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EXISTENCE AND NONEXISTENCE OF GLOBAL SOLUTIONS OF SOME NON-LOCAL DEGENERATE PARABOLIC SYSTEMS

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Abstract. This paper establishes a new criterion for global existence and nonexistence of positive solutions of the non-local degenerate parabolic system

$$\begin{aligned} u_t &= v^p\left(\Delta u + a\int_\Omega v dx\right), \\ v_t &= u^q\left(\Delta v + b\int_\Omega u dx\right), \quad x\in\Omega, t>0, \end{aligned}$$

with homogeneous Dirichlet boundary conditions, where $\Omega\subset\mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$ and p,q,a,b are positive constants. For all initial data, it is proved that there exists a global positive solution iff $\int_\Omega \varphi(x)dx \le 1/\sqrt{ab}$, where $\varphi(x)$ is the unique positive solution of the linear elliptic problem $-\Delta\varphi(x)=1, x\in\Omega; \varphi(x)=0, x\in\partial\Omega$.

1. Introduction

In [1], the authors investigate the global existence and nonexistence of positive solutions of the strongly coupled degenerate parabolic system

$$\begin{aligned} u_t &= v^p(\Delta u + au), \\ v_t &= u^q(\Delta v + bv), \quad x\in\Omega, t>0, \end{aligned} \tag{1.1}$$

with homogeneous Dirichlet boundary conditions. It is shown that there exists a global positive solution if and only if $\lambda_1 \ge \min\{a,b\}$, where λ_1 is the first Dirichlet eigenvalue for the Laplacian on Ω.

In this paper, we study a new parabolic system with a non-local source

$$\begin{aligned} u_t &= v^p\left(\Delta u + a\int_\Omega v dx\right), \\ v_t &= u^q\left(\Delta v + b\int_\Omega u dx\right), && x\in\Omega, t>0, \\ u(x,t) &= v(x,t) = 0, && x\in\partial\Omega, t>0, \\ u(x,0) &= u_0(x), \quad v(x,0)=v_0(x), && x\in\Omega, \end{aligned} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$ and p,q,a,b are positive constants.

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Over the past several years, a variety of non-local parabolic equations were studied by many authors (see [2]–[10] and references therein). In particular, some authors [8]–[10] studied a class of non-local degenerate parabolic equations which arise in a model of population that communicates through chemical means.

In order to motivate the main result for system (1.2), we recall a classical result of Galaktionov et al. (see [12], [13]) for the system

$$\begin{aligned} u_t &= \Delta u^{\nu+1} + v^p, \\ v_t &= \Delta v^{\mu+1} + u^q, \quad x \in \Omega, t > 0 \end{aligned} \tag{1.3}$$

with homogeneous Dirichlet boundary conditions. It is shown that if $pq < (1+\mu)(1+\nu)$, every solution of (1.3) is global, while if $pq > (1+\mu)(1+\nu)$, there are solutions that blow up and others that are global. In the critical case where $p = 1+\mu$, $q = 1+\nu$, they proved that:

(1) If $\lambda_1 > 1$, all solutions of (1.3) are global.
(2) If $\lambda_1 < 1$, there are no nontrivial global solutions of (1.3).

Their results show that the first eigenvalue λ_1 plays a crucial role in the critical case $pq = (1+\mu)(1+\nu)$ (see also [14], [15]).

Similar results have also been obtained for the scalar equation

$$u_t = u^p(\Delta u + u).$$

It was shown that there exists a unique positive solution which blows up in finite time if $\lambda_1 < 1$ and exists globally if $\lambda_1 \geq 1$ (see [16]–[18] and the references therein). But, for system (1.2), it seems that λ_1 no longer takes action. Motivated by these results, in this paper we will establish a new criterion for global existence and nonexistence of solutions for system (1.2).

Throughout this paper, the initial values and the boundary $\partial\Omega$ are assumed to satisfy

$$\begin{aligned} &\partial\Omega \in C^{2+\alpha}, \\ &u_0(x),\ v_0(x) \in C^1(\overline{\Omega}),\quad u_0(x),\ v_0(x) > 0 \text{ in } \Omega, \\ &u_0(x) = v_0(x) = 0, \quad \partial u_0/\partial n, \partial v_0/\partial n < 0 \text{ on } \partial\Omega. \end{aligned} \tag{H1}$$

Definition 1.1. A positive solution of the system (1.2) is a vector function $(u,v) \in C(\overline{\Omega} \times [0,T^*)) \cap C^{2,1}(\Omega \times (0,T^*))$, positive in $\Omega \times (0,T^*)$ and satisfying (1.2), where T^* is the maximal existence time of the solution. If $T^* = \infty$, we say (u,v) is global.

In our considerations a crucial role is played by

$$\mu = \int_\Omega \varphi(x)dx, \tag{1.4}$$

where $\varphi(x)$ is the unique positive solution of the following linear elliptic problem

$$-\Delta\varphi(x) = 1,\ x \in \Omega;\ \varphi(x) = 0,\ x \in \partial\Omega. \tag{1.5}$$

Then, let us state our main result.

Theorem 1.2. *Assume that* (H1) *holds. Then there exists a global positive solution of* (1.2) *iff* $\mu^2 \leq 1/(ab)$.

We are also interested in another non-local degenerate parabolic system, which is of the form

$$\begin{aligned} u_t &= v^p\left(\Delta u + a\int_\Omega u dx\right), \\ v_t &= u^q\left(\Delta v + b\int_\Omega v dx\right), \quad x\in\Omega, t>0, \end{aligned} \tag{1.6}$$

with similar initial-boundary conditions as in (1.2). For system (1.6), we get a different criterion as follows.

Theorem 1.3. *Assume that* (H1) *holds. Then there exists a global positive solution of* (1.6) *iff* $1/\mu \geq \min\{a,b\}$.

The result shows that for system (1.6), it is not λ_1 but $1/\mu$ that plays a crucial role. We will not discuss (1.6) in detail since it can be easily proved by combining the present arguments with those in [1].

Remark 1.4. Combining the arguments in [1] and in the present paper, we can show that $\lambda_1^2 \geq ab$ is the critical condition of system

$$u_t = v^p(\Delta u + av), \qquad v_t = u^q(\Delta v + bu).$$

We will not give the proof here, since this paper is concerned about the non-local problem.

This paper is organized as follows. Section 2 establishes the local theory. Section 3 gives the proof of the main result.

2. Local existence

Set $Q_T = \Omega\times(0,T]$, $S_T = \partial\Omega\times(0,T]$ for $0<T<\infty$. We first give a maximum principle for non-local systems, of which the proof is standard, and omit its proof.

Proposition 2.1. *Suppose that* $w_1(x,t), w_2(x,t) \in C(\overline{Q_T})\cap C^{2,1}(Q_T)$ *satisfy*

$$\begin{aligned} & w_{1t} - d_1\Delta w_1 \geq c_{11}w_1 + c_{12}w_2 + c_{13}w_1w_2 \\ & \quad + c_{14}\int_\Omega c_{15}w_1(x,t)dx + c_{16}\int_\Omega c_{17}w_2(x,t)dx, \\ & w_{2t} - d_2\Delta w_2 \geq c_{21}w_1 + c_{22}w_2 + c_{23}w_1w_2 \\ & \quad + c_{24}\int_\Omega c_{25}w_1(x,t)dx + c_{26}\int_\Omega c_{27}w_2(x,t)dx, && (x,t)\in Q_T, \\ & w_1(x,t)\geq 0,\ w_2(x,t)\geq 0, && (x,t)\in S_T, \\ & w_1(x,0)\geq 0,\ w_2(x,0)\geq 0, && x\in\Omega, \end{aligned}$$

where $d_i(x,t)$, $c_{ij}(x,t)$ $(i=1,2; j=1,\dots,7)$ *are bounded functions and*

$$d_1,\ d_2,\ c_{12},\ c_{21},\ c_{1j},\ c_{2j}\geq 0,\ j=4,\dots,7 \quad in \quad Q_T.$$

Then $w_j(x,t)\geq 0$ *on* $\overline{Q_T}$.

Proposition 2.2. *Let* $(\tilde u,\tilde v)\in C(\overline{Q_T})\cap C^{2,1}(Q_T)$ *and* $(\bar u,\bar v)\in C(\overline{Q_T})\cap C^{2,1}(Q_T)$ *be a nonnegative subsolution and a nonnegative supersolution of* (1.2)*, respectively. Assume that* $(\bar u,\bar v)\geq\delta>0$ *and either*

$$\Delta\tilde u + a\int_\Omega \tilde v dx \geq 0, \quad \Delta\tilde v + b\int_\Omega \tilde u dx \geq 0 \tag{2.1}$$

or

$$\Delta\bar{u} + a\int_\Omega \bar{v}dx \geq 0, \quad \Delta\bar{v} + b\int_\Omega \bar{u}dx \geq 0 \tag{2.2}$$

hold. Then $(\tilde{u},\tilde{v}) \leq (\bar{u},\bar{v})$ *on* $\overline{Q_T}$ *if* $(\tilde{u}_0,\tilde{v}_0) \leq (\bar{u}_0,\bar{v}_0)$.

Proof. This proposition is a direct consequence of Proposition 2.1. □

Next, in this section, we will give the local existence of the solution for system (1.2) by the same method utilized in [1]. For system (1.2) we introduce, for $n = 1, 2, \ldots$, the following regularized system:

$$\begin{aligned} u_{nt} &= v_n^p\left(\Delta u_n + a\int_\Omega v_n dx\right), \\ v_{nt} &= u_n^q\left(\Delta v_n + b\int_\Omega u_n dx\right), && x\in\Omega, t>0, \\ u_n(x,t) &= v_n(x,t) = 1/n, && x\in\partial\Omega, t>0, \\ u_n(x,0) &= u_0(x)+1/n, \quad v_n(x,0) = v_0(x)+1/n, && x\in\Omega. \end{aligned} \tag{2.3}$$

By a similar discussion as in [7], under (H1), we can show that (2.3) has a classical solution (u_n, v_n) with $u_n, v_n \geq 1/n$, defined on $\overline{\Omega}\times[0,T_n^*)$, where T_n^* is the maximal existence time.

Now we construct a uniform upper bound for (u_n, v_n). Consider the ordinary differential equation

$$\begin{aligned} H'(t) &= \hat{a}(H(t))^{\hat{p}}, \\ H(0) &= \max\{\max_{x\in\overline{\Omega}} u_0(x)+1, \max_{x\in\overline{\Omega}} v_0(x)+1\}, \end{aligned} \tag{2.4}$$

where $\hat{a} = \max\{a|\Omega|, b|\Omega|\}$, $\hat{p} = \max\{p+1, q+1\}$. Obviously, there exists $T_0 > 0$ such that (2.4) has a non-decreasing solution $H(t) > 0$ on $[0, T_0]$; namely, $0 < H(0) \leq H(t) \leq H(T_0) < \infty$. Using Proposition 2.2 for system (2.3), we obtain the following lemma.

Lemma 2.3. *There exist* T_0 *and an a priori bound* $H(t)$ *depending only on* $u_0, v_0, \hat{a}$ *and* $\hat{p}$ *such that for all* $n \geq 1$ *the solution of* (2.3) *satisfies* $u_n, v_n \leq H(t)$ *on* $\overline{Q_{T_0}}$.

Denote by $\lambda_1 > 0$ and $\phi(x)$ the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem

$$-\Delta\phi(x) = \lambda\phi(x),\ x\in\Omega;\ \phi(x) = 0,\ x\in\partial\Omega.$$

It is well known that $\phi(x)$ may be normalized as $\phi(x) > 0$ in Ω and $\max_\Omega \phi(x) = 1$. Thus, by Proposition 2.1, we have

Lemma 2.4. *Let* $h(x,t) = ke^{-\rho t}\phi(x)$, *where* k *is small such that* $u_0, v_0 \geq k\phi(x)$ *and* $\rho = \max\{\lambda_1(H(T_0))^p, \lambda_1(H(T_0))^q\}$. *Then for all* $n \geq 1$, *it holds that* $u_n, v_n \geq h(x,t)$ *in* $\overline{Q_{T_0}}$.

In proving there exists a positive solution of (1.2), we still need the following regularity lemma, whose proof is similar to [1, Lemma 2.3].

Lemma 2.5. $u_n, v_n \in V_2^{1,0}(Q_{T_0})$ *(see* [19, *p.* 6]*).*

Then by the so-called extension method (for details see [1]), we have that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\lim_{i\to\infty}(u_{n_i}, v_{n_i}) = (u, v) \quad \text{in} \quad C^{2,1}(Q_{T_0}). \tag{2.5}$$

Similarly, we can show that u, v are continuous at any point $(y,t), y \in \partial\Omega$ and $u(y,t) = 0$ (see [16], [20]), and continuous up to $\{t = 0\}$ (see [21], [22]).

Let T^* be the supremum over T_0 for which (u, v) exists on $(0, T_0)$. Thus, we have

Theorem 2.6. *Assume that* (H1) *holds. Then there exists a positive solution of* (1.2) *on* $(0, T^*)$. *Moreover, if* $T^* < \infty$, *then*

$$\limsup_{t\to T^*} \|u(x,t)\|_{L^\infty} = +\infty \quad \textit{or} \quad \limsup_{t\to T^*} \|v(x,t)\|_{L^\infty} = +\infty.$$

Remark 2.7. Obviously, all discussions of this section are applicable to system (1.6).

3. Proof of the main result

In order to prove the main result, we give an auxiliary lemma first. Let G be a bounded smooth domain of $\mathbb{R}^N$. Consider the problem

$$\begin{aligned} w_t &= dw^r\left(\Delta w + a_0\int_G w dx\right), && x \in G, t > 0,\\ w(x,t) &= c, && x \in \partial G, t \geq 0,\\ w(x,0) &= c, && x \in G, \end{aligned} \tag{3.1}$$

where $0 < r < 1$ and a_0, c, d are positive constants. By the standard method (see [7], [10]), it follows that (3.1) has a unique classical solution $w(x,t)$ and $w(x,t) \geq c$. Denote by $\varphi_0(x)$ the unique positive solution of the linear elliptic problem

$$-\Delta\varphi_0(x) = 1, \quad x \in G; \quad \varphi_0(x) = 0, \quad x \in \partial G.$$

Set $\mu_0 = \int_G \varphi_0(x)dx$. Thus, we have

Lemma 3.1. *If* $\mu_0 > 1/a_0$, *then the positive solution of* (3.1) *blows up in finite time.*

Proof. Set $F(t) = \int_G w^{1-r}\varphi_0 dx$; then

$$\begin{aligned} \frac{1}{1-r}F'(t) &= d\left(\int_G \Delta w\varphi_0 dx + a_0\int_G w dx\int_G \varphi_0 dx\right)\\ &\geq d(a_0\mu_0 - 1)\int_G w dx\\ &\geq d(a_0\mu_0 - 1)\left(\int_G w\varphi_0 dx\right)\Big/ M, \end{aligned} \tag{3.2}$$

where $M = \max_{x\in\overline{G}}\{\varphi_0(x)\}$. Letting $z = w^{1-r}$ in (3.2) yields

$$\int_G z_t(x,t)\varphi_0 dx \geq d(1-r)(a_0\mu_0 - 1)\left(\int_G z^{1/(1-r)}\varphi_0 dx\right)\Big/ M.$$

Since $\frac{1}{1-r} > 1$, by the Jensen inequality, it follows that

$$\int_G z_t(x,t)\varphi_0 dx \geq d(1-r)(a_0\mu_0 - 1)(\mu_0)^{-r/(1-r)}\left(\int_G z\varphi_0 dx\right)^{1/(1-r)}\Big/ M.$$

That is,

$$F'(t) \geq C_0(F(t))^{1/(1-r)},$$

where $C_0 = d(1-r)(a_0\mu_0 - 1)(\mu_0)^{-r/(1-r)}/M > 0$. In view of $1/(1-r) > 1$ and $F(0) > 0$, it follows that there exists $T < \infty$ such that $\lim_{t\to T} F(t) = +\infty$, and hence $w(x,t)$ blows up in finite time. □

Lemma 3.2. *Assume that* (H1) *holds. Then there exist positive constants* k_1, k_2 *such that* $u(x,t) \geq k_1\varphi, v(x,t) \geq k_2\varphi$ *for* $(x,t) \in \overline{\Omega} \times [0,T^*)$ *if* $\mu^2 \geq 1/(ab)$.

Proof. From (H1), since $\mu^2 \geq 1/(ab)$ we see that there exist positive constants k_1 and k_2 such that

$$u_0(x) \geq k_1\varphi(x), \quad v_0(x) \geq k_2\varphi(x), \quad x \in \overline{\Omega}, \tag{3.3}$$

and

$$a\mu \geq k_1/k_2 \geq 1/(b\mu). \tag{3.4}$$

Let $w(x,t) = u(x,t) - k_1\varphi(x)$, $s(x,t) = v(x,t) - k_2\varphi(x)$. Then we obtain, by (3.4), for any $T \in (0,T^*)$,

$$\begin{aligned} w_t = u_t &= v^p\left(\Delta u + a\int_\Omega v dx\right) \\ &= v^p\left(\Delta w + a\int_\Omega s dx\right) + v^p(-k_1 + ak_2\mu) \\ &\geq v^p\left(\Delta w + a\int_\Omega s dx\right), \\ s_t &\geq u^q\left(\Delta s + b\int_\Omega w dx\right), \qquad x \in \Omega, 0 < t \leq T, \\ w(x,t) &= s(x,t) = 0, \qquad x \in \partial\Omega, 0 < t \leq T. \end{aligned} \tag{3.5}$$

By Proposition 2.1, it follows from (3.3) and (3.5) that $w \geq 0, s \geq 0$ and hence $u \geq k_1\varphi, v \geq k_2\varphi$ on $\overline{\Omega} \times [0,T]$. The arbitrariness of T shows that the result holds. □

Lemma 3.3. *Assume that* (H1) *holds. Then no global solution of* (1.2) *exists if* $\mu^2 > 1/(ab)$.

Proof. Denote by $\varphi_1(x)$ the unique positive solution of the linear elliptic problem

$$-\Delta\varphi_1(x) = 1, \quad x \in \Omega_1; \quad \varphi_1(x) = 0, \quad x \in \partial\Omega_1.$$

Here $\Omega_1 \subset\subset \Omega$. Since the function $U := \varphi - \varphi_1 \geq 0$ is harmonic in Ω_1 and satisfies $U \leq \varphi$ on $\partial\Omega_1$, we have $\|\varphi - \varphi_1\|_\infty \leq \|\varphi\|_{L^\infty(\partial\Omega_1)}$ by the maximum principle. By the continuity of φ it follows that $\|\varphi - \varphi_1\|_\infty \to 0$, as $\mathrm{dist}(\partial\Omega_1, \partial\Omega) \to 0$. Let $\mu_1 = \int_{\Omega_1} \varphi_1(x)dx$. The above discussion implies, in particular, $\mu_1 \to \mu$, as $\mathrm{dist}(\partial\Omega_1, \partial\Omega) \to 0$.

Therefore, in view of $\mu^2 > 1/(ab)$, we can choose a smooth sub-domain $\Omega_1 \subset\subset \Omega$ such that $\mu_1^2 > 1/(ab)$. Denote

$$\delta = \frac{1}{2}\min\{k_1 \min_{\overline{\Omega_1}} \varphi, k_2 \min_{\overline{\Omega_1}} \varphi\}.$$

Then $\delta > 0$ and

$$u(x,t) \geq 2\delta, \quad v(x,t) \geq 2\delta, \quad \forall (x,t) \in \overline{\Omega_1} \times [0,T^*),$$

by Lemma 3.2. Then (u,v) in $\Omega_1 \times (0,T^*)$ satisfies

$$
\begin{aligned}
u_t &= v^p\left(\Delta u + a\int_\Omega v dx\right)\\
&\geq v^p\left(\Delta u + a\int_{\Omega_1} v dx\right),\\
v_t &\geq u^q\left(\Delta v + b\int_{\Omega_1} u dx\right), \quad x\in\Omega_1,\ t\in(0,T^*),\\
u(x,t)&\geq 2\delta,\quad v(x,t)\geq 2\delta, \qquad x\in\partial\Omega_1,\ t\in(0,T^*),\\
u(x,0)&\geq 2\delta,\quad v(x,0)\geq 2\delta, \qquad x\in\ \Omega_1.
\end{aligned}
\tag{3.6}
$$

Now, we consider the system

$$
\begin{aligned}
\underline{u}_t &= \underline{v}^p\left(\Delta\underline{u} + a\int_{\Omega_1} \underline{v} dx\right),\\
\underline{v}_t &= \underline{u}^q\left(\Delta\underline{v} + b\int_{\Omega_1} \underline{u} dx\right), \quad x\in\Omega_1, t>0,\\
\underline{u}(x,t)&=f(t),\ \underline{v}(x,t)=g(t), \qquad x\in\partial\Omega_1, t>0,\\
\underline{u}(x,0)&=\underline{v}(x,0)=\delta, \qquad x\in\Omega_1,
\end{aligned}
\tag{3.7}
$$

where $f(t), g(t)$ satisfy

$$
\begin{aligned}
&f(t),\ g(t)\in C^\infty([0,\infty)),\ f'(t),\ g'(t)>0,\ f(t),\ g(t)\leq 2\delta,\\
&f(0)=g(0)=\delta,\ f'(0)=a|\Omega_1|\delta^{p+1},\ g'(0)=b|\Omega_1|\delta^{q+1}.
\end{aligned}
$$

A similar discussion as in [7] shows that there exists a unique classical solution $(\underline{u},\underline{v})\in C^{2+\beta,1+\beta/2}(\overline{\Omega_1}\times[0,T_1))$ for some $\beta\in(0,1)$, where T_1 is the maximal existence time, and

$$
\underline{u},\underline{v}\geq\delta \quad \text{in}\ \ \overline{\Omega_1}\times[0,T_1). \tag{3.8}
$$

Since the initial data is a subsolution of (3.7), we have $\underline{u}_t,\underline{v}_t\geq 0$ in $\overline{\Omega_1}\times[0,T_1)$ and hence

$$
\Delta\underline{u} + a\int_{\Omega_1}\underline{v}dx\geq 0,\quad \Delta\underline{v} + b\int_{\Omega_1}\underline{u}dx\geq 0\ \ \text{in}\ \ \overline{\Omega_1}\times[0,T_1). \tag{3.9}
$$

Thus from Proposition 2.2, we have $T_1\geq T^*$ and

$$
u(x,t)\geq\underline{u}(x,t),\quad v(x,t)\geq\underline{v}(x,t)\ \ \text{in}\ \ \overline{\Omega_1}\times[0,T^*).
$$

Therefore, it suffices to show that $(\underline{u},\underline{v})$ blows up in finite time, because if so, its upper bound (u,v) does exist up to a finite time T^*.

By (3.8) and (3.9), we have

$$
\begin{aligned}
\underline{u}_t &\geq \delta^{p-r}\underline{v}^r\left(\Delta\underline{u} + a\int_{\Omega_1}\underline{v}dx\right),\\
\underline{v}_t &\geq \delta^{q-r}\underline{u}^r\left(\Delta\underline{v} + b\int_{\Omega_1}\underline{u}dx\right) \qquad \text{in}\quad \Omega_1\times(0,T_1)
\end{aligned}
\tag{3.10}
$$

with the corresponding initial and boundary conditions and $0<r<1$.

By use of $\mu_1^2>1/(ab)$, there exist positive constants l_1,l_2 with $l_1,l_2>1$, and l such that

$$
a\mu_1>\frac{l_1}{l_2}>\frac{1}{b\mu_1},\ \mu_1>\frac{1}{l}>\frac{l_1}{al_2},\ \mu_1>\frac{1}{l}>\frac{l_2}{bl_1}. \tag{3.11}
$$

Choose

$$d = \min\{\delta^{p-r}, \delta^{q-r}\}, \quad \gamma = \min\{1/l_1, 1/l_2\}. \tag{3.12}$$

Denote by $z(x,t)$ the unique positive solution of the problem

$$\begin{aligned} z_t &= dz^r\left(\Delta z + l\int_{\Omega_1} z dx\right), && x \in \Omega_1, t > 0, \\ z(x,t) &= \gamma\delta, && x \in \partial\Omega_1, t \geq 0, \\ z(x,0) &= \gamma\delta, && x \in \Omega_1, \end{aligned} \tag{3.13}$$

where l, d, γ satisfy (3.11) and (3.12). By Lemma 3.1, it follows that $z(x,t)$ blows up in finite time $T_0 < \infty$. Moreover, $z_t \geq 0$, i.e., $\Delta z + l\int_{\Omega_1} z dx \geq 0$, since the initial data is a subsolution of (3.13). Let

$$w(x,t) = l_1 z(x,t), \qquad s(x,t) = l_2 z(x,t).$$

Thus, from (3.11)–(3.13) and $l_1, l_2 > 1$, we have

$$\begin{aligned} & w_t - \delta^{p-r}s^r\left(\Delta w + a\int_{\Omega_1} s dx\right) = l_1 dz^r\left(\Delta z + l\int_{\Omega_1} z dx\right) \\ & \quad - l_1\delta^{p-r}(l_2 z)^r\left(\Delta z + (al_2/l_1)\int_{\Omega_1} z dx\right) \leq 0, \\ & s_t - \delta^{q-r}w^r\left(\Delta s + b\int_{\Omega_1} w dx\right) \leq 0, \qquad x \in \Omega_1, 0 < t < T_0, \\ & w(x,t) = l_1\gamma\delta \leq \delta,\ s(x,t) = l_2\gamma\delta \leq \delta, \quad x \in \partial\Omega_1,\ 0 \leq t < T_0, \\ & w(x,0) = l_1\gamma\delta \leq \delta,\ s(x,0) = l_2\gamma\delta \leq \delta, \quad x \in \Omega_1. \end{aligned} \tag{3.14}$$

By use of Proposition 2.2, it follows from (3.8), (3.10), (3.14) and $\Delta z + l\int_{\Omega_1} z dx \geq 0$ that

$$(\underline{u}, \underline{v}) \geq (l_1 z, l_2 z) \ \text{ in } \ \Omega_1 \times (0, T_1).$$

Hence $(\underline{u}, \underline{v})$ blows up in finite time since $z(x,t)$ does. Therefore, (u,v) exists no later than $T_0 < \infty$. This completes the proof. □

Lemma 3.4. *Assume that* (H1) *holds. Then the positive solution* (u,v) *of* (1.2) *defined by* (2.5) *is global if* $\mu^2 \leq 1/(ab)$.

Proof. Applying $\mu^2 \leq 1/(ab)$ and (H1) we see that there exist large positive constants K_1 and K_2 such that

$$a\mu \leq K_1/K_2 \leq 1/(b\mu) \tag{3.15}$$

and

$$u_0(x) \leq K_1\varphi(x), \qquad v_0(x) \leq K_2\varphi(x), \qquad \forall x \in \overline{\Omega}. \tag{3.16}$$

Let $W(x,t) = K_1\varphi(x) - u(x,t)$, $S(x,t) = K_2\varphi(x) - v(x,t)$. Then, from (3.15), we obtain, for any $T \in (0, T^*)$,

$$\begin{aligned} W_t = -u_t &= -v^p\left(\Delta u + a\int_\Omega v dx\right) \\ &= v^p\left(\Delta W + a\int_\Omega S dx\right) + v^p(K_1 - aK_2\mu) \\ &\geq v^p\left(\Delta W + a\int_\Omega S dx\right), \\ S_t &\geq u^q\left(\Delta S + b\int_\Omega W dx\right), \quad x \in \Omega, 0 < t \leq T, \\ W(x,t) &= S(x,t) = 0, \qquad\qquad x \in \partial\Omega, 0 < t \leq T. \end{aligned} \tag{3.17}$$

By Proposition 2.1, it follows from (3.16) and (3.17) that $W \geq 0, S \geq 0$ and hence $u \leq K_1\varphi, v \leq K_2\varphi$ on $\overline{\Omega} \times [0,T]$. The arbitrariness of T shows that $u \leq K_1\varphi, v \leq K_2\varphi$ on $\overline{\Omega} \times [0,T^*)$. Therefore, the solution (u,v) of (1.2) defined by (2.5) exists globally. □

From Lemma 3.3 and Lemma 3.4, it follows that Theorem 1.2 holds.

Remark 3.5. From Lemma 3.2 and Lemma 3.4, we have that if $\mu^2 = 1/(ab)$, there exist positive constants k_1, k_2, K_1 and K_2 such that $k_1\varphi \leq u(x,t) \leq K_1\varphi$, $k_2\varphi \leq u(x,t) \leq K_2\varphi$ for $x \in \overline{\Omega}$ and $t > 0$.

Remark 3.6. Theorem 1.3 for system (1.6) can be proved by combining the present arguments (for system (1.2)) with those in [1] (for system (1.1)).

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