

## ON QUASI-AFFINE TRANSFORMS OF READ'S OPERATOR

THOMAS SCHLUMPRECHT AND VLADIMIR G. TROITSKY

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ABSTRACT. We show that C. J. Read's example of an operator  $T$  on  $\ell_1$  which does not have any non-trivial invariant subspaces is not the adjoint of an operator on a predual of  $\ell_1$ . Furthermore, we present a bounded diagonal operator  $D$  such that even though  $D^{-1}$  is unbounded, the operator  $D^{-1}TD$  is a bounded operator on  $\ell_1$  with invariant subspaces, and is adjoint to an operator on  $c_0$ .

### 1. INTRODUCTION

In this note we deal with the Invariant Subspace Problem, the problem of the existence of a closed non-trivial invariant subspace for a given bounded operator on a Banach space. The problem was solved in the positive for certain classes of operators (see [RR73, AAB98] for details), however in the mid-seventies P. Enflo [Enf76, Enf87] constructed an example of a continuous operator on a Banach space with no invariant subspaces, thus answering the Invariant Subspace Problem for general Banach spaces in the negative. In [Read85] C. J. Read presented an example of a bounded operator  $T$  on  $\ell_1$  with no invariant subspace. Recently V. Lomonosov suggested that every adjoint operator has an invariant subspace. In the first part of this note we show that the Read operator  $T$  is not an adjoint of any bounded operator defined on some predual of  $\ell_1$ .

Suppose that  $A$  has a non-trivial invariant (or a hyperinvariant) subspace, and suppose that  $B$  is similar to  $A$ , that is,  $B = CAC^{-1}$  for some invertible operator  $C$ . Clearly,  $B$  also has a non-trivial invariant (respectively hyperinvariant) subspace. Moreover, it is known (see [RR73, Theorem 6.19]) that if  $A$  has a hyperinvariant subspace and  $B$  is quasi-similar to  $A$  (that is,  $CA = BC$  and  $AD = DB$ , where  $C$  and  $D$  are two bounded one-to-one operators with dense range), then  $B$  also has a hyperinvariant subspace. To our knowledge it is still unknown whether or not  $A$  has a non-trivial invariant subspace if and only if  $B$  has a non-trivial invariant subspace, assuming  $A$  and  $B$  are quasi-similar.

Recall (cf. [Sz-NF68]) that an operator  $A$  is said to be a *quasi-affine transform of  $B$*  if  $CA = BC$ , for some injective operator  $C$  with dense range. In the second part of this paper we construct an injective diagonal operator  $D$  on  $\ell_1$  such that even though  $D^{-1}$  is unbounded, the operator  $S = D^{-1}TD$  ( $T$  being Read's operator)

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is bounded and has an invariant subspace. Thus, we show that a quasi-affine transform of an operator with no non-trivial invariant subspace might have a non-trivial invariant subspace. Furthermore,  $S$  is the adjoint of a bounded operator on  $c_0$ .

Although we prove our statement for a specific choice of  $D$ , it is true for a much more general choice, and it seems to be true for any diagonal operator  $D$  that  $S = D^{-1}TD$  has a non-trivial invariant subspace, whenever  $S$  is an adjoint of an operator on  $c_0$ . More generally, the following question is of interest in view of the above-mentioned conjecture by V. Lomonosov.

**Question.** Does every quasi-affine transform of Read’s operator, which is an adjoint of an operator on  $c_0$ , have a non-trivial invariant subspace?

We introduce the following notations. Following [Read86] we denote by  $F$  the vector space of all eventually vanishing scalar sequences, and by  $(f_i)$  the standard unit vector basis of  $F$ . For an  $x = \sum a_i f_i \in F$ , we define the *support of  $x$*  to be the set  $\{i \in \mathbb{N} : a_i \neq 0\}$  and denote it by  $\text{supp}(x)$ . The linear span of some subset  $A$  of a vector space is denoted by  $\text{lin } A$ .

2. READ’S OPERATOR IS NOT ADJOINT

We begin by reminding the reader of the construction of the operator  $T$  in [Read85, Read86]. It depends on a strictly increasing sequence  $\mathbf{d} = (a_1, b_1, a_2, b_2, \dots)$  of positive integers which has to be chosen to be *sufficiently rapidly increasing*. Also let  $a_0 = 1, v_0 = 0$ , and  $v_n = n(a_n + b_n)$  for  $n \geq 1$ .

Read’s operator  $T$  is defined by prescribing the orbit  $(e_i)_{i \geq 0}$  of the first basis element  $f_0$ .

**Definition 2.1.** There is a unique sequence  $(e_i)_{i=0}^\infty \subset F$  with the following properties:

- (0)  $f_0 = e_0$ ;
- (A) if integers  $r, n$ , and  $i$  satisfy  $0 < r \leq n, i \in [0, v_{n-r}] + ra_n$ , we have
 
$$f_i = a_{n-r}(e_i - e_{i-ra_n});$$
- (B) if integers  $r, n$ , and  $i$  satisfy  $1 \leq r < n, i \in (ra_n + v_{n-r}, (r + 1)a_n)$ , (respectively,  $1 \leq n, i \in (v_{n-1}, a_n)$ ), then
 
$$f_i = 2^{(h-i)/\sqrt{a_n}} e_i, \text{ where } h = (r + \frac{1}{2})a_n \text{ (respectively, } h = \frac{1}{2}a_n);$$
- (C) if integers  $r, n$ , and  $i$  satisfy  $1 \leq r \leq n, i \in [r(a_n + b_n), na_n + rb_n]$ , then
 
$$f_i = e_i - b_n e_{i-b_n};$$
- (D) if integers  $r, n$ , and  $i$  satisfy  $0 \leq r < n, i \in (na_n + rb_n, (r + 1)(a_n + b_n))$ , then
 
$$f_i = 2^{(h-i)/\sqrt{b_n}} e_i, \text{ where } h = (r + \frac{1}{2})b_n.$$

Indeed, since  $f_i = \sum_{j=0}^i \lambda_{ij} e_j$  for each  $i \geq 0$  and  $\lambda_{ii}$  is always nonzero, this linear relation is invertible. Further,

$$\text{lin}\{e_i \mid i = 1, \dots, n\} = \text{lin}\{f_i \mid i = 1, \dots, n\} \text{ for every } n \geq 0.$$

In particular, all  $e_i$  are linearly independent and also span  $F$ . Then Read defines  $T: F \rightarrow F$  to be the unique linear map such that  $Te_i = e_{i+1}$ . Read proves that  $T$  can be extended to a bounded operator on  $\ell_1$  with no invariant subspaces provided  $\mathbf{d}$  increases sufficiently rapidly.

**Proposition 2.2.** *T is not the adjoint of an operator S : X → X where X is a Banach space whose dual is isometric to ℓ<sub>1</sub>.*

*Proof.* Assume that our claim is not true. Then there is a local convex topology τ on ℓ<sub>1</sub> so that

- (a) τ is weaker than the norm topology of ℓ<sub>1</sub>;
- (b) B(ℓ<sub>1</sub>) is sequentially compact with respect to τ;
- (c) if (x<sub>n</sub>) ⊂ ℓ<sub>1</sub> converges with respect to τ to x, then  $\liminf_{n \rightarrow \infty} \|x_n\| \geq \|x\|$ ;
- (d) T is continuous with respect to τ.

Note that with respect to any predual X of ℓ<sub>1</sub> the weak\* topology has properties (a)–(d). Let s ∈ ℕ be fixed, and n > s. Then  $f_{(n-s)a_n} = a_s(e_{(n-s)a_n} - e_0)$  by (A) above. It follows that  $T^{v_s+1}f_{(n-s)a_n} = a_s(e_{(n-s)a_n+v_s+1} - e_{v_s+1})$ . Further, it follows from (B) that  $e_{(n-s)a_n+v_s+1}$  equals  $2^{(1+v_s-\frac{1}{2}a_n)/\sqrt{a_n}}f_{(n-s)a_n+v_s+1}$  and converges to zero in norm (and, hence, in τ) as n → ∞. Therefore

$$(1) \quad \tau\text{-}\lim_{n \rightarrow \infty} T^{v_s+1}f_{(n-s)a_n} = -a_s e_{v_s+1} = T^{v_s+1}(-a_s e_0).$$

Notice that  $T^{v_s+1}$  is τ-continuous and one-to-one because its null space is T-invariant. By sequential compactness of B(ℓ<sub>1</sub>), the sequence  $f_{(n-s)a_n}$  must have a τ-convergent subsequence. Then, by (1), the limit point has to be  $-a_s e_0$ . Since that argument applies to any subsequence, we deduce that

$$(2) \quad \tau\text{-}\lim_{n \rightarrow \infty} f_{(n-s)a_n} = -a_s e_0.$$

Since  $\|f_{(n-s)a_n}\| = 1$  for each n and s while  $\|a_s e_0\| = a_s > 1$ , this contradicts (2). □

*Remark.* The statement of the theorem remains valid if we consider an equivalent norm on ℓ<sub>1</sub>. Indeed, suppose  $\frac{1}{K}\|\cdot\| \leq \|\cdot\| \leq K\|\cdot\|$ . Then  $\|f_{(n-s)a_n}\| \leq K$  for each n and s, but since  $\lim_{n \rightarrow \infty} a_n = \infty$ , we can choose a<sub>s</sub> in (2) so that  $\|a_s e_0\| > K$ .

### 3. AN ADJOINT OPERATOR WITH INVARIANT SUBSPACES OF THE FORM D<sup>-1</sup>TD

Define a sequence of positive reals (d<sub>i</sub>) as follows:

$$(3) \quad d_i = \begin{cases} \frac{1}{r} & \text{if } ra_m \leq i \leq ra_m + v_{m-r} \text{ for some } 0 < r \leq m, \\ 1 & \text{otherwise.} \end{cases}$$

Let D be the diagonal operator with diagonal (d<sub>i</sub>), that is,  $Df_i = d_i f_i$  for every i. Define  $S = D^{-1}TD$ . Clearly, S is defined on F. Once we write S in matrix form it will be clear that it is bounded on F and, therefore, can be extended to ℓ<sub>1</sub>. Let  $\hat{e}_i = D^{-1}e_i$ , in particular  $\hat{e}_0 = e_0$ . Then  $S\hat{e}_i = D^{-1}Te_i = \hat{e}_{i+1}$ , so that the sequence ( $\hat{e}_i$ ) is the orbit of e<sub>0</sub> under S.

Next, we examine Definition 2.1 to represent the f<sub>i</sub>'s in terms of  $\hat{e}_i$ 's.

( $\hat{0}$ )  $f_0 = e_0 = \hat{e}_0$ ;

( $\hat{A}$ ) if i satisfies  $i \in [0, v_{n-r}] + ra_n$  for some  $0 < r \leq n$ , then

$$f_i = d_i D^{-1}f_i = d_i D^{-1}(a_{n-r}(e_i - e_{i-ra_n})) = \frac{a_{n-r}}{r}(\hat{e}_i - \hat{e}_{i-ra_n});$$

( $\hat{B}$ ) if integers r, n, and i satisfy  $1 \leq r < n$ ,  $i \in (ra_n + v_{n-r}, (r+1)a_n)$ , (respectively,  $1 \leq n$ ,  $i \in (v_{n-1}, a_n)$ ), then

$$f_i = d_i D^{-1}f_i = 2^{(h-i)/\sqrt{a_n}}\hat{e}_i, \text{ where } h = (r + \frac{1}{2})a_n \text{ (respectively, } h = \frac{1}{2}a_n);$$

( $\widehat{C}$ ) if integers  $r, n$ , and  $i$  satisfy  $1 \leq r \leq n, i \in [r(a_n + b_n), na_n + rb_n]$ , then

$$f_i = d_i D^{-1} f_i = \hat{e}_i - b_n \hat{e}_{i-b_n};$$

( $\widehat{D}$ ) if integers  $r, n$ , and  $i$  satisfy  $0 \leq r < n, i \in (na_n + rb_n, (r + 1)(a_n + b_n))$ , then

$$f_i = d_i D^{-1} f_i = 2^{(h-i)/\sqrt{b_n}} \hat{e}_i, \text{ where } h = (r + \frac{1}{2})b_n.$$

We see that it differs from Definition 2.1 only in case ( $\widehat{A}$ ). Now we can actually write the matrix of  $S$ :

$$Sf_i = \begin{cases} 2^{(1-\frac{1}{2}a_1)/\sqrt{a_1}} f_1 & \text{if } i = 0, \\ f_{i+1} & \text{if } i \in [0, v_{n-r}) + ra_n, \\ & \text{with } r = 1, 2, \dots, n, \\ f_{i+1} & \text{if } i \in [r(a_n + b_n), na_n + rb_n), \\ & \text{with } r = 1, 2, \dots, n, \\ 2^{1/\sqrt{a_n}} f_{i+1} & \text{if } i \in (ra_n + v_{n-r}, (r + 1)a_n - 1), \\ & \text{with } r = 1, 2, \dots, n - 1 \\ & \text{or } i \in (v_{n-1}, a_n - 1), \\ 2^{1/\sqrt{b_n}} f_{i+1} & \text{if } i \in (na_n + rb_n, (r + 1)(a_n + b_n) - 1) \\ & \text{with } r = 0, 1, \dots, n - 1, \\ \frac{a_{n-r}}{r} (\varepsilon_1 f_{i+1} - \varepsilon_2 f_{v_{n-r}+1}) & \text{if } i = ra_n + v_{n-r}, \\ & \text{with } r = 1, 2, \dots, n, \\ \text{where} & \\ \varepsilon_2 = 2^{(1+v_{n-r}-\frac{1}{2}a_{n-r+1})/\sqrt{a_{n-r+1}}} & \\ \varepsilon_1 = 2^{(1+v_{n-r}-\frac{1}{2}a_n)/\sqrt{a_n}} & \text{if } r < n \text{ and} \\ \varepsilon_1 = 2^{(1+na_n-\frac{1}{2}b_n)/\sqrt{b_n}} & \text{if } r = n, \\ 2^{(1-\frac{1}{2}a_n)/\sqrt{a_n}} [f_0 + \frac{(r+1)f_{i+1}}{a_{n-r-1}}] & \text{if } i = (r + 1)a_n - 1 \\ & \text{with } r = 0, 1, \dots, n - 1, \\ \varepsilon_1 f_{i+1} - b_n \varepsilon_2 f_{i+1-b_n} & \text{if } i = na_n + rb_n \\ \text{where} & \text{with } r = 1, 2, \dots, n, \\ \varepsilon_2 = 2^{(1+na_n-\frac{1}{2}b_n)/\sqrt{b_n}} & \\ \varepsilon_1 = 2^{(1+na_n-\frac{1}{2}b_n)/\sqrt{b_n}} & \text{if } r < n, \text{ and} \\ \varepsilon_1 = 2^{(v_n+1-\frac{1}{2}a_{n+1})/\sqrt{a_{n+1}}} & \text{if } r = n, \\ 2^{-((r+1)a_n+\frac{1}{2}b_n-1)/\sqrt{b_n}} & \text{if } i = (r + 1)(a_n + b_n) - 1 \\ \cdot \left[ \sum_{j=0}^r b_n^j f_{i-jb_n+1} \right. & \\ \left. + b_n^{r+1} (f_0 + \frac{(r+1)f_{(r+1)a_n}}{a_{n-r-1}}) \right] & \text{with } r = 0, 1, \dots, n - 1. \end{cases}$$

Inspecting the matrix line by line we observe that, assuming  $(a_n)$  and  $(b_n)$  are increasing sufficiently rapidly, it follows that  $\|S\| \leq 2$ . Again by inspecting each line of the matrix, we deduce that if  $f_j^*$  is the  $j$ -th coordinate functional on  $\ell_1$ ,  $j \geq 0$ , it follows that  $\lim_{i \rightarrow \infty} f_j^*(S(f_i)) = 0$ . In other words, the rows of the matrix converge to zero. Therefore  $S$  is the adjoint of a linear bounded operator on  $c_0$ .

**Theorem 3.1.** *S has a non-trivial closed invariant subspace.*

We shall show that  $S$  has an invariant subspace by producing a vector  $x_\infty$  such that the linear span of the orbit of  $x_\infty$  stays away from  $e_0$ , hence its closure is a non-trivial  $S$ -invariant subspace.

We will introduce the following notations.

First we choose two sequences of positive integers  $(m_i)$  and  $(r_i)$  as follows. Let  $m_0 \geq 2$  be arbitrary, put  $r_0 = 1$ . Once  $m_i$  and  $r_i$  are defined, choose  $r_{i+1} \in \mathbb{N}$  so that

$$(4) \quad r_{i+1} \in [a_{m_i-1} \cdot \max_{\ell \leq v_{m_i-1}} \|\hat{e}_\ell\|, 1 + a_{m_i-1} \cdot \max_{\ell \leq v_{m_i-1}} \|\hat{e}_\ell\|]$$

and let

$$(5) \quad m_{i+1} = m_i + r_{i+1}.$$

Define an increasing sequence  $(j_i)$  of positive integers inductively: pick any

$$(6) \quad j_0 \in [r_0 a_{m_0}, r_0 a_{m_0} + v_{m_0-r_0}],$$

and once  $j_i$  is defined, put

$$(7) \quad j_{i+1} = j_i + r_i b_{m_i} + r_{i+1} a_{m_{i+1}}.$$

Finally, for each  $i \geq 0$  define

$$(8) \quad p_i = \prod_{k=0}^i b_{m_k}^{-r_k},$$

$$(9) \quad z_i = f_{j_i+r_i b_{m_i}} + b_{m_i} f_{j_i+(r_i-1)b_{m_i}} + \dots + b_{m_i}^{r_i-1} f_{j_i+b_{m_i}} + \frac{r_{i+1} f_{j_{i+1}}}{a_{m_i}},$$

$$(10) \quad x_i = p_{i-1} \hat{e}_{j_i}.$$

We note the following easy-to-prove properties for our choices.

**Proposition 3.2.** *For each  $i \geq 0$  the following statements hold:*

- (a)  $j_i \in [r_i a_{m_i}, r_i a_{m_i} + v_{m_i-r_i}]$ ;
- (b)  $x_{i+1} = x_i + p_i z_i$ , and thus  $x_i = \hat{e}_{j_0} + \sum_{k=0}^{i-1} p_k z_k$ ;
- (c) if  $i$  and  $i + \ell$  both belong to  $[r a_n, r a_n + v_{n-r}]$  or if they both belong to  $[r(a_n + b_n), n a_n + r b_n]$ , then  $S^\ell f_i = f_{i+\ell}$ ;
- (d) if  $\ell < m_i a_{m_i} - j_i$ , then  $\min \text{supp } S^\ell z_k \geq j_i + b_{m_i}$  whenever  $k \geq i$ .

*Proof.* (a) The proof is by induction. For  $i = 0$  the required inclusion follows from the choice of  $j_0$ , and if this condition holds for  $j_i$ , then

$$\begin{aligned} j_{i+1} &= j_i + r_i b_{m_i} + r_{i+1} a_{m_{i+1}} \\ &\in [r_i a_{m_i} + r_i b_{m_i} + r_{i+1} a_{m_{i+1}}, r_i a_{m_i} + v_{m_i-r_i} + r_i b_{m_i} + r_{i+1} a_{m_{i+1}}] \\ &\subseteq [r_{i+1} a_{m_{i+1}}, r_{i+1} a_{m_{i+1}} + m_i(a_{m_i} + b_{m_i})] = [r_{i+1} a_{m_{i+1}}, r_{i+1} a_{m_{i+1}} + v_{m_i}]. \end{aligned}$$

(b) First note that by using  $(\widehat{D})$  we obtain for a  $i \in [r(a_n + b_n), n a_n + r b_n]$ , with  $1 \leq r \leq n$  in  $\mathbb{N}$ , that

$$\begin{aligned} (11) \quad \hat{e}_i &= b_n \hat{e}_{i-b_n} + f_i \\ &= b_n^2 \hat{e}_{i-2b_n} + b_n f_{i-b_n} + f_i \\ &\vdots \\ &= b_n^r \hat{e}_{i-rb_n} + b_n^{r-1} f_{i-(r-1)b_n} + \dots + b_n f_{i-b_n} + f_i. \end{aligned}$$

Note that  $j_i + r_i b_{m_i} \in [r_i(a_{m_i} + b_{m_i}), m_i a_{m_i} + r_i b_{m_i}]$ . By using first  $(\widehat{A})$  and then (11) we obtain

$$\begin{aligned} \hat{e}_{j_{i+1}} &= \hat{e}_{j_i+r_i b_{m_i}+r_{i+1} a_{m_i}} \\ &= \hat{e}_{j_i+r_i b_{m_i}} + \frac{r_{i+1}}{a_{m_i}} f_{j_i+r_i b_{m_i}+r_{i+1} a_{m_i}} \\ &= b_{m_i}^{r_i} \hat{e}_{j_i} + b_{m_i}^{r_i-1} f_{j_i+b_{m_i}} + \dots + b_{m_i} f_{j_i+(r_i-1)b_{m_i}} + \frac{r_{i+1}}{a_{m_i}} f_{j_i+r_i b_{m_i}+r_{i+1} a_{m_i}} \\ &= b_{m_i}^{r_i} \hat{e}_{j_i} + z_i. \end{aligned}$$

Thus,  $x_{i+1} = p_i \hat{e}_{j_{i+1}} = p_{i-1} \hat{e}_{j_i} + p_i z_i = x_i + p_i z_i$ .

(c) If  $i$  and  $i + \ell$  are both in  $[ra_n, ra_n + v_{n-r}]$ , it follows from  $(\widehat{A})$  that

$$S^\ell(f_i) = \frac{a_{n-r}}{r} S^\ell(\hat{e}_i - \hat{e}_{i-ra_n}) = \frac{a_{n-r}}{r} (\hat{e}_{i+\ell} - \hat{e}_{i-ra_n+\ell}) = f_{i+\ell}.$$

The second part of (c) can be deduced in a similar way using  $(\widehat{C})$ .

(d) First note that for  $k \geq i$  it follows that (recall that  $m_k \geq m_0 \geq 2$ )

$$m_k a_{m_k} - j_k > (m_k - r_k - 1)a_{m_k} = (m_{k-1} - 1)a_{m_k} \geq m_{k-1} a_{m_{k-1}} - j_{k-1}.$$

We can therefore assume that  $k = i$ . Furthermore, note that for any  $1 \leq r \leq r_i$  it follows that

$$r(a_{m_i} + b_{m_i}) \leq j_i + r b_{m_i} \leq j_i + r b_{m_i} + \ell \leq m_i a_{m_i} + r b_{m_i}$$

and

$$\begin{aligned} r_{i+1} a_{m_{i+1}} &\leq j_{i+1} \leq j_{i+1} + \ell \leq j_{i+1} + m_i a_{m_i} - j_i \\ &= r_{i+1} a_{m_{i+1}} + r_i b_{m_i} + m_i a_{m_i} \\ &\leq r_{i+1} a_{m_{i+1}} + v_{m_i} \\ &= r_{i+1} a_{m_{i+1}} + v_{m_{i+1}-r_{i+1}}. \end{aligned}$$

Therefore the claim follows from the definition of  $z_i$ , (9) and part (c). □

Notice that

$$\|z_i\| = 1 + b_{m_i} + b_{m_i}^2 + \dots + b_{m_i}^{r_i-1} + \frac{r_{i+1}}{a_{m_i}} \leq m_i b_{m_i}^{r_i-1} + \frac{r_{i+1}}{a_{m_i}}.$$

Further, since  $p_i \leq \frac{1}{b_{m_i}^{r_i}}$ , we have

$$\|p_i z_i\| \leq \frac{m_i}{b_{m_i}} + \frac{r_{i+1}}{a_{m_i} b_{m_i}^{r_i}}.$$

The series  $\sum_{i=0}^\infty \frac{m_i}{b_{m_i}}$  converges because  $(b_i)$  increases sufficiently rapidly. Secondly, it follows from the definition of  $(r_i)$  that

$$a_{m_i}^{-1} r_{i+1} \leq a_{m_i}^{-1} [1 + a_{m_{i-1}} \cdot \max_{\ell \leq v_{m_{i-1}}} \|\hat{e}_\ell\|].$$

Thus, again since  $(b_i)$  is increasing fast enough, it follows that the series

$$\sum_{i=0}^\infty \frac{r_{i+1}}{a_{m_i} b_{m_i}^{r_i}}$$

converges. Therefore the  $\sum_{i=0}^\infty p_i z_i$  converges, and the following definition is justified.

**Definition 3.3.** Define  $x_\infty = \lim_i x_i = \lim_i p_{i-1} \hat{e}_{j_i} = \hat{e}_{j_0} + \sum_{i=0}^\infty p_i z_i$ .

Now we can state and prove the key result for proving Theorem 3.1.

**Lemma 3.4.** *There exists a constant  $C > 0$  such that  $\text{dist}(y, e_0) \geq C$  for every  $i$  and every vector of the form  $y = \sum_{j=j_i}^{m_i a_{m_i}} \gamma_j \hat{e}_j$ .*

*Proof.* Let  $C = \inf \left\{ \text{dist}(y, e_0) \mid y = \sum_{j=j_0}^{m_0 a_{m_0}} \gamma_j \hat{e}_j \right\}$ . Since the infimum is taken over a finite-dimensional set, it must be attained at some  $y_0$ . However since all  $\hat{e}_j$  are linear independent, it follows that  $C = \text{dist}(y_0, e_0) > 0$ .

We shall prove the statement of the lemma by induction on  $i$ . The way we defined  $C$  guarantees that the base of the induction holds. Suppose  $y = \sum_{j=j_i}^{m_i a_{m_i}} \gamma_j \hat{e}_j$ . Write  $y = y_1 + y_2 + y_3$ , where

$$y_1 = \sum_{j=j_i}^{r_i a_{m_i} + v_{m_i-1}} \gamma_j \hat{e}_j, \quad y_2 = \sum_{r=r_i+1}^{m_i} \sum_{j=r a_{m_i}}^{r a_{m_i} + v_{m_i-r}} \gamma_j \hat{e}_j, \quad \text{and} \quad y_3 = \sum_{r=r_i}^{m_i-1} \sum_{j=r a_{m_i} + v_{m_i-r+1}}^{(r+1)a_{m_i}-1} \gamma_j \hat{e}_j.$$

Notice that by  $(\widehat{B})$

$$y_3 = \sum_{r=r_i}^{m_i-1} \sum_{j=r a_{m_i} + v_{m_i-r+1}}^{(r+1)a_{m_i}-1} \gamma_j 2^{-(r+\frac{1}{2}-j)/\sqrt{a_{m_i}}} f_j,$$

so that  $\text{supp } y_3 \subseteq \bigcup_{r=r_i}^{m_i-1} (r a_{m_i} + v_{m_i-r}, (r+1)a_{m_i})$ . Furthermore, using  $(\widehat{A})$ , we write  $y_2 = y'_2 + y''_2$  where

$$y'_2 = \sum_{r=r_i+1}^{m_i} \sum_{j=r a_{m_i}}^{r a_{m_i} + v_{m_i-r}} \gamma_j \hat{e}_{j-r a_{m_i}} = \sum_{r=r_i+1}^{m_i} \sum_{j=0}^{v_{m_i-r}} \gamma_{j+r a_{m_i}} \hat{e}_j$$

$$\text{and} \quad y''_2 = \sum_{r=r_i+1}^{m_i} \sum_{j=r a_{m_i}}^{r a_{m_i} + v_{m_i-r}} \frac{\gamma_j r}{a_{m_i-r}} f_j.$$

Therefore,

$$\text{supp}(y_1 + y_2) \subseteq [0, r_i a_{m_i} + v_{m_i-1}] \cup \bigcup_{r=r_i+1}^{m_i} [r a_{m_i}, r a_{m_i} + v_{m_i-r}].$$

One observes that the vectors  $y_1 + y_2$  and  $y_3$  have disjoint supports; it follows that  $\text{dist}(y, e_0) \geq \text{dist}(y_1 + y_2, e_0)$ .

Furthermore,

$$\|y'_2\| = \left\| \sum_{r=r_i+1}^{m_i} \sum_{j=r a_{m_i}}^{r a_{m_i} + v_{m_i-r}} \gamma_j \hat{e}_{j-r a_{m_i}} \right\| \leq \sum_{r=r_i+1}^{m_i} \sum_{j=r a_{m_i}}^{r a_{m_i} + v_{m_i-r}} |\gamma_j| \cdot \max_{k \leq v_{m_i-1}-1} \|\hat{e}_k\|.$$

By choice of  $(r_i)$  (4), we have  $\max_{k \leq v_{m_i-1}-1} \|\hat{e}_k\| \leq \frac{r_i}{a_{m_i-r_i-1}} \leq \frac{r}{a_{m_i-r}}$  when  $r_i < r \leq m_i$ .

This yields

$$\|y'_2\| \leq \left\| \sum_{r=r_i+1}^{m_i} \sum_{j=r a_{m_i}}^{r a_{m_i} + v_{m_i-r}} \frac{\gamma_j r}{a_{m_i-r}} f_j \right\| = \|y''_2\|.$$

Since the support of  $y_2''$  is disjoint from that of  $y_1 + y_2'$  and doesn't contain 0, we have

$$\begin{aligned} \text{dist}(y_1, e_0) &\leq \text{dist}(y_1 + y_2', e_0) + \|y_2'\| \\ &= \text{dist}(y_1 + y_2' + y_2'', e_0) - \|y_2''\| + \|y_2'\| \\ &\leq \text{dist}(y_1 + y_2, e_0) \leq \text{dist}(y, e_0). \end{aligned}$$

It is left to show that  $\text{dist}(y_1, e_0) \geq C$ . Since  $j_i \geq r_i a_{m_i}$ , it follows from  $(\widehat{A})$  that  $y_1 = y_1' + y_1''$  where

$$y_1' = \sum_{j=j_i}^{r_i a_{m_i} + v_{m_{i-1}}} \gamma_j \hat{e}_{j-r_i a_{m_i}} \quad \text{and} \quad y_1'' = \sum_{j=j_i}^{r_i a_{m_i} + v_{m_{i-1}}} \frac{\gamma_j r}{a_{m_i-r_i}} f_j.$$

Since  $j_i = j_{i-1} + r_{i-1} b_{m_{i-1}} + r_i a_{m_i}$ , we have  $y_1' = \sum_{j=j_{i-1} + r_{i-1} b_{m_{i-1}}}^{v_{m_{i-1}}} \beta_j \hat{e}_j$ , where  $\beta_j = \gamma_{j+r_i a_{m_i}}$ . In particular this means that  $\text{supp } y_1' \subseteq [0, v_{m_{i-1}}]$ , while  $\min \text{supp } y_1'' \geq j_i \geq r_i a_{m_i}$ . Thus, the supports are disjoint, which yields  $\text{dist}(y_1, e_0) \geq \text{dist}(y_1', e_0)$ .

Split the index set of  $y_1'$  into two disjoint subsets: let

$$\begin{aligned} A &= [j_{i-1} + r_{i-1} b_{m_{i-1}}, v_{m_{i-1}}] \cap \bigcup_{r=r_{i-1}}^{m_{i-1}} (m_{i-1} a_{m_{i-1}} + r b_{m_{i-1}}, (r+1)(a_{m_{i-1}} + b_{m_{i-1}})), \\ B &= [j_{i-1} + r_{i-1} b_{m_{i-1}}, v_{m_{i-1}}] \cap \bigcup_{r=r_{i-1}}^{m_{i-1}} [r(a_{m_{i-1}} + b_{m_{i-1}}), m_{i-1} a_{m_{i-1}} + r b_{m_{i-1}}]. \end{aligned}$$

Write  $y_1' = z_a + z_b$  where  $z_a = \sum_{j \in A} \beta_j \hat{e}_j$  and  $z_b = \sum_{j \in B} \beta_j \hat{e}_j$ . For  $j \in A$  we have  $\hat{e}_j = 2^{((r+1/2)b_{m_{i-1}} - j)/\sqrt{b_{m_{i-1}}}} f_j$ , so that  $\text{supp } z_a \subseteq A$ . In view of (11) we can write  $z_b = z_b' + z_b''$ , where

$$z_b' = \sum_{j \in B} \sum_{k=0}^{r-1} \beta_j b_{m_{i-1}}^k f_{j-kb_{m_{i-1}}} \quad \text{and} \quad z_b'' = \sum_{j \in B} \beta_j b_{m_{i-1}}^r \hat{e}_{j-rb_{m_{i-1}}}.$$

We first note that  $\text{supp } z_b' \subseteq B$  and determine the support of  $z_b''$  as follows. If  $j \in B$ , then  $j \geq j_{i-1} + r_{i-1} b_{m_{i-1}}$  and  $j \in [r(a_{m_{i-1}} + b_{m_{i-1}}), m_{i-1} a_{m_{i-1}} + r b_{m_{i-1}}]$  for some  $r \in [r_{i-1}, m_{i-1}]$ . If  $r = r_{i-1}$ , then  $j - r b_{m_{i-1}} \geq j_{i-1}$ . If  $r > r_{i-1}$ , then  $j - r b_{m_{i-1}} \geq r a_{m_{i-1}} > r_{i-1} a_{m_{i-1}} + v_{m_{i-2}} \geq j_{i-1}$  by (7). We see that  $z_b''$  is a linear combination of  $\hat{e}_j$ 's with  $j_{i-1} \leq j \leq m_{i-1} a_{m_{i-1}}$ . Hence its support is contained in  $[0, m_{i-1} a_{m_{i-1}}]$  and, therefore, is disjoint from that of  $z_a$  and  $z_b'$ . It follows that  $\text{dist}(y, e_0) \geq \text{dist}(y_1', e_0) \geq \text{dist}(z_b'', e_0)$ . Finally,  $\text{dist}(z_b'', e_0) \geq C$  by the induction hypothesis.  $\square$

*Proof of Theorem 3.1.* We will prove that the linear span of the orbit of  $x_\infty$  under  $S$  is at least distance  $C$  from  $e_0$ , hence its closure is a non-trivial invariant subspace for  $S$ . Consider a linear combination  $\sum_{\ell=0}^N \alpha_\ell S^\ell x_\infty$ . It follows from (7) that the sequence  $(m_i a_{m_i} - j_i)$  is unbounded, so that  $N < m_i a_{m_i} - j_i$  for some  $i \geq 0$ . Recall that  $x_\infty = x_i + \sum_{k=i}^\infty p_k z_k$ ; then

$$\sum_{\ell=0}^N \alpha_\ell S^\ell x_\infty = \sum_{s=0}^N \alpha_\ell S^\ell x_i + \sum_{\ell=0}^N \sum_{k=i}^\infty \alpha_\ell S^\ell (p_k z_k).$$

Notice that the two sums have disjoint supports, and the support of the second one does not contain 0. Indeed, since  $x_i = p_{i-1}\hat{e}_{j_i}$ , then  $S^\ell x_i = p_{i-1}\hat{e}_{j_i+\ell}$  for  $\ell = 1, \dots, N$ . Furthermore,

$$j_i \leq j_i + \ell \leq j_i + N < j_i + (m_i a_{m_i} - j_i) = m_i a_{m_i}.$$

It follows that  $\sum_{\ell=0}^N S^\ell x_i$  is a linear combination of  $\hat{e}_j$ 's with  $j_i \leq j \leq m_i a_{m_i}$ . In particular, its support is contained in  $[0, m_i a_{m_i}]$ . On the other hand, Proposition 3.2 (d) implies that

$$\min \operatorname{supp} \left( \sum_{\ell=0}^N \sum_{k=i}^{\infty} S^\ell(p_k z_k) \right) \geq j_i + b_{m_i}.$$

Therefore, by Lemma 3.4

$$\operatorname{dist} \left( \sum_{\ell=0}^N S^\ell x_\infty, e_0 \right) \geq \operatorname{dist} \left( \sum_{\ell=0}^N S^\ell x_i, e_0 \right) \geq C.$$

□

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DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TEXAS 77843  
*E-mail address*: schlump@math.tamu.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA T6G 2G1  
*E-mail address*: vtroitsky@math.ualberta.ca