x, D) \circ \cdots \circ P_j(x, D), j = 1, \ldots, N.$
From the multi-quasi-ellipticity of the system \((P_j)^N_{j=1}\) and following the proof of Lemma 3.4 of [6], we obtain

**Lemma 1.** There exist \(p_0 > 0\) and \(C_1 > 0, \forall \epsilon \in [0, \frac{1}{10}]\) (\(v(n)\) denote the number of elements of \(\mathcal{K} \cap [0, n]\)), \(\exists C_2(\epsilon) > 0, \forall \delta \in [0, 1], \forall \rho > 0, B_{\rho+\delta} \subset B_{\rho_0}, \forall u \in C^\infty(B_{\rho_0}), \forall p \geq n,

\begin{equation}
|u|_{p+1,B_{\rho_0}} \leq C_1 \left( \sum_{j=1}^{N} |P_j^n(x, D)u|_{p-n+1,B_{\rho+\delta}} + \epsilon |u|_{p+1,B_{\rho+\delta}} + (\epsilon \delta)^{-n\mu} |u|_{p-n+1,B_{\rho+\delta}} + \sum_{h=0}^{p} \left( \frac{(p+1)!}{h!} \right)^{\nu} C_2(\epsilon)^{p+1-h} |u|_{h,B_{\rho+\delta}} \right),
\end{equation}

and for \(p \leq n\), we have

\begin{equation}
|u|_{p+1,B_{\rho_0}} \leq C_2 \left( \sum_{j=1}^{N} |P_j^n(x, D)u|_{p-n+1,B_{\rho+\delta}} + \epsilon |u|_{p+1,B_{\rho+\delta}} + (\epsilon \delta)^{-(p+1)n} |u|_{0,B_{\rho+\delta}} \right).
\end{equation}

Let \(\lambda > 0\) and \(R > 0\). For \(p \in \mathbb{N}\), we set

\[
\sigma_p(u, \lambda) = (pl)^{-\lambda^p} \sup_{R/2 \leq \rho < R} (R - \rho)^{pl} |u|_{p,B_\rho}.
\]

**Lemma 2.** Let \(\rho_0\) be as in the Lemma 1 and let \(0 < R < 1\) such that \(B_R \subset B_{\rho_0}\). Then there exists \(\lambda_0 > 0\) \((\lambda_0\) depends only on \(R\) and \((P_j)^N_{j=1}, \forall u \in C^\infty(B_{\rho_0}), \forall \lambda \geq \lambda_0, \forall p \geq n,

\begin{equation}
\sigma_{p+1}(u, \lambda) \leq [(p-n+2) \cdots (p+1)]^{-\lambda^p} \sum_{j=1}^{N} \sigma_{p-n+1}(P_j^n u, \lambda) + \sum_{h=0}^{p} \sigma_h(u, \lambda),
\end{equation}

and for \(p \leq n-1,

\begin{equation}
\sigma_{p+1}(u, \lambda) \leq (p+1)!^{-\lambda^p} \sum_{j=1}^{N} \sigma_0(P_j^{p+1} u, \lambda) + \sigma_0(u, \lambda).
\end{equation}

**Proof.** Let \(p \geq n\), multiply both sides of (2.1) by \((p+1)!^{-\lambda^p} \lambda^{-p-1} (R - \rho)^{pl}\), put \(\delta = \frac{R - \rho}{p-n+2}\) and then taking the sum over \(\rho \in [R/2, R]\), we obtain

\[
\sigma_{p+1}(u, \lambda) \leq C_1 \left( I_1 + \epsilon I_2 + \epsilon^{-n\mu} I_3 + I_4 \right),
\]

where \(I_1, I_2, I_3\) and \(I_4\) are such that

\[
I_1 \leq \sum_{j=1}^{N} \left( \frac{(p-n+1)!}{(p+1)!} \right)^{\nu} \frac{e^\mu}{\lambda^p} \sigma_{p-n+1}(P_j^n u, \lambda),
I_2 \leq (2^\mu e^\mu)^{\nu} \sigma_{p+1}(u, \lambda),
I_3 \leq \frac{\epsilon^\mu}{\lambda^p} \sigma_{p-n+1}(u, \lambda),
I_4 \leq \frac{\epsilon^\mu C_2(\epsilon)}{\lambda} \sum_{h=0}^{p} \left( \frac{C_2(\epsilon)}{\lambda} \right)^{p-h} \sigma_h(u, \lambda).
\]
By a suitable choice of $\varepsilon$, we find
\[
\sigma_{p+1}(u, \lambda) \leq \frac{(p-n+1)!}{(p+1)!} s^p \tilde{C}_1 \sum_{j=1}^N \sigma_{p-n+1}(P_j^n u, \lambda) + \frac{\tilde{C}_2}{\lambda^n} \sigma_{p-n+1}(u, \lambda)
+ \frac{\tilde{C}_3}{\lambda} \sum_{h=0}^p \left( \frac{\tilde{C}_4}{\lambda} \right)^{p-h} \sigma_h(u, \lambda).
\]
It suffices to take $\lambda_0 = \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3 + \tilde{C}_4$ to get (2.3). For the inequality (2.4) we multiply both sides of inequality (2.2) by $\frac{(R-p)(p+1)}{(p+1)!} \lambda^n$, take $\delta = \frac{R-p}{2}$ and then we follow the same procedure for obtaining (2.3).

\[\square\]

**Lemma 3.** Let $\rho_0$, $R$ and $\lambda_0$ be as in Lemma 2. Then for any $u \in C^\infty(B_{\rho_0})$, $\forall \lambda \geq \lambda_0$, $\forall p \in \mathbb{N}$, we have
\[
\sigma_{p+1}(u, \lambda) \leq 2^{p+1} \sigma_0(u, \lambda) + \sum_{l=1}^{p+1} 2^{p+1-l} C_l \left( \frac{1}{l!} \right)^{s^p} \sum_{1 \leq i_1, \ldots, i_l \leq N} \sigma_0(P_{i_1} \ldots P_{i_l} u, \lambda).
\]

**Proof.** It is obtained by recurrence over $p$. $\square$

Our first result is the following theorem, which generalizes the results of [6], [7] and [8] to systems.

**Theorem 2.** Let $\Omega$ be an open subset of $\mathbb{R}^n$, $s \geq 1$ and $(P_j(x, D))_{j=1}^N$ be a system of linear differential operators with $C^{\theta,s}(\Omega)$ coefficients. Then
\[
(P_j)_{j=1}^N \text{ is multi-quasi-elliptic in } \Omega \Rightarrow G^s(\Omega, (P_j)_{j=1}^N) \subset G^{F,s}(\Omega).
\]

**Proof.** It is sufficient to check (1.1) in a neighborhood of every point $x$ of $\Omega$. Let us assume $x$ is the origin. Then there exist $\rho_0$, $\lambda_0$ and $R$ such that the precedent lemmas hold. Let $u \in G^s(\Omega, (P_j)_{j=1}^N)$. Then there is $C_1 > 0$ such that
\[
\sigma_0(P_{i_1} \ldots P_{i_l} u, \lambda_0) \leq C_1^{l+1} (l!)^{s^p}, \forall l \in \mathbb{N},
\]
hence from (2.5), we obtain
\[
\sigma_{p+1}(u, \lambda_0) \leq C_1 (2 + N C_1)^{p+1}, \forall p \in \mathbb{N},
\]
which gives
\[
|u|_{p+1, B_{R/2}} \leq (p+1)! s^p C_2^{(p+1)\mu+1}, \forall p \in \mathbb{N}.
\]
Following the same steps as in [6] we obtain
\[
|u|_{k, B_{R/2}} \leq C_3^{k\mu+1} (k+1)^{s^p}.
\]
Consequently from (2.7) it is easy, as in [6], to obtain the estimate (1.1). $\square$

### 3. NECESSARY CONDITION

In this section we prove the converse of Theorem 2. For this aim we need a characterization of the multi-quasi-ellipticity of the system $(P_j(x, D))_{j=1}^N$, known in the case of a scalar operator; see [4].
Proposition 1. A system \( (P_j)_{j=1}^N \), satisfying 1) and 2) of Definition 2, is multi-quasi-elliptic in \( \Omega \) if and only if for any \( x \in \Omega, \forall q \in Q \),
\[
\sum_{j=1}^N |P_{jq}(x, \xi)| \neq 0, \quad \forall \xi \in \mathbb{R}^n, \xi_1 \ldots \xi_n \neq 0,
\]
where \( P_{jq} \) is the \( q \)-quasi-homogenous part of \( P_j \), i.e.
\[
P_{jq}(x, \xi) = \sum_{(\alpha, q)=1} a_{jq}(x) \xi^\alpha.
\]

Theorem 3. Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) and \( P_j(x, D) \), \( j = 1, \ldots, N \), be differential operators with \( G^{\alpha, \sigma}(\Omega) \) coefficients. If \( s > \sigma \geq 1 \), then
\[
G^s(\Omega, (P_j)_{j=1}^N) \subset G^{F,s}(\Omega) \Rightarrow (P_j)_{j=1}^N \text{ is multi-quasi-elliptic in } \Omega.
\]

Proof. Assume that the system \( (P_j)_{j=1}^N \) is not multi-quasi-elliptic. Then there exist \( x_0 \in \Omega, q \in Q \) and \( \xi_0 \in S^{n-1}, \xi_0, 1 \ldots \xi_n \neq 0 \), such that
\[
P_{jq}(x_0, \xi_0) = 0, \quad \forall j = 1, \ldots, N.
\]
We construct a function \( u \in G^s(\Omega, (P_j)_{j=1}^N) \) such that \( u \notin G^{F,s}(\Omega) \), which contradicts the hypothesis. Put \( \eta = \frac{1 - s/\mu}{s} \), and choose \( \varepsilon \) satisfying
\[
0 < \varepsilon \leq \frac{\mu(s-\sigma)}{2\mu s - \sigma} < \frac{1}{2} \quad \text{and} \quad \varepsilon < \min_{(\beta, q)<1} \mu(1 - (\beta, q)).
\]
Let \( \delta > 0 \) such that the ball \( B_0 = B(x_0, 2\delta) \) is relatively compact in \( \Omega \) and \( \varphi \in G^{q, \sigma, \mu}(\mathbb{R}^n) \) with compact support in \( B(0, 2\delta) \) and \( \varphi(x) \equiv 1 \) in \( B(0, \delta) \). The desired function is defined by
\[
u(x) = \int_1^{+\infty} \varphi[r^q(x - x_0)] e^{-r^n} e^{i(x - x_0, r^n \xi_0)} dr,
\]
where \( r^n x = (r^q x_1, r^q x_2, \ldots, r^q x_n) \).

Following [5] and [8] it is easy to show that \( u \notin G^{F,s}(U) \) for any neighborhood \( U \) of \( x_0 \).

Let us verify that \( u \in G^s(\Omega, (P_j)_{j=1}^N) \). Since the coefficients of the operators \( P_j \) are in \( G^{\alpha, \sigma}(\Omega) \subset G^{q, \sigma, \mu}(\Omega) \), then \( \exists M > 0, \forall \alpha \in \mathbb{Z}_+^n, \forall \beta \in \mathbb{Z}_+^n, \forall x \in B_0, \forall r \geq 1, \forall j = 1, \ldots, N \), such that
\[
|\left(D_{x}^{\beta} P_{j}^{(\alpha)}(x, r^n \xi_0)\right)| \leq M|\beta|+1 |\Gamma((\beta, q)+1)|^{\sigma \mu} r^{1-(\alpha, q)}.
\]
On the other hand in view of (3.1) it is easy to obtain \( \forall \delta > 0, \exists C_1 > 0, \forall r \geq 1, \forall x \in \Omega, |x - x_0| < 2\delta r^{-\varepsilon/\mu}, \forall j = 1, \ldots, N \),
\[
|P_{j}(x, r^n \xi_0)| \leq C_1 r^{1-\varepsilon/\mu}.
\]
Now we need a convenient form of \( P_{i_k} \ldots P_{i_0} u \), for any integer \( k \geq 1 \). The generalized Leibniz formula \( P_{j}(x, D)(uv) = \sum_{\alpha} \frac{1}{\alpha!} P_{j}^{(\alpha)} u D^{\alpha} v \) gives
\[
P_{i_k} \ldots P_{i_0} u(x) = \int_1^{+\infty} A_{i_k \ldots i_0}(x, r) e^{-r^n} e^{i(x - x_0, r^n \xi_0)} dr,
\]
where $1 \leq i_1 \leq N$, for any integer $l \leq k$, $P_{i_0}$ designs the identity operator, and

$$(3.4) \quad \begin{cases} A_{i_0}(x, r) = \varphi [r^{\varepsilon_0} (x - x_0)], \\ A_{i_k+1, i_k \ldots i_0}(x, r) = \sum_{(\alpha, q) \leq 1} \frac{1}{\alpha!} D_{i_k+1}^{(\alpha)} (x, r^q \xi_0) D_x A_{i_k \ldots i_0}(x, r). \end{cases}$$

To complete the proof we need the following

**Lemma 4.** \( \exists L > 0, \exists L_0 > 0, \exists C_0 > 0, \forall k \in \mathbb{Z}_+, \forall \gamma \in \mathbb{Z}_+^n, \forall x \in B_0, \forall r \geq 1, \)

$$|D_x^2 A_{i_k \ldots i_0}(x, r)| \leq C_0 (L_0 r^\varepsilon)^{(\gamma, q)} L^k (r^{(1-\varepsilon/\mu)k} [\Gamma((\gamma, q) + 1)]^{\sigma \mu} \right)$$

$$(3.5) \quad + [\Gamma((\gamma, q) + k + 1)]^{\sigma \mu} r^{k(2-1/\mu)}.$$ 

**Proof.** It is obtained by recurrence over $k$. In fact for $k = 0$, the estimate (3.5) means $\varphi \in G_0^{0, \sigma \mu} (\mathbb{R}^n)$. So suppose that the estimate (3.5) holds up to the order $k$ and let us check it at the order $k + 1$. Set $\lambda = r^{1-\varepsilon/\mu}$ and $\tau = r^{(2-1/\mu)}$. Then the estimate (3.5) is written as

$$|D_x^2 A_{i_k \ldots i_0}(x, r)| \leq C_0 (L_0 r^\varepsilon)^{(\gamma, q)} L^k S(k, \gamma),$$

where

$$S(k, \beta) = \lambda^k [\Gamma(\beta, q) + 1]^{\sigma \mu} + [\Gamma(\beta, q) + k + 1)]^{\sigma \mu} r^k.$$ 

Let $\omega = \min_{1 \leq j \leq n} q_j$. Then we have

$$\lambda^{1-(\alpha, q)} \tau^{(\alpha, q)} S(k, \beta + \alpha) \leq 2^{\omega+1} S(k + 1, \beta), \quad (\alpha, q) \leq 1.$$ 

From (3.4), we have

$$|D_x^2 A_{i_k+1 \ldots i_0}(x, r)| \leq I_1 + I_2 + I_3,$$

where

$$I_1 = |P_{i_k+1}(x, r^q \xi_0)| |D_x^2 A_{i_k \ldots i_0}(x, r)|,$$

$$I_2 = \sum_{\beta \leq \gamma} \left( \begin{array}{c} \gamma \\ \beta \end{array} \right) |D_x^{\gamma-\beta} P_{i_k+1}(x, r^q \xi_0)| |D_x^\beta A_{i_k \ldots i_0}(x, r)|,$$

$$I_3 = \sum_{0 < (\alpha, q) \leq 1} \sum_{\beta \leq \gamma} \frac{1}{\alpha!} \left( \begin{array}{c} \gamma \\ \beta \end{array} \right) |D_x^{\gamma-\beta} P_{i_k+1}^{(\alpha)}(x, r^q \xi_0)| |D_x^{\alpha+\beta} A_{i_k \ldots i_0}(x, r)|.$$ 

Since $A_{i_k \ldots i_0}$ are functions of compact supports in $B(x_0, 2\delta r^{-\varepsilon/\mu})$, and according to (3.3) and (3.6), we have

$$(3.7) \quad I_1 \leq 2^{\omega+1} C_1 C_0 (L_0 r^\varepsilon)^{(\gamma, q)} S(k + 1, \gamma) L^k.$$ 

The estimates (3.2) and (3.6) give

$$I_2 \leq \sum_{\beta \leq \gamma} \left( \begin{array}{c} \gamma \\ \beta \end{array} \right) [\Gamma((\gamma - \beta, q) + 1)]^{\sigma \mu} M^{(\gamma - \beta, q) + 1} r^\varepsilon C_0 (L_0 r^\varepsilon)^{(\beta, q)}$$

$$\cdot 2^{\omega+1} S(k + 1, \beta) L^k.$$ 

On the other hand, using properties of the gamma function, we have

$$(3.8) \quad \left( \begin{array}{c} \gamma \\ \beta \end{array} \right) [\Gamma((\gamma - \beta, q) + 1)]^{\sigma \mu} S(k, \beta) \leq C_2^{\sigma \mu (\gamma - \beta, q)} S(k, \gamma).$$
Thus we obtain
\[
I_2 \leq \frac{nMC_2^{\sigma \mu}}{L_0 r^\varepsilon} \sum_{\beta \geq 0} \left( \frac{MC_2^{\sigma \mu}}{L_0 r^\varepsilon} \right)^{\beta q} r^\varepsilon 2^{\delta \varepsilon + 1} MC_0 (L_0 r^\varepsilon)^{(\gamma, q)} S (k + 1, \gamma) L^k.
\]
Set \( C_3 = \sum_{\alpha \geq 0} \left( \frac{1}{2} \right)^{(\alpha, q)} \), take \( L_0 \geq 2MC_2^{\sigma \mu} \) and \( r \geq 1 \), and then
\[
(3.9) \quad I_2 \leq \frac{nMC_2^{\sigma \mu}}{L_0} C_3 2^{\delta \varepsilon + 1} MC_0 (L_0 r^\varepsilon)^{(\gamma, q)} S (k + 1, \gamma) L^k.
\]
Finally in view of (3.2)
\[
I_3 \leq \sum_{0 < (\alpha, q) \leq 1} \sum_{\beta \leq \gamma} \left( \frac{\beta}{\gamma} \right) [\Gamma(\gamma - \beta, q) + 1]^{\sigma \mu} M^{\gamma - \beta + 1} r^{1 - (\alpha, q)} \times C_0 (L_0 r^\varepsilon)^{|\beta + \alpha|} S (k, \beta + \alpha) L^k.
\]
For any \( \alpha \in \mathbb{Z}^+ \), \( 0 < (\alpha, q) \leq 1 \), we have \( r^{1 - (\alpha, q) + \varepsilon(\alpha, q)} \leq \lambda^{1 - (\alpha, q) + \varepsilon(\alpha, q)} \), which gives, with (3.6) and (3.8),
\[
I_3 \leq \sum_{0 < (\alpha, q) \leq 1} \sum_{0 < (\alpha, q) \leq 1} \left( \frac{MC_2^{\sigma \mu}}{L_0 r^\varepsilon} \right)^{|\gamma|} 2^{\delta \varepsilon + 1} MC_0 L_0^{[\alpha]} (L_0 r^\varepsilon)^{|\gamma|} S (k + 1, \gamma) L^k.
\]
Put \( C_4 = \sum_{0 < (\alpha, q) \leq 1} L_0^{(\alpha, q)} \). Then we obtain
\[
(3.10) \quad I_3 \leq 2^{\delta \varepsilon + 1} MC_4 C_3 C_0 (L_0 r^\varepsilon)^{(\gamma, q)} S (k + 1, \gamma) L^k.
\]
If we choose
\[
L \geq 2^{\delta \varepsilon + 1} \left( C_1 + \frac{nM^2 C_2^{\sigma \mu}}{L_0} - C_3 + MC_3 C_4 \right),
\]
we get, from (3.7), (3.9) and (3.10),
\[
I_1 + I_2 + I_3 \leq C_0 (L_0 r^\varepsilon)^{(\gamma, q)} S (k + 1, \gamma) L^{k+1},
\]
which means that (3.5) holds at the order \( k + 1 \).

**End of the Proof of Theorem 3.** Applying the last lemma for \( \gamma = 0 \), we find
\[
(3.11) \quad |A_{i_k, \ldots, i_0} (x, r)| \leq C_0 L^k \left( r^{(1 - \varepsilon/\mu)k} + (k!)^{\sigma \mu} \epsilon(2 - 1/\mu) \right).
\]
Thus we obtain
\[
|A_{i_k, \ldots, i_0} (x, r)| \leq C_0 L^k (2s \mu)^{k \mu s} (k!)^{\mu s} \left[ \exp \left( \frac{\eta^\mu}{2} \right) + \exp \left( \frac{\eta^\prime}{2} \right) \right],
\]
where \( \eta^\prime = \frac{\varepsilon(2 - 1/\mu)}{\mu(s - \sigma)} \leq \eta = \frac{1 - \varepsilon/\mu}{\mu s} \), since \( \varepsilon \leq \mu(s - \sigma) / 2\mu s - \sigma \).
Therefore
\[
|P_{i_k, \ldots, i_0} u (x)| \leq 2C_0 L^k (k!)^{\mu s} \int_1^{+\infty} \exp \left( - \frac{\eta^\prime}{2} r \right) dr \\
\leq C^{k+1} (k!)^{\mu s},
\]
which means that \( u \in G^s \left( \Omega, (P_j)_j^{N} \right) \).
4. Consequences

A first consequence of Theorem 2 is a result on Gevrey-hypoellipticity for multi-quasi-elliptic systems.

**Corollary 1.** Under the assumptions of Theorem 2, the following propositions are equivalent:

(i) \( u \in D'(\Omega) \), \( P_j u \in G^{p,s}(\Omega) \), \( \forall j = 1, \ldots, N \).

(ii) \( u \in G^{p,s}(\Omega) \).

The theorems of this work unify the results of Bolley-Camus [1] and Métivier [5] in the homogeneous case, the results of Zanghirati [7] and [8] in the scalar quasi-homogenous case and generalize them to quasi-homogenous systems.

**Corollary 2.** Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) and \( \sigma > s \geq 1 \), and let \( (P_j)_{j=1}^N \) be a system of linear differential operators with coefficients in \( G^{q,\sigma}(\Omega) \). Then \( (P_j)_{j=1}^N \) is \( q \)-quasi-elliptic in \( \Omega \) \( \iff \) \( G^s \left( \Omega, (P_j)_{j=1}^N \right) \subset G^{q,s}(\Omega) \).

**References**


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