EXISTENCE AND NONEXISTENCE OF GLOBAL SOLUTIONS
OF SOME NON-LOCAL DEGENERATE PARABOLIC SYSTEMS

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Abstract. This paper establishes a new criterion for global existence and nonexistence of positive solutions of the non-local degenerate parabolic system

\begin{align*}
    u_t &= v^p \left( \Delta u + a \int_{\Omega} v \, dx \right), \\
    v_t &= u^q \left( \Delta v + b \int_{\Omega} u \, dx \right),
\end{align*}

with homogeneous Dirichlet boundary conditions, where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with a smooth boundary \( \partial \Omega \) and \( p, q, a, b \) are positive constants. For all initial data, it is proved that there exists a global positive solution iff

\[ \int_{\Omega} \varphi(x) \, dx \leq \frac{1}{\sqrt{ab}}, \]

where \( \varphi(x) \) is the unique positive solution of the linear elliptic problem \(-\Delta \varphi(x) = 1, x \in \Omega; \varphi(x) = 0, x \in \partial \Omega.\)

1. Introduction

In [1], the authors investigate the global existence and nonexistence of positive solutions of the strongly coupled degenerate parabolic system

\begin{align}
    u_t &= v^p(\Delta u + au), \\
    v_t &= u^q(\Delta v + bv), \quad x \in \Omega, t > 0, \label{eq:1.1}
\end{align}

with homogeneous Dirichlet boundary conditions. It is shown that there exists a global positive solution if and only if \( \lambda_1 \geq \min\{a, b\} \), where \( \lambda_1 \) is the first Dirichlet eigenvalue for the Laplacian on \( \Omega. \)

In this paper, we study a new parabolic system with a non-local source

\begin{align}
    u_t &= v^p \left( \Delta u + a \int_{\Omega} v \, dx \right), \\
    v_t &= u^q \left( \Delta v + b \int_{\Omega} u \, dx \right), \quad x \in \Omega, t > 0, \label{eq:1.2}
\end{align}

\[ u(x, t) = v(x, t) = 0, \quad x \in \partial \Omega, t > 0, \]

\[ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega, \]

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with a smooth boundary \( \partial \Omega \) and \( p, q, a, b \) are positive constants.

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Over the past several years, a variety of non-local parabolic equations were studied by many authors (see [2]–[10] and references therein). In particular, some authors [8]–[10] studied a class of non-local degenerate parabolic equations which arise in a model of population that communicates through chemical means.

In order to motivate the main result for system (1.2), we recall a classical result of Galaktionov et al. (see [12], [13]) for the system
\begin{align*}
  u_t &= \Delta u^{\nu+1} + v^p, \\
  v_t &= \Delta v^{\mu+1} + u^q,
\end{align*}
(1.3)
with homogeneous Dirichlet boundary conditions. It is shown that if \( pq < (1 + \mu)(1 + \nu) \), every solution of (1.3) is global, while if \( pq > (1 + \mu)(1 + \nu) \), there are solutions that blow up and others that are global. In the critical case where \( p = 1 + \mu, q = 1 + \nu \), they proved that:

1. If \( \lambda_1 > 1 \), all solutions of (1.3) are global.
2. If \( \lambda_1 < 1 \), there are no nontrivial global solutions of (1.3).

Their results show that the first eigenvalue \( \lambda_1 \) plays a crucial role in the critical case \( pq = (1 + \mu)(1 + \nu) \) (see also [14], [15]).

Similar results have also been obtained for the scalar equation
\[ u_t = u^p(\Delta u + u). \]
It was shown that there exists a unique positive solution which blows up in finite time if \( \lambda_1 < 1 \) and exists globally if \( \lambda_1 \geq 1 \) (see [16]–[18] and the references therein).

But, for system (1.2), it seems that \( \lambda_1 \) no longer takes action. Motivated by these results, in this paper we will establish a new criterion for global existence and nonexistence of solutions for system (1.2).

Throughout this paper, the initial values and the boundary \( \partial \Omega \) are assumed to satisfy
\begin{align*}
  \partial \Omega &\in C^{2+\alpha}, \\
  (H1) &\quad u_0(x), v_0(x) \in C^1(\overline{\Omega}), \quad u_0(x), \ v_0(x) > 0 \text{ in } \Omega, \\
  &\quad u_0(x) = v_0(x) = 0, \quad \partial u_0/\partial n, \partial v_0/\partial n < 0 \text{ on } \partial \Omega.
\end{align*}

**Definition 1.1.** A positive solution of the system (1.2) is a vector function \((u, v) \in C(\overline{\Omega} \times [0, T^*)) \cap C^{2,1}(\Omega \times (0, T^*))\), positive in \( \Omega \times (0, T^*) \) and satisfying (1.2), where \( T^* \) is the maximal existence time of the solution. If \( T^* = \infty \), we say \((u, v)\) is global.

In our considerations a crucial role is played by
\[ \mu = \int_{\Omega} \varphi(x)dx, \]
(1.4)
where \( \varphi(x) \) is the unique positive solution of the following linear elliptic problem
\[ -\Delta \varphi = 1, \quad x \in \Omega; \quad \varphi = 0, \quad x \in \partial \Omega. \]

Then, let us state our main result.

**Theorem 1.2.** Assume that (H1) holds. Then there exists a global positive solution of (1.2) iff \( \mu^2 \leq 1/(ab) \).
We are also interested in another non-local degenerate parabolic system, which is of the form
\[
\begin{align*}
    u_t &= v^p \left( \Delta u + a \int_\Omega u \, dx \right), \\
    v_t &= u^q \left( \Delta v + b \int_\Omega v \, dx \right), \quad x \in \Omega, t > 0,
\end{align*}
\]
with similar initial-boundary conditions as in (1.2). For system (1.6), we get a different criterion as follows.

**Theorem 1.3.** Assume that (H1) holds. Then there exists a global positive solution of (1.6) iff \( 1/\mu \geq \min\{a, b\} \).

The result shows that for system (1.6), it is not \( \lambda_1 \) but \( 1/\mu \) that plays a crucial role. We will not discuss (1.6) in detail since it can be easily proved by combining the present arguments with those in [1].

**Remark 1.4.** Combining the arguments in [1] and in the present paper, we can show that \( \lambda_1^2 \geq ab \) is the critical condition of system
\[
\begin{align*}
    u_t &= v^p (\Delta u + av), \\
    v_t &= u^q (\Delta v + bu).
\end{align*}
\]

We will not give the proof here, since this paper is concerned about the non-local problem.

This paper is organized as follows. Section 2 establishes the local theory. Section 3 gives the proof of the main result.

## 2. LOCAL EXISTENCE

Set \( Q_T = \Omega \times (0, T], S_T = \partial \Omega \times (0, T] \) for \( 0 < T < \infty \). We first give a maximum principle for non-local systems, of which the proof is standard, and omit its proof.

**Proposition 2.1.** Suppose that \( w_1(x, t), w_2(x, t) \in C(\overline{Q_T}) \cap C^{2,1}(Q_T) \) satisfy
\[
\begin{align*}
    w_1_t - d_1 \Delta w_1 &\geq c_{11} w_1 + c_{12} w_2 + c_{13} w_1 w_2 \\
    &+ c_{14} \int_\Omega c_{15} w_1(x, t) \, dx + c_{16} \int_\Omega c_{17} w_2(x, t) \, dx, \\
    w_2_t - d_2 \Delta w_2 &\geq c_{21} w_1 + c_{22} w_2 + c_{23} w_1 w_2 \\
    &+ c_{24} \int_\Omega c_{25} w_1(x, t) \, dx + c_{26} \int_\Omega c_{27} w_2(x, t) \, dx, \quad (x, t) \in Q_T, \\
    w_1(x, t) &\geq 0, \quad w_2(x, t) \geq 0, \quad (x, t) \in S_T, \\
    w_1(x, 0) &\geq 0, \quad w_2(x, 0) \geq 0, \quad x \in \Omega,
\end{align*}
\]
where \( d_i(x, t), c_{ij}(x, t) \ (i = 1, 2; j = 1, \ldots, 7) \) are bounded functions and
\[
d_1, d_2, c_{12}, c_{21}, c_{1j}, c_{2j} \geq 0, \quad j = 4, \ldots, 7 \quad \text{in} \quad Q_T.
\]
Then \( w_j(x, t) \geq 0 \) on \( \overline{Q_T} \).

**Proposition 2.2.** Let \((\bar{u}, \bar{v}) \in C(\overline{Q_T}) \cap C^{2,1}(Q_T)\) and \((\tilde{u}, \tilde{v}) \in C(\overline{Q_T}) \cap C^{2,1}(Q_T)\) be a nonnegative subsolution and a nonnegative supersolution of (1.2), respectively. Assume that \( (\bar{u}, \bar{v}) \geq \delta > 0 \) and either
\[
\begin{align*}
    \Delta \bar{u} + a \int_\Omega \bar{v} \, dx &\geq 0, \\
    \Delta \tilde{u} + b \int_\Omega \tilde{v} \, dx &\geq 0.
\end{align*}
\]
or
\begin{equation}
\Delta \bar{u} + a \int_{\Omega} \bar{v}dx \geq 0, \quad \Delta \bar{v} + b \int_{\Omega} \bar{u}dx \geq 0
\end{equation}
hold. Then \((\bar{u}, \bar{v}) \leq (\bar{u}, \bar{v})\) on \(\overline{Q_T}\) if \((\bar{u}_0, \bar{v}_0) \leq (\bar{u}_0, \bar{v}_0)\).

**Proof.** This proposition is a direct consequence of Proposition 2.1.

Next, in this section, we will give the local existence of the solution for system (1.2) by the same method utilized in [1]. For system (1.2) we introduce, for \(n = 1, 2, \ldots\), the following regularized system:
\begin{equation}
\begin{align*}
&u_{nt} = v_n^p \left( \Delta u_n + a \int_{\Omega} v_n dx \right), \\
v_{nt} = u_n^q \left( \Delta v_n + b \int_{\Omega} u_n dx \right), \\
u_n(x, t) = v_n(x, t) = 1/n, \\
u_n(x, 0) = u_0(x) + 1/n, \\v_n(x, 0) = v_0(x) + 1/n, \
x \in \Omega, t > 0,
\end{align*}
\end{equation}
By a similar discussion as in [7], under (H1), we can show that (2.3) has a classical solution \((u_n, v_n)\) with \(u_n, v_n \geq 1/n\), defined on \(\overline{\Omega} \times [0, T^*_n]\), where \(T^*_n\) is the maximal existence time.

Now we construct a uniform upper bound for \((u_n, v_n)\). Consider the ordinary differential equation
\begin{equation}
\begin{align*}
H'(t) &= \hat{a}(H(t))^\hat{p}, \\
H(0) &= \max\{\max_{x \in \overline{\Omega}} u_0(x) + 1, \max_{x \in \overline{\Omega}} v_0(x) + 1\},
\end{align*}
\end{equation}
where \(\hat{a} = \max\{a|\Omega|, b|\Omega|\}\), \(\hat{p} = \max\{p + 1, q + 1\}\). Obviously, there exists \(T_0 > 0\) such that (2.3) has a non-decreasing solution \(H(t) > 0\) on \([0, T_0]\); namely, \(0 < H(0) \leq H(t) \leq H(T_0) < \infty\). Using Proposition 2.2 for system (2.3), we obtain the following lemma.

**Lemma 2.3.** There exist \(T_0\) and an a priori bound \(H(t)\) depending only on \(u_0, v_0, \hat{a}\) and \(\hat{p}\) such that for all \(n \geq 1\) the solution of (2.3) satisfies \(u_n, v_n \leq H(t)\) on \(\overline{Q_{T_0}}\).

Denote by \(\lambda_1 > 0\) and \(\phi(x)\) the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem
\[-\Delta \phi(x) = \lambda \phi(x), \quad x \in \Omega; \quad \phi(x) = 0, \quad x \in \partial \Omega.\]
It is well known that \(\phi(x)\) may be normalized as \(\phi(x) > 0\) in \(\Omega\) and \(\max_\Omega \phi(x) = 1\). Thus, by Proposition 2.1 we have

**Lemma 2.4.** Let \(h(x, t) = ke^{-\rho t}\phi(x)\), where \(k\) is small such that \(u_0, v_0 \geq k\phi(x)\) and \(\rho = \max\{\lambda_1(H(T_0))^\rho, \lambda_1(H(T_0))\}\). Then for all \(n \geq 1\), it holds that \(u_n, v_n \geq h(x, t)\) in \(\overline{Q_{T_0}}\).

In proving there exists a positive solution of (1.2), we still need the following regularity lemma, whose proof is similar to [1] Lemma 2.3.

**Lemma 2.5.** \(u_n, v_n \in V_2^{1, 0}(Q_{T_0})\) (see [19, p. 6]).
Then by the so-called extension method (for details see [1]), we have that there exists a subsequence \( \{n_i\} \) of \( \{n\} \) such that

\[
\lim_{i \to \infty} (u_{n_i}, v_{n_i}) = (u, v) \quad \text{in} \quad C^{2,1}(Q_{T_0}).
\]

Similarly, we can show that \( u, v \) are continuous at any point \((y, t), y \in \partial \Omega \) and \( u(y, t) = 0 \) (see [16], [20]), and continuous up to \( \{t = 0\} \) (see [21], [22]).

Let \( T^* \) be the supremum over \( T_0 \) for which \((u, v)\) exists on \((0, T_0)\). Thus, we have

**Theorem 2.6.** Assume that (H1) holds. Then there exists a positive solution of (1.2) on \((0, T^*)\). Moreover, if \( T^* < \infty \), then

\[
\limsup_{t \to T^*} \|u(x, t)\|_{L^\infty} = +\infty \quad \text{or} \quad \limsup_{t \to T^*} \|v(x, t)\|_{L^\infty} = +\infty.
\]

**Remark 2.7.** Obviously, all discussions of this section are applicable to system (1.6).

3. **Proof of the main result**

In order to prove the main result, we give an auxiliary lemma first. Let \( G \) be a bounded smooth domain of \( \mathbb{R}^N \). Consider the problem

\[
\begin{align*}
  w_t &= dw^r \left( \Delta w + a_0 \int_G w \, dx \right), \quad x \in G, t > 0, \\
  w(x, t) &= c, \quad x \in \partial G, t \geq 0, \\
  w(x, 0) &= c, \quad x \in G,
\end{align*}
\]

where \( 0 < r < 1 \) and \( a_0, c, d \) are positive constants. By the standard method (see [7], [10]), it follows that (3.1) has a unique classical solution \( w(x, t) \) and \( w(x, t) \geq c \).

Denote by \( \varphi_0(x) \) the unique positive solution of the linear elliptic problem

\[
-\Delta \varphi_0(x) = 1, \quad x \in G; \quad \varphi_0(x) = 0, \quad x \in \partial G.
\]

Set \( \mu_0 = \int_G \varphi_0(x) dx \). Thus, we have

**Lemma 3.1.** If \( \mu_0 > 1/a_0 \), then the positive solution of (3.1) blows up in finite time.

**Proof.** Set \( F(t) = \int_G w^{1-r} \varphi_0 dx \); then

\[
\frac{1}{1-r} F'(t) = d \left( \int_G \Delta w \varphi_0 dx + a_0 \int_G w dx \int_G \varphi_0 dx \right)
\]

\[
\geq d(a_0 \mu_0 - 1) \int_G w dx
\]

\[
\geq d(a_0 \mu_0 - 1) \left( \int_G w \varphi_0 dx \right) / M,
\]

where \( M = \max_{x \in G} \{ \varphi_0(x) \} \). Letting \( z = w^{1-r} \) in (3.2) yields

\[
\int_G z_t(x, t) \varphi_0 dx \geq d(1 - r)(a_0 \mu_0 - 1) \left( \int_G z^{1/(1-r)} \varphi_0 dx \right) / M.
\]

Since \( \frac{1}{1-r} > 1 \), by the Jensen inequality, it follows that

\[
\int_G z_t(x, t) \varphi_0 dx \geq d(1 - r)(a_0 \mu_0 - 1)(\mu_0)^{r/(1-r)} \left( \int_G z \varphi_0 dx \right)^{1/(1-r)} / M.
\]
That is,
\[ F'(t) \geq C_0 (F(t))^{1/(1-r)}, \]
where \( C_0 = d(1-r)(\alpha_0 \mu_0 - 1)/(\mu_0)^{r/(1-r)} / M > 0 \). In view of \( 1/(1-r) > 1 \) and \( F(0) > 0 \), it follows that there exists \( T < \infty \) such that \( \lim_{t \to T} F(t) = +\infty \), and hence \( w(x,t) \) blows up in finite time. \( \square \)

**Lemma 3.2.** Assume that (H1) holds. Then there exist positive constants \( k_1, k_2 \) such that \( u(x,t) \geq k_1 \varphi, v(x,t) \geq k_2 \varphi \) for \( (x,t) \in \overline{\Omega} \times [0,T^*) \) if \( \mu^2 \geq 1/(ab) \).

**Proof.** From (H1), since \( \mu^2 \geq 1/(ab) \) we see that there exist positive constants \( k_1 \) and \( k_2 \) such that
\[
(3.3) \quad u_0(x) \geq k_1 \varphi(x), \quad v_0(x) \geq k_2 \varphi(x), \quad x \in \overline{\Omega},
\]
and
\[
(3.4) \quad \alpha \mu \geq k_1/k_2 \geq 1/(b\mu). \tag{3.4}
\]
Let \( w(x,t) = u(x,t) - k_1 \varphi(x), s(x,t) = v(x,t) - k_2 \varphi(x) \). Then we obtain, by (3.4), for any \( T \in (0,T^*) \),
\[
w_t = u_t = v^p \left( \Delta u + a \int_{\Omega} v \, dx \right)
= v^p \left( \Delta w + a \int_{\Omega} s \, dx \right) + v^p (-k_1 + ak_2\mu)
\geq v^p \left( \Delta w + a \int_{\Omega} s \, dx \right),
\]
\[
s_t \geq v^q \left( \Delta s + b \int_{\Omega} w \, dx \right), \quad x \in \Omega, 0 < t \leq T,
\]
\[ w(x,t) = s(x,t) = 0, \quad x \in \partial \Omega, 0 < t \leq T. \]

By Proposition 2.1 it follows from (3.3) and (3.4) that \( w \geq 0, s \geq 0 \) and hence \( u \geq k_1 \varphi, v \geq k_2 \varphi \) on \( \overline{\Omega} \times [0,T) \). The arbitrariness of \( T \) shows that the result holds. \( \square \)

**Lemma 3.3.** Assume that (H1) holds. Then no global solution of (1.2) exists if \( \mu^2 > 1/(ab) \).

**Proof.** Denote by \( \varphi_1(x) \) the unique positive solution of the linear elliptic problem
\[
-\Delta \varphi_1(x) = 1, \quad x \in \Omega_1; \quad \varphi_1(x) = 0, \quad x \in \partial \Omega_1.
\]
Here \( \Omega_1 \subset \subset \Omega \). Since the function \( U := \varphi - \varphi_1 \geq 0 \) is harmonic in \( \Omega_1 \) and satisfies \( U \leq \varphi \) on \( \partial \Omega_1 \), we have \( \| \varphi - \varphi_1 \|_{\infty} \leq \| \varphi \|_{L^\infty(\partial \Omega_1)} \) by the maximum principle. By the continuity of \( \varphi \) it follows that \( \| \varphi - \varphi_1 \|_{\infty} \to 0 \), as \( \text{dist}(\partial \Omega_1, \partial \Omega) \to 0 \).

Let \( \mu_1 = \int_{\Omega_1} \varphi_1(x) \, dx. \) The above discussion implies, in particular, \( \mu_1 \to \mu \), as \( \text{dist}(\partial \Omega_1, \partial \Omega) \to 0 \).

Therefore, in view of \( \mu^2 > 1/(ab) \), we can choose a smooth sub-domain \( \Omega_1 \subset \subset \Omega \) such that \( \mu_1^2 > 1/(ab) \).

Denote
\[
\delta = \frac{1}{2} \min \{ k_1 \min_{\overline{\Omega}_1} \varphi, k_2 \min_{\overline{\Omega}_1} \varphi \}.
\]
Then \( \delta > 0 \) and
\[
u(x,t) \geq 2\delta, \quad \forall (x,t) \in \overline{\Omega}_1 \times [0,T^*),
\]

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by Lemma 3.2. Then \((u, v)\) in \(\Omega_1 \times (0, T^*)\) satisfies

\[
u_t = v^p \left( \Delta u + a \int_{\Omega} v \, dx \right) \geq v^p \left( \Delta u + a \int_{\Omega_1} v \, dx \right),
\]

\[
u_t \geq u^q \left( \Delta v + b \int_{\Omega} u \, dx \right), \quad x \in \Omega_1, t \in (0, T^*),
\]

\[
u(u, T) = \int_{\Omega_1} u \, dx = 0,
\]

(3.6)

\[
u(x, 0) \geq 2\delta, \quad \nu(x, T) \geq 2\delta, \quad x \in \Omega_1, t \in (0, T^*),
\]

\[
u(x, 0) \geq 2\delta, \quad \nu(x, 0) \geq 2\delta, \quad x \in \Omega_1.
\]

Now, we consider the system

\[
u = u^p \left( \Delta u + a \int_{\Omega_1} v \, dx \right),
\]

\[
u = u^q \left( \Delta v + b \int_{\Omega_1} u \, dx \right), \quad x \in \Omega_1, t > 0,
\]

(3.7)

\[
u(u, t) = f(t), \quad \nu(v, t) = g(t), \quad x \in \partial \Omega_1, t > 0,
\]

where \(f(t), g(t)\) satisfy

\[f(t), g(t) \in C^\infty([0, \infty)), f'(t), g'(t) > 0, f(t), g(t) \leq 2\delta,\]

\[f(0) = g(0) = \delta, f'(0) = a|\Omega_1|\delta^{p+1}, g'(0) = b|\Omega_1|\beta^{q+1}.\]

A similar discussion as in [7] shows that there exists a unique classical solution \((u, v) \in C^{2+\beta, 1+\beta/2}(\Omega_1 \times [0, T_1])\) for some \(\beta \in (0, 1)\), where \(T_1\) is the maximal existence time, and

\[u, v \geq \delta \quad \text{in} \quad \overline{\Omega_1} \times [0, T_1].\]

Since the initial data is a subsolution of (3.7), we have \(u^*, v^* \geq 0\) in \(\overline{\Omega_1} \times [0, T_1]\) and hence

\[\Delta u + a \int_{\Omega_1} v \, dx \geq 0, \quad \Delta v + b \int_{\Omega_1} u \, dx \geq 0 \quad \text{in} \quad \overline{\Omega_1} \times [0, T_1].\]

(3.9)

Thus from Proposition 2.2 we have \(T_1 \geq T^*\) and

\[u(x, t) \geq u^*(x, t), \quad v(x, t) \geq v^*(x, t) \quad \text{in} \quad \overline{\Omega_1} \times [0, T^*].\]

Therefore, it suffices to show that \((u, v)\) blows up in finite time, because if so, its upper bound \((u, v)\) does exist up to a finite time \(T^*\).

By (3.8) and (3.9), we have

\[
u^* \geq \delta^{p-r} v^* \left( \Delta u + a \int_{\Omega_1} v \, dx \right),
\]

\[
u^* \geq \delta^{q-r} u^* \left( \Delta v + b \int_{\Omega_1} u \, dx \right) \quad \text{in} \quad \Omega_1 \times (0, T_1)
\]

with the corresponding initial and boundary conditions and \(0 < r < 1\).

By use of \(\mu_1^2 > 1/(ab)\), there exist positive constants \(l_1, l_2, l\) with \(l_1, l_2 > 1\), and \(l\) such that

\[a \mu_1 > \frac{l_1}{l_2} > \frac{1}{b \mu_1}, \quad \mu_1 > \frac{1}{l}, \quad \frac{l_1}{l_2} > \frac{1}{l_1}, \quad \mu_1 > \frac{l}{b \mu_1}.\]

(3.11)
Choose
\begin{equation}
(3.12) \quad d = \min\{\delta^p - \gamma, \delta^q - \gamma\}, \quad \gamma = \min\{1/l_1, 1/l_2\}.
\end{equation}

Denote by $z(x, t)$ the unique positive solution of the problem
\begin{equation}
(3.13) \quad z_t = dz^r \left( \Delta z + l \int_{\Omega_1} z \, dx \right), \quad x \in \Omega_1, t > 0,
\end{equation}
\begin{equation}
z(x, t) = \gamma \delta, \quad x \in \partial \Omega_1, t \geq 0,
\end{equation}
\begin{equation}
z(x, 0) = \gamma \delta, \quad x \in \Omega_1,
\end{equation}
where $l, d, \gamma$ satisfy (3.11) and (3.12). By Lemma 3.1, it follows that $z(x, t)$ blows up in finite time $T_0 < \infty$. Moreover, $z_t \geq 0$, i.e., $\Delta z + l \int_{\Omega_1} z \, dx \geq 0$, since the initial data is a subsolution of (3.13). Let
\begin{equation}
w(x, t) = l_1 z(x, t), \quad s(x, t) = l_2 z(x, t).
\end{equation}
Thus, from (3.11)–(3.13) and $l_1, l_2 > 1$, we have
\begin{equation}
w_t - \delta^p - \gamma \left( \Delta w + a \int_{\Omega_1} \, s \, dx \right) = l_1 dz^r \left( \Delta z + l \int_{\Omega_1} z \, dx \right)
\end{equation}
\begin{equation}
- l_1 \delta^p - \gamma (l_2 z)^r \left( \Delta z + (al_2/l_1) \int_{\Omega_1} z \, dx \right) \leq 0,
\end{equation}
\begin{equation}
s_t - \delta^q - \gamma \left( \Delta s + b \int_{\Omega_1} \, w \, dx \right) \leq 0, \quad x \in \Omega_1, 0 < t < T_0,
\end{equation}
\begin{equation}
w(x, t) = l_1 \gamma \delta \leq \delta, \ s(x, t) = l_2 \gamma \delta \leq \delta, \ x \in \partial \Omega_1, 0 \leq t < T_0,
\end{equation}
\begin{equation}
w(x, 0) = l_1 \gamma \delta \leq \delta, \ s(x, 0) = l_2 \gamma \delta \leq \delta, \ x \in \Omega_1.
\end{equation}
By use of Proposition 2.2, it follows from (3.8), (3.10), (3.11) and $\Delta z + l \int_{\Omega_1} z \, dx \geq 0$ that
\begin{equation}
(u, v) \geq (l_1 z, l_2 z) \text{ in} \ \Omega_1 \times (0, T_1).
\end{equation}
Hence $(u, v)$ blows up in finite time since $z(x, t)$ does. Therefore, $(u, v)$ exists no later than $T_0 < \infty$. This completes the proof. \hfill \Box

**Lemma 3.4.** Assume that (H1) holds. Then the positive solution $(u, v)$ of (1.2) defined by (2.5) is global if $\mu^2 \leq 1/(ab)$.

**Proof.** Applying $\mu^2 \leq 1/(ab)$ and (H1) we see that there exist large positive constants $K_1$ and $K_2$ such that
\begin{equation}
(3.15) \quad a \mu \leq K_1 \leq 1/(b \mu)
\end{equation}
and
\begin{equation}
(3.16) \quad u_0(x) \leq K_1 \varphi(x), \quad v_0(x) \leq K_2 \varphi(x), \quad \forall x \in \Omega.
\end{equation}
Let \( W(x, t) = K_1 \varphi(x) - u(x, t) \), \( S(x, t) = K_2 \varphi(x) - v(x, t) \). Then, from (3.15), we obtain, for any \( T \in (0, T^*) \),
\[
W_t = -u_t = -v^p \left( \Delta u + a \int_\Omega v \, dx \right) \\
= v^p \left( \Delta W + a \int_\Omega S \, dx \right) + v^p (K_1 - aK_2 \mu) \\
\geq v^p \left( \Delta W + a \int_\Omega S \, dx \right),
\]
(3.17)
\[
S_t \geq u^q \left( \Delta S + b \int_\Omega W \, dx \right), \quad x \in \Omega, 0 < t \leq T,
\]
\[
W(x, t) = S(x, t) = 0, \quad x \in \partial \Omega, 0 < t \leq T.
\]
By Proposition 2.1, it follows from (3.16) and (3.17) that \( W \geq 0, S \geq 0 \) and hence \( u \leq K_1 \varphi, v \leq K_2 \varphi \) on \( \Omega \times [0, T] \). The arbitrariness of \( T \) shows that \( u \leq K_1 \varphi, v \leq K_2 \varphi \) on \( \Omega \times [0, T^*) \). Therefore, the solution \((u, v)\) of (1.2) defined by (2.5) exists globally.

From Lemma 3.3 and Lemma 3.4 it follows that Theorem 1.2 holds.

**Remark 3.5.** From Lemma 3.2 and Lemma 3.4, we have that if \( \mu^2 = 1/(ab) \), there exist positive constants \( k_1, k_2, K_1 \) and \( K_2 \) such that \( k_1 \varphi \leq u(x, t) \leq K_1 \varphi, k_2 \varphi \leq v(x, t) \leq K_2 \varphi \) for \( x \in \Omega \) and \( t > 0 \).

**Remark 3.6.** Theorem 1.3 for system (1.6) can be proved by combining the present arguments (for system (1.2)) with those in [1] (for system (1.1)).

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