

ON BIFURCATION POINTS OF A COMPLEX POLYNOMIAL

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ABSTRACT. Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial of degree d . Assume that the set $\tilde{K}_\infty(f) = \{y \in \mathbb{C} : \text{there is a sequence } x_l \rightarrow \infty \text{ s.t. } f(x_l) \rightarrow y \text{ and } \|df(x_l)\| \rightarrow 0\}$ is finite. We prove that the set $\tilde{K}(f) = K_0(f) \cup \tilde{K}_\infty(f)$ of generalized critical values of f (hence in particular the set of bifurcation points of f) has at most $(d-1)^n$ points. Moreover, $\#\tilde{K}_\infty(f) \leq (d-1)^{n-1}$. We also compute the set $\tilde{K}(f)$ effectively.

1. INTRODUCTION

Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial mapping. It is well-known that f is a fibration outside a finite set. The smallest such set is called *the bifurcation set of f* ; we denote it by $B(f)$. It can be proved that the set $K_0(f)$, the set of critical values of f , is contained in $B(f)$. But in general the set $B(f)$ is bigger than $K_0(f)$. It also contains the set $B_\infty(f)$ of bifurcation points at infinity. Briefly speaking the set $B_\infty(f)$ consists of points at which f is not a locally trivial fibration at infinity (i.e., outside a compact set). In the paper [8] we have estimated the number of points in sets $B(f)$ and $B_\infty(f)$. The aim of this paper is to obtain a better estimation, but only for a special class of polynomials (this class coincides with the class of all polynomials for $n = 1, 2$ only). Let

$$\tilde{K}_\infty(f) = \{y \in \mathbb{C} : \text{there is a sequence } x_l \rightarrow \infty \text{ s.t. } f(x_l) \rightarrow y \text{ and } \|df(x_l)\| \rightarrow 0\}.$$

If $c \notin \tilde{K}_\infty(f)$, then we say that f satisfies *Fedoryuk's condition* at c . This set has been studied in [2] and [10]. It is well-known ([10]) that $B_\infty(f) \subset \tilde{K}_\infty(f)$. In particular $B(f) \subset \tilde{K}(f) = K_0(f) \cup \tilde{K}_\infty(f)$. Moreover, if $n = 2$ we have $B_\infty(f) = \tilde{K}_\infty(f)$ and $B(f) = \tilde{K}(f)$ (see [4], [5], [9]). In this paper we give a sharp estimation of the numbers $\#\tilde{K}_\infty(f)$ and $\#\tilde{K}(f)$ (and hence also the numbers $\#B_\infty(f)$ and $\#B(f)$), provided $\#\tilde{K}_\infty(f) < \infty$. We also give an effective method to compute the set $\tilde{K}(f)$. Our main result is:

Theorem 1.1. *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial of degree $d > 0$. Assume that the set $\tilde{K}_\infty(f)$ is finite. Let $a = \#\tilde{K}_\infty(f)$ and $b = \#\tilde{K}(f)$. Then:*

- 1) $(d-1)a + b \leq d(d-1)^{n-1}$,
- 2) $a \leq (d-1)^{n-1}$ and $b \leq (d-1)^n$,

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- 3) if $\tilde{K}_\infty(f) \neq \emptyset$, then $b \leq \max\{1, (d-1)^n - d + 1\}$,
- 4) if e denotes the number of isolated critical points of f , then $a + e \leq (d-1)^n$ and $da + e \leq d(d-1)^{n-1}$,
- 5) moreover, if $a > 0$, then $a + e \leq \max\{1, (d-1)^n - d + 1\}$.

Corollary 1.1. *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial of degree $d > 0$. If $\#\tilde{K}_\infty(f) = (d-1)^{n-1}$, then $\tilde{K}(f) = \tilde{K}_\infty(f)$ and f has no isolated critical points.*

Proof. Indeed, we have $(d-1)a + b \leq d(d-1)^{n-1}$ and $a \leq b$, hence $a = b$. Moreover, since $da + e \leq d(d-1)^{n-1}$, we obtain $e = 0$. □

Remark 1.1. Let us note that for $n = 2$ the set $\tilde{K}_\infty(f)$ is always finite and $B_\infty(f) = \tilde{K}_\infty(f)$ (see [4], [5], [9]). In particular, for $n = 2$ we recover a well-known fact ([3], [9]) that $\#B_\infty(f) \leq d-1$. Moreover, we get a sharp estimation of numbers $\#B(f)$ and $\#B_\infty(f)$ in the class of all polynomials $f \in \mathbb{C}[x, y]$ of degree d .

2. PRELIMINARIES

Let us recall that a mapping $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is *not proper* at a point $y \in \mathbb{C}^m$ if there is no neighborhood U of y such that $f^{-1}(\bar{U})$ is compact. In other words, f is not proper at y if there is a sequence $x_l \rightarrow \infty$ such that $f(x_l) \rightarrow y$. Let S_f denote the set of points at which the mapping f is not proper. We have the following characterization of the set S_f (see [6], [7]):

Theorem 2.1. *Let $F = (F_1, \dots, F_m) : \mathbb{C}^n \rightarrow \mathbb{C}^m$ be a generically-finite polynomial mapping. Then the set S_F is an algebraic subset of \mathbb{C}^m and it is either empty or it has pure dimension $n - 1$. Moreover, if $n = m$ we have*

$$\deg S_F \leq \frac{(\prod_{i=1}^n \deg F_i) - \mu(F)}{\min_{1 \leq i \leq n} \deg F_i},$$

where $\mu(F)$ denotes the geometric degree of F (i.e., it is a number of points in a generic fiber of F).

In the proof of Theorem 1.1 we need the following technical lemmas. The first lemma follows from the Bezout theorem in the version of Vogel.

Lemma 2.1. *Let A be an irreducible algebraic subvariety of \mathbb{C}^N and let H be a linear subspace of \mathbb{C}^N . Assume that the set $H \cap A = \{x_1, \dots, x_r\}$ is finite. Then $\deg A \geq r$. More precisely, if germ \mathbf{A}_{x_i} have m_i irreducible components, for $i = 1, \dots, r$, then $\deg A \geq \sum_{i=1}^r m_i$.*

The next lemma is:

Lemma 2.2. *Let $B \subset A$ be algebraic subsets of \mathbb{C}^{N+1} , $\dim B < \dim A = n$. Let L be a line and M a linear subspace of \mathbb{C}^{N+1} , which contains L , $\dim M = n$. Assume that $L \not\subset B$. Then there exists a linear projection $p : \mathbb{C}^{N+1} \rightarrow M$ such that p restricted to A is finite and $L \not\subset p(B)$. In particular p is proper on A .*

Proof. Take a point $a \in L \setminus B$. Let Λ be the Zariski closure of the cone $\bigcup \overline{ax}$, $x \in B$. It is easy to see that $\dim \Lambda \leq n$. Let H_∞ be the hyperplane at infinity of $\mathbb{C} \times \mathbb{C}^N$. For any $Z \subset \mathbb{C}^N$ denote by \tilde{Z} the projective closure of Z . Observe that

$$\dim H_\infty \cap (\tilde{\Lambda} \cup \tilde{\Gamma} \cap \tilde{M}) \leq n - 1.$$

Thus, there is a projective subspace $Q \subset H_\infty$ of dimension $N - n$, which is disjoint with $(\tilde{A} \cup \tilde{A} \cap \tilde{M})$. Denote by $p_Q : \mathbb{P}^{N+1} \setminus Q \rightarrow \tilde{M}$ the linear projection determined by the subspace Q .

Now, let $p : \mathbb{C}^{N+1} \rightarrow M$ be the restriction of p_Q to \mathbb{C}^{N+1} . It is easily seen that p has desired properties, i.e., $p : A \rightarrow M$ is a finite mapping and $a \notin L \cap p(B)$. \square

Lemma 2.3. *Let $\Phi = (\phi_1, \dots, \phi_n) : \mathbb{C}^n \rightarrow \mathbb{C}^m$ be a polynomial mapping with $\deg \phi_i = d$, for $i = 1, \dots, m$. Let $r = \dim \Phi(\mathbb{C}^n)$. Assume that there is a variety W , which contains \mathbb{C}^n as a dense subset and a polynomial proper mapping $\bar{\Phi} : W \rightarrow \mathbb{C}^n$, such that $\Phi = \text{res}_{\mathbb{C}^n} \bar{\Phi}$. Let q be a maximal number of connected components of fibers of $\bar{\Phi}$. Then $q \leq d^r$. Moreover, if $r = n$ and the mapping Φ is not proper, then $q \leq d^n - d$.*

Proof. First, taking the normalization we can assume that the variety W is normal. Let $\Gamma = \text{cl}(\Phi(\mathbb{C}^n))$ and let $p : \Gamma \rightarrow \mathbb{C}^r$ be a finite linear projection. Take $\Phi' = p \circ \Phi$. If q' denotes a maximal number of connected components of fibers of $\bar{\Phi}'$, then it is easy to see that $q' \geq q$. Moreover, if a projection p is sufficiently general, then we have $\Phi' = (\phi'_1, \dots, \phi'_r)$, where $\deg \phi'_i = d$, for $i = 1, \dots, r$.

Consequently we can assume that $\Phi' = \Phi$, i.e., that the mapping Φ is a dominant mapping. By Bezout's Theorem we have that a generic fiber of Φ has at most d^r irreducible components. It implies that a generic fiber of the mapping $\bar{\Phi}$ also has at most d^r irreducible components. By the Stein Factorization Theorem there exist a normal variety S , and regular surjective mappings $p : W \rightarrow S$, $q : S \rightarrow \mathbb{C}^r$, such that $\bar{\Phi} = q \circ p$, where p has only connected fibers and q is finite. Moreover, it is easy to see that the geometric degree $\mu(q)$ of the mapping q is estimated by d^r . Since varieties S, \mathbb{C}^r are normal and the mapping q is finite, we have that every fiber of the mapping q has at most d^r points. Consequently, we obtain that every fiber of $\bar{\Phi}$ has at most d^r connected components.

Now assume that $r = n$ and the mapping Φ is not proper. In particular $S_\Phi \neq \emptyset$. By Theorem 2.1 we get that the geometric degree $\mu(\Phi)$ of the mapping Φ is estimated by $d^n - d(\deg S_\Phi) \leq d^n - d$. In particular a generic (and consequently every) fiber of $\bar{\Phi}$ has at most $d^n - d$ connected components. \square

3. ESTIMATIONS

Now we can pass to the proof of Theorem 1.1. In fact we prove slightly more general results. Let $a = \#\tilde{K}_\infty(f)$ and $b = \#\tilde{K}(f)$. We begin with:

Theorem 3.1. *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial of degree $d > 0$. Assume that the set $\tilde{K}_\infty(f)$ is finite. Let $\Phi = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ and let $r = \dim \Phi(\mathbb{C}^n)$. Then $b \leq (d - 1)^r$. Moreover, if $\tilde{K}_\infty(f) \neq \emptyset$, then we have better estimation $b \leq \max\{1, (d - 1)^n - d + 1\}$. Finally, if e denotes the number of isolated critical points of f , then $a + e \leq (d - 1)^n$. If $\tilde{K}_\infty(f) \neq \emptyset$, then we have better estimation $a + e \leq \max\{1, (d - 1)^n - d + 1\}$.*

Proof. For $n = 1$ the theorem is obviously true. Let $n > 1$. Consider the polynomial mapping $\Phi = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) : \mathbb{C}^n \rightarrow \mathbb{C}^n$. Let $r = \dim \Phi(\mathbb{C}^n)$. It is well-known (see e.g., [7]) that there is a normal variety W , which contains \mathbb{C}^n as a dense subset and a polynomial proper mapping $\bar{\Phi} : W \rightarrow \mathbb{C}^n$, such that $\Phi = \text{res}_{\mathbb{C}^n} \bar{\Phi}$. By a proper modification of W we can assume that the mapping f has a regular extension $\bar{f} : W \rightarrow \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. Let $A = \bar{\Phi}^{-1}(0)$. It is easy to see that

$\tilde{K}(f) = \bar{f}(A) \setminus \{\infty\}$. It means that $\#\tilde{K}(f)$ is estimated by the number of connected components of the set A . Consequently, by Lemma 2.3 we have that $b \leq (d - 1)^r$.

Moreover, if $\tilde{K}_\infty(f) \neq \emptyset$ and $r = n$, then we have better estimation $b \leq (d - 1)^n - d + 1$. If $r < n$, then $b \leq (d - 1)^r \leq \max\{1, (d - 1)^n - d + 1\}$. Finally, if e denotes the number of isolated critical points of f , then $a + e \leq (d - 1)^n$ and again, if $\tilde{K}_\infty(f) \neq \emptyset$, then $a + e \leq \max\{1, (d - 1)^n - d + 1\}$. \square

Corollary 3.1. *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial of degree $d > 0$. Assume that the set $\tilde{K}_\infty(f)$ is finite. Then*

$$b \leq (d - 1)^n.$$

Theorem 3.2. *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial of degree $d > 0$. Assume that the set $\tilde{K}_\infty(f)$ is finite. Then*

$$(d - 1)a + b \leq d(d - 1)^{n-1}.$$

Proof. For $n = 1$ the theorem is obviously true. Let $n > 1$. Let us define a polynomial mapping $\Psi : \mathbb{C}^n \rightarrow \mathbb{C} \times \mathbb{C}^n$ by

$$\Psi = \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

Denote $\Gamma = \Psi(\mathbb{C}^n)$, and by $\bar{\Gamma}$ its Zariski closure. Let $r = \dim \bar{\Gamma}$. Consider the line $L := \mathbb{C} \times \{(0, \dots, 0)\} \subset \mathbb{C} \times \mathbb{C}^n$. We further identify this line with a copy of \mathbb{C} . By definition of $\tilde{K}(f)$ we have

$$\tilde{K}(f) = L \cap \bar{\Gamma}.$$

We further identify this line with a copy of \mathbb{C} . We have two possibilities:

- 1) $r = n$, i.e., Ψ is a generically finite mapping,
- 2) $r < n$, i.e., Ψ is not a generically-finite mapping.

Let us consider case 1). By the definition of $\tilde{K}_\infty(f)$ and Ψ we have

$$\tilde{K}_\infty(f) = L \cap S_\Psi,$$

where S_Ψ denotes the set of points at which the mapping Ψ is not proper. Recall that by the assumption the set $\tilde{K}_\infty(f)$ is finite, hence also $\#L \cap S_\Psi < \infty$. Choose a linear space M of dimension n , which contains the line L . Lemma 2.2 applied to $A = \bar{\Gamma}$ and $B = S_\Psi$ yields a projection $p : \mathbb{C}^{n+1} \rightarrow M$ which is finite on $\bar{\Gamma}$ and such that $L \not\subset p(S_\Psi)$. Denote $X = p(S_\Psi)$. Then $\tilde{K}_\infty(f) \subset X$ and $L \not\subset X$. Since p is proper on $\bar{\Gamma}$, we obtain that $X = S_F$, where $F = p \circ \Psi$.

Moreover, we have $F_i = a_{i0}f + \sum_{k=1}^n a_{ik} \frac{\partial f}{\partial x_k}$. If we take a projection p to be sufficiently general, then by a linear change of coordinates

$$T(x_1, \dots, x_n) = (x_1, x_2 - (a_{20}/a_{10})x_1, \dots, x_n - (a_{n0}/a_{10})x_1),$$

we get that $T \circ F = (F_1, \dots, F_n)$, where $\deg F_1 = d$, $\deg F_i = d - 1$ for $i > 1$. Hence we can assume that $F = (F_1, \dots, F_n)$, where $\deg F_1 = d$, $\deg F_i = d - 1$ for $i > 1$.

Let us estimate the geometric degree $\mu(F)$ of F . We have $\mu(F) = \mu(p \circ \Psi) \geq \mu(\text{res}_{\bar{\Gamma}} p) = \deg \bar{\Gamma}$. Let us estimate the degree of $\bar{\Gamma}$. Consider a linear subspace $H = L = \mathbb{C} \times \{0, \dots, 0\}$ and take $A = \bar{\Gamma}$. It is easy to see that $H \cap A = \tilde{K}(f)$. By Lemma 2.1 we have $\deg A \geq b$, consequently $\mu(F) \geq b$. Now Theorem 2.1 yields that the degree of the variety $X \subset M$ is bounded by $(d(d - 1)^{n-1} - b)/(d - 1)$. So, the set $X \cap L$ has no more than $(d(d - 1)^n - b)/(d - 1)$ points. Finally we obtain that $a \leq (d(d - 1)^n - b)/(d - 1)$ and that $(d - 1)a + b \leq d(d - 1)^n$.

Now let us consider case 2). It is easy to see that $a = b$. Choose a linear space $M \cong \mathbb{C}^{r+1}$, which contains the line L . Lemma 2.2 applied to $A = \bar{\Gamma}$ and $B = S_\Psi$ yields a projection $p : \mathbb{C}^{n+1} \rightarrow M$ which is finite on $\bar{\Gamma}$ and such that $L \not\subset p(\bar{\Gamma})$. Denote $X = p(\bar{\Gamma})$. Then $\tilde{K}(f) \subset X$ and $L \not\subset X$. Let $F = (F_0, F_1, \dots, F_r) = p \circ \Psi$. We have $F_i = a_{i0}f + \sum_{k=1}^n a_{ik} \frac{\partial f}{\partial x_k}$, where $i = 0, 1, \dots, r$. Moreover, we can assume that $F_0 = f$. By a linear change of coordinates $T(x_0, \dots, x_n) = (x_0, x_1 - a_{10}x_1, \dots, x_n - a_{n0}x_1)$ we get that

$$T \circ F = \left(f, \sum_{k=1}^n a_{k1} \frac{\partial f}{\partial x_k}, \dots, \sum_{k=1}^n a_{kr} \frac{\partial f}{\partial x_k} \right).$$

In particular we can assume that $F = (f, F_1, \dots, F_r)$. Take a mapping $\Lambda : \mathbb{C}^r \ni (t_1, \dots, t_r) \rightarrow (\sum_{k=1}^r a_{1k}t_k, \dots, \sum_{k=1}^r a_{rk}t_k)$. Taking a projection p (and hence values a_{ij}) sufficiently general, we can assume that the linear subspace $\Lambda(\mathbb{C}^r)$ meets the fiber $F^{-1}(0)$ in the finite and non-empty set. This means that a mapping $G := F \circ \Lambda : \mathbb{C}^r \rightarrow X$ is generically-finite, in particular it must be dominant. By the construction we have $G = (g, \frac{\partial g}{\partial t_1}, \dots, \frac{\partial g}{\partial t_r})$, where $g = f \circ \Lambda$. Moreover, $\tilde{K}(f) \subset X \cap L = \overline{G(\mathbb{C}^r)} \cap L = \tilde{K}(g)$ and we can use Theorem 3.1. Consequently $b \leq (d-1)^r \leq (d-1)^{n-1}$. Since $a = b$, we have $(d-1)a + b = db \leq d(d-1)^{n-1}$. This finishes the proof of Theorem 3.2. \square

We can summarize our results as:

Corollary 3.2. *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial of degree $d > 0$. Assume that the set $\tilde{K}_\infty(f)$ is finite. Let $\Phi = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ and $r = \dim \Phi(\mathbb{C}^n)$. If $r = n$, then $a \leq (d-1)^{n-1}$ and $b \leq (d-1)^n$. If $r < n$, then $a = b \leq (d-1)^r$.*

Proof. Indeed, if $r < n$, then it is easy to see that $a = b$ (there is no isolated critical points) and the corollary follows from Theorem 3.1. Let $r = n$. We have $(d-1)a + b \leq d(d-1)^{n-1}$ and $a \leq b$. Consequently $da \leq d(d-1)^{n-1}$ and finally $a \leq (d-1)^{n-1}$. Moreover, $b \leq (d-1)^n$ by Theorem 3.1. \square

Our last result is the following:

Theorem 3.3. *Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial of degree $d > 0$. Assume that the set $\tilde{K}_\infty(f)$ is finite. If f has e isolated critical points, then*

$$da + e \leq d(d-1)^{n-1}.$$

Proof. The proof goes along similar lines as the first part of the proof of Theorem 3.1. If $e = 0$, the result follows from Corollary 3.2. Hence, we can assume that $e > 0$, in particular we can assume that the mapping Ψ (we take the notation from the proof of Theorem 3.2) is generically finite. Let us consider mappings p, F and set X as above. Note that X is exactly the set of points at which the mapping $F = p \circ \Psi$ is not proper. As above we can assume that $F = (F_1, \dots, F_n)$, where $\deg F_1 = d$, $\deg F_i = d-1$ for $i > 1$.

Now let us estimate the geometric degree $\mu(F)$ of F more precisely. We have $\mu(F) = \mu(p \circ \Psi) = \mu(\Psi)\mu(\text{res}_{\bar{\Gamma}} p) = \mu(\Psi) \deg \bar{\Gamma}$. Let us estimate the degree of $\bar{\Gamma}$. Consider a linear subspace $H = L = \mathbb{C} \times \{0, 0, \dots, 0\}$ and take $A = \bar{\Gamma}$. It is easy to see that $H \cap A = \tilde{K}(f) := \{a_1, \dots, a_k\}$. Let m_i denote the number of irreducible components of (an analytic) germ $\mathbf{A}_{a_i} = \bigcup_{j=1}^{m_i} \mathbf{B}_{ij}$. By Lemma 2.1 we have $\deg A \geq \sum_{i=1}^k m_i$. Let c be an isolated critical point of f . We say that c lies over an

irreducible component \mathbf{B}_{ij} of germ \mathbf{A}_{a_i} if there is a small ball U around c , such that $\Psi(U) \subset \mathbf{B}_{ij}$. It is easy to see that every critical point lies over some \mathbf{B}_{ij} and for a fixed component \mathbf{B}_{ij} , there is at most $\mu(\Psi)$ critical points which lie over it. In particular $e \leq (\sum_{i=1}^k m_i)\mu(\Psi) \leq (\deg A)\mu(\Psi) = \mu(F)$. In fact, if we also consider the points at infinity, which correspond to asymptotic values, we have stronger inequality

$$e + a \leq \left(\sum_{i=1}^k m_i\right)\mu(\Psi) \leq (\deg A)\mu(\Psi) = \mu(F).$$

Now the degree of the variety $X \subset M$ is bounded by $(d(d-1)^{n-1} - e - a)/(d-1)$ by Theorem 2.1. So, the set $X \cap L$ has no more than $(d(d-1)^n - e - a)/(d-1)$ points. Finally we obtain that $a \leq (d(d-1)^n - e - a)/(d-1)$ and that $da + e \leq d(d-1)^n$. \square

Example 3.1 (see [8]). We show that our estimate is sharp to both $\tilde{K}_\infty(f)$ and $B_\infty(f)$. More precisely, we have:

For every $d > 0$ there are polynomials $g_n \in \mathbb{C}[x_1, \dots, x_n]$; $n = 1, 2, \dots$, and $f_n \in \mathbb{C}[x_1, \dots, x_n]$; $n = 2, 3, \dots$, of degree d , with finite sets $\tilde{K}_\infty(g_n)$ and $\tilde{K}_\infty(f_n)$ such that:

- 1) $\#\tilde{K}(g_n) = \#B(g_n) = (d-1)^n$;
- 2) $\#\tilde{K}_\infty(f_n) = \#B_\infty(f_n) = (d-1)^{n-1}$.

First we construct a polynomial g_n . Let us consider a polynomial of one variable $h(t) := t^d/d - t$ and take

$$g_n = \sum_{i=1}^n A_i h(x_i),$$

where numbers A_i are sufficiently general. It is easy to check that $\#K_0(g_n) = (d-1)^n$. Put $f_n(x_1, \dots, x_n) := g_{n-1}(x_1, \dots, x_{n-1})$. It is easy to see that $K_0(g_{n-1}) = \tilde{K}_\infty(f_n) = B_\infty(f_n)$ and consequently $\#\tilde{K}_\infty(f_n) = \#B_\infty(f_n) = (d-1)^{n-1}$.

Remark 3.1. It is worth mentioning that the set $\tilde{K}(f)$ can be computed effectively. In particular we are in a position to effectively check whether the set $\tilde{K}_\infty(f)$ is finite. Indeed, let us recall that $\Psi = (f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) = (\psi_1, \dots, \psi_{n+1})$, $\Gamma = \Psi(\mathbb{C}^n)$, $L = \mathbb{C} \times \{0, \dots, 0\}$ and $\tilde{K}(f) = L \cap \bar{\Gamma}$. Hence, it is enough to produce equations for the hypersurface $\bar{\Gamma}$. It can be done by using the Gröbner bases techniques.

Let us consider the ideal I given by polynomials $\{y_i - \psi_i(x)\}_{i=1, \dots, n+1}$ in the ring

$$R = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_{n+1}].$$

In R we consider the lexicographic order, i.e., $x_1 > x_2 > \dots > x_n > y_1 > \dots > y_{n+1}$. Now compute a Gröbner basis \mathcal{A} of the ideal I in R and then take $\mathcal{B} = \mathcal{A} \cap \mathbb{C}[y_1, \dots, y_{n+1}]$. It is a standard fact that \mathcal{B} is the Gröbner basis of the ideal $I(\bar{\Gamma})$ of the hypersurface $\bar{\Gamma}$. Consequently, we have $\tilde{K}(f) = \{y_1 \in \mathbb{C} : h(y_1, 0, \dots, 0) = 0, \text{ for every } h \in \mathcal{B}\}$. In particular, the set $\tilde{K}_\infty(f)$ is finite iff there exists a polynomial $h \in \mathcal{B}$, such that $h(y_1, 0, \dots, 0) \neq 0$.

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