

INTERPOLATION FOR MULTIPLIERS ON REPRODUCING KERNEL HILBERT SPACES

VLADIMIR BOLOTNIKOV

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ABSTRACT. All solutions of a tangential interpolation problem for contractive multipliers between two reproducing kernel Hilbert spaces of analytic vector-valued functions are characterized in terms of certain positive kernels. In a special important case when the spaces consist of analytic functions on the unit ball of \mathbb{C}^d and the reproducing kernels are of the form $(1 - \langle z, w \rangle^{-1})I_p$ and $(1 - \langle z, w \rangle)^{-1}I_q$, the characterization leads to a parametrization of the set of all solutions in terms of a linear fractional transformation.

1. INTRODUCTION

A Hilbert space \mathcal{H} of $\mathbb{C}^{p \times 1}$ -valued functions which are defined on a domain $\Omega \in \mathbb{C}^d$ is said to be a *reproducing kernel Hilbert space* if there exists a $\mathbb{C}^{p \times p}$ -valued function $K(z, \omega)$ such that for every point $\omega \in \Omega$ and every vector $c \in \mathbb{C}^p$, the function $K_w c := K(\cdot, \omega)c$ belongs to \mathcal{H} and $\langle f, K_w c \rangle_{\mathcal{H}} = c^* f(\omega)$ for every function $f \in \mathcal{H}$. The function $K(z, \omega)$ turns to be positive on Ω in the sense that $\sum_{j, \ell=1}^n c_j^* K(z^{(j)}, z^{(\ell)}) c_{\ell} \geq 0$ for every choice of an integer n , of vectors $c_1, \dots, c_n \in \mathbb{C}^p$ and of points $z^{(1)}, \dots, z^{(n)} \in \Omega$ or, equivalently, if the Hermitian block matrix with ℓj -th entry $K(z^{(j)}, z^{(\ell)})$ is positive semidefinite. This property will be denoted by $K(z, \omega) \succeq 0$. The function $K(z, \omega)$ is, furthermore, uniquely defined (as is easily verified), and is called *the reproducing kernel* of \mathcal{H} . The fundamental result of Aronszajn [4] states that for every positive kernel K on Ω , there is a unique reproducing kernel Hilbert space $\mathcal{H}(K)$ with K as its reproducing kernel. Moreover, the set \mathcal{H}_0 consisting of functions of the form $\sum K(\cdot, w_j)c_j$, where $\{c_j\}$ and $\{w_j\}$ are finite sequences in \mathbb{C}^p and Ω , respectively, is a dense linear manifold in $\mathcal{H}(K)$. In what follows we shall write $K_w(z)$ rather than $K(z, w)$ if the last function will be considered as a function of z with a fixed point $w \in \Omega$.

Let $K^{(1)}(z, w)$ and $K^{(2)}(z, w)$ be two positive kernels on Ω , which are respectively, $\mathbb{C}^{q \times q}$ - and $\mathbb{C}^{p \times p}$ -valued and let $\mathcal{H}(K^{(1)})$ and $\mathcal{H}(K^{(2)})$ be the corresponding reproducing kernel Hilbert spaces. A $\mathbb{C}^{p \times q}$ -valued function S defined on Ω is called a *contractive multiplier* from $\mathcal{H}(K^{(1)})$ to $\mathcal{H}(K^{(2)})$ if the multiplication operator $\mathbf{M}_S : \mathcal{H}(K^{(1)}) \rightarrow \mathcal{H}(K^{(2)})$, defined by

$$(1.1) \quad \mathbf{M}_S(f(z)) = S(z)f(z),$$

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is a contraction (if $K^{(1)} = K^{(2)}$, then S is called a contractive multiplier on $\mathcal{H}(K^{(1)})$). The latter means that $I_{\mathcal{H}(K^{(2)})} - \mathbf{M}_S \mathbf{M}_S^* \geq 0$ and is equivalent to

$$(1.2) \quad K_S(z, w) := K^{(2)}(z, w) - S(z)K^{(1)}(z, w)S(w)^* \succeq 0 \quad (z, w \in \Omega).$$

The set of all contractive multipliers S from $\mathcal{H}(K^{(1)})$ to $\mathcal{H}(K^{(2)})$ will be denoted by $\mathcal{S}(K^{(1)}, K^{(2)})$.

In this paper we shall focus on the following interpolation

Problem 1.1. Given functions f_1, \dots, f_n in $\mathcal{H}(K^{(1)})$ and h_1, \dots, h_n in $\mathcal{H}(K^{(2)})$ find necessary and sufficient conditions which insure the existence of a function $S \in \mathcal{S}(K^{(1)}, K^{(2)})$ such that

$$(1.3) \quad (\mathbf{M}_S^* h_j)(z) = f_j(z), \quad j = 1, \dots, n.$$

We shall make frequent use of notations

$$(1.4) \quad H(z) = [h_1(z) \ \dots \ h_n(z)] \quad \text{and} \quad F(z) = [f_1(z) \ \dots \ f_n(z)],$$

which allows us to rewrite interpolation conditions (1.3) in a more compact form as

$$(1.5) \quad (\mathbf{M}_S^* H)(z) = F(z).$$

Note that the tangential Nevanlinna–Pick problem in the class $\mathcal{S}(K^{(1)}, K^{(2)})$ is a particular case of Problem 1.1. Indeed, a simple computation shows that

$$(\mathbf{M}_S^* K_w^{(2)})(z) = K_w^{(1)}(z)S(w)^*,$$

and thus a special choice of $h_j = K_{w_j}^{(2)} c_j$ and $f_j = K_{w_j}^{(1)} d_j$ in (1.3) leads to the left-sided interpolation conditions

$$K_{w_j}^{(1)}(z)S(w_j)^* c_j \equiv K_{w_j}^{(1)}(z) d_j, \quad j = 1, \dots, n.$$

Under the further assumption that $K^{(1)}$ is not degenerate (i.e., $K^{(1)}(z, z) > 0$ for all $z \in \Omega$), the latter conditions are equivalent to the classical Nevanlinna–Pick conditions

$$S(w_j)^* c_j = d_j, \quad j = 1, \dots, n.$$

In Section 2 all the solutions S of Problem 1.1 are characterized in terms of certain positive kernels constructed from the interpolation data. In Section 3 we consider a particular case of Problem 1.1 for multipliers on multivariable analogues of the Hardy space \mathbf{H}_2 of the unit disk, studied in [5]. For this case, the general result (Theorem 2.4) leads to a parametrization of the set of all solutions in terms of a linear fractional transformation. The Nevanlinna–Pick problem in this setting (see Remark 3.1) has been considered in [6].

2. THE MAIN RESULT

As mentioned above, for a function $S \in \mathcal{S}(K^{(1)}, K^{(2)})$ the kernel $K_S(z, w)$ defined in (1.2) is positive on Ω . The corresponding reproducing kernel Hilbert space will be referred to as $\mathcal{H}(S)$.

The original characterization of $\mathcal{H}(S)$, as the space of all elements functions $f \in \mathcal{H}(K^{(2)})$ such that

$$(2.1) \quad \kappa(f) := \sup_{g \in \mathcal{H}(K^{(1)})} \left\{ \|f + Sg\|_{\mathcal{H}(K^{(2)})}^2 - \|g\|_{\mathcal{H}(K^{(1)})}^2 \right\}$$

is finite and $\|f\|_{\mathcal{H}(S)}^2 = \kappa(f)$, is due to de Branges and Rovnyak [8].

On the other hand, the general complementation theory (see, e.g., [15, Ch.1]) applied to the contractive operator \mathbf{M}_S provides the characterization of $\mathcal{H}(S)$ as the operator range

$$(2.2) \quad \mathcal{H}(S) = \text{Range}(I - \mathbf{M}_S \mathbf{M}_S^*)^{\frac{1}{2}} \text{ with } \|(I - \mathbf{M}_S \mathbf{M}_S^*)^{\frac{1}{2}} h\|_{\mathcal{H}(S)} = \|(I - \pi)h\|_{\mathcal{H}_{\mathcal{H}(K^{(2)})}},$$

where π denotes the orthogonal projection onto $\text{Ker}(I - \mathbf{M}_S \mathbf{M}_S^*)^{\frac{1}{2}}$.

To state the main theorem we need some preliminary results. The first one follows immediately from the characterization (2.2) upon setting $h = (I - \mathbf{M}_S \mathbf{M}_S^*)^{\frac{1}{2}} f$.

Lemma 2.1. *Let $S \in \mathcal{S}(K^{(1)}, K^{(2)})$ and $f \in \mathcal{H}(K^{(2)})$. Then $(I_{\mathcal{H}(K^{(2)})} - \mathbf{M}_S \mathbf{M}_S^*)f$ belongs to $\mathcal{H}(S)$ and*

$$(2.3) \quad \|(I - \mathbf{M}_S \mathbf{M}_S^*)f\|_{\mathcal{H}(S)}^2 = \langle (I - \mathbf{M}_S \mathbf{M}_S^*)f, f \rangle_{\mathcal{H}(K^{(2)})}.$$

Let K be a positive $\mathbb{C}^{N \times N}$ -valued kernel on Ω and let $\mathcal{H}(K)$ be the corresponding reproducing kernel Hilbert space consisting of \mathbb{C}^N -valued vector functions. The usage of matrix-valued functions with the columns in $\mathcal{H}(K)$ prompts us to introduce (besides the standard inner product) the following bilinear form:

$$(2.4) \quad [X, Y]_{\mathcal{H}(K)} = \left(\langle x_\ell, y_j \rangle_{\mathcal{H}(K)} \right)_{\ell=1, \dots, n}^{j=1, \dots, m}$$

which makes sense for every pair of functions

$$X(z) = [x_1(z) \dots x_n(z)] \in (\mathcal{H}(K))^{1 \times n}, \quad Y(z) = [y_1(z) \dots y_m(z)] \in (\mathcal{H}(K))^{1 \times m},$$

which are respectively, $\mathbb{C}^{N \times n}$ - and $\mathbb{C}^{N \times m}$ -valued.

Remark 2.2. The form (2.4) can be viewed as the matrix representation of the operator $\mathbf{M}_Y^* \mathbf{M}_X : \mathbb{C}^n \rightarrow \mathbb{C}^m$ with respect to the standard basis, where $\mathbf{M}_X : \mathbb{C}^n \rightarrow \mathcal{H}(K)$ and $\mathbf{M}_Y : \mathbb{C}^m \rightarrow \mathcal{H}(K)$ are the multiplication operators given by

$$\mathbf{M}_X c = X(z)c \quad \text{and} \quad \mathbf{M}_Y d = Y(z)d.$$

The next preliminary lemma characterizes $\mathcal{H}(K)$ in terms of positive kernels (see [9, Theorem 2.2] for scalar-valued kernels and [2, Lemma 2.2] for the matrix case):

Lemma 2.3. *A nonzero vector-valued function f defined on Ω belongs to $\mathcal{H}(K)$ and satisfies $\|f\|_{\mathcal{H}(K)}^2 \leq \gamma$ if and only if the kernel $K(z, w) - \gamma^{-1} f(z) f(w)^*$ is positive on Ω .*

The next theorem characterizes all the solutions S of Problem 1.1 in terms of positive kernels and in terms of the reproducing kernel Hilbert spaces $\mathcal{H}(S)$. The first develops Potapov’s method (which characterizes the solutions of an interpolation problem in terms of a related fundamental matrix inequality [11]), and the second is related to reproducing kernel methods in interpolation theory [10].

Theorem 2.4. *Let $H \in (\mathcal{H}(K^{(2)}))^{1 \times n}$ and $F \in (\mathcal{H}(K^{(1)}))^{1 \times n}$ be as in (1.4), let S be a $p \times q$ matrix-valued function which is analytic in Ω , let K_S be defined by (1.2) and let*

$$(2.5) \quad P := [H, H]_{\mathcal{H}(K^{(2)})} - [F, F]_{\mathcal{H}(K^{(1)})}$$

and

$$(2.6) \quad B(z) = H(z) - S(z)F(z).$$

Then the following statements are equivalent:

- (1) S is a solution to Problem 1.1.
- (2) For every choice of $x \in \mathbb{C}^n$, the function $B(z)x$ belongs to the space $\mathcal{H}(S)$ and

$$(2.7) \quad \|Bx\|_{\mathcal{H}(S)}^2 = x^*Px.$$

- (3) The following kernel is positive on Ω :

$$(2.8) \quad \mathbf{K}(z, w) := \begin{bmatrix} P & B(w)^* \\ B(z) & K_S(z, w) \end{bmatrix} \succeq 0.$$

- (4) The following operator

$$(2.9) \quad \mathbf{P} := \begin{bmatrix} P & \mathbf{M}_B^* \\ \mathbf{M}_B & I - \mathbf{M}_S \mathbf{M}_S^* \end{bmatrix} : \begin{bmatrix} \mathbb{C}^n \\ \mathcal{H}(K^{(2)}) \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{C}^n \\ \mathcal{H}(K^{(1)}) \end{bmatrix}$$

is positive semidefinite.

Proof. (1) \Rightarrow (2). Let S be a solution to Problem 1.1. Then $\mathbf{M}_S \mathbf{M}_S^* \leq I_{\mathcal{H}(K^{(2)})}$ and (1.5) is in force. Substituting (1.5) into the right-hand side of (2.5) and (2.6) we get

$$(2.10) \quad P = [H, H]_{\mathcal{H}(K^{(2)})} - [\mathbf{M}_S^* H, \mathbf{M}_S^* H]_{\mathcal{H}(K^{(1)})} = [(I - \mathbf{M}_S \mathbf{M}_S^*)H, H]_{\mathcal{H}(K^{(2)})}$$

and

$$(2.11) \quad B(z) = H(z) - S(z)(\mathbf{M}_S^* H)(z) = (\{I - \mathbf{M}_S \mathbf{M}_S^*\} H)(z).$$

Since $Hx \in \mathcal{H}(K^{(2)})$ for every $x \in \mathbb{C}^n$, the last formula implies, by Lemma 2.1, that $Bx \in \mathcal{H}(S)$. Finally, by (2.3) and (2.10),

$$\|Bx\|_{\mathcal{H}(S)}^2 = \|(I - \mathbf{M}_S \mathbf{M}_S^*)Hx\|_{\mathcal{H}(S)}^2 = \langle (I - \mathbf{M}_S \mathbf{M}_S^*)Hx, Hx \rangle_{\mathcal{H}(K^{(2)})} = x^*Px.$$

- (2) \Rightarrow (3). By Lemma 2.3, equality (2.7) implies

$$K_S(z, w) - (x^*Px)^{-1}B(z)xx^*B(w)^* \succeq 0 \quad (z, w \in \Omega)$$

for every vector $x \in \mathbb{C}^n$ such that $Px \neq 0$. The last inequality is obviously equivalent to

$$(2.12) \quad \begin{bmatrix} x^*Px & x^*B(w)^* \\ B(z)x & K_S(z, w) \end{bmatrix} \succeq 0 \quad (z, w \in \Omega).$$

If $Px = 0$, then (2.7) implies $B(z)x \equiv 0$, and thus (2.12) is in force as well. Thus, (2.12) holds for every $x \in \mathbb{C}^n$, which is equivalent to (2.8).

- (3) \Rightarrow (4). By the reproducing kernel property,

$$(\mathbf{M}_S^* K_w^{(2)})(z) = [\mathbf{M}_S^* K_w^{(2)}, K_z^{(1)}]_{\mathcal{H}(K^{(1)})} = [K_w^{(2)}, SK_z^{(1)}]_{\mathcal{H}(K^{(2)})} = K_w^{(1)}(z)S(w)^*$$

and therefore,

$$(2.13) \quad \left[(I - \mathbf{M}_S \mathbf{M}_S^*)K_w^{(2)}, K_z^{(2)} \right]_{\mathcal{H}(K^{(2)})} = K_w^{(2)}(z) - S(z)K_w^{(1)}(z)S(w)^* = K_S(z, w),$$

which shows, in particular, that the kernel K_S is positive. Fix a vector $f \in \mathbb{C}^n \oplus \mathcal{H}(K^{(2)})$ of the form

$$(2.14) \quad f = \sum_{j=1}^r \begin{bmatrix} c_j \\ K_{w^{(j)}}^{(2)} d_j \end{bmatrix} \quad (c_j \in \mathbb{C}^n, d_j \in \mathbb{C}^p, w^{(j)} \in \Omega).$$

By (2.13),

$$\left\langle (I - \mathbf{M}_S \mathbf{M}_S^*) K_{w^{(\ell)}}^{(2)} d_\ell, K_{w^{(j)}}^{(2)} d_j \right\rangle_{\mathcal{H}(K^{(2)})} = d_j^* K_S(w^{(j)}, w^{(\ell)}) d_\ell$$

and by the reproducing kernel property,

$$\left\langle \mathbf{M}_B c_\ell, K_{w^{(j)}}^{(2)} d_j \right\rangle_{\mathcal{H}(K^{(2)})} = d_j^* B(w^{(j)}) c_\ell.$$

Using the two last equalities and taking into account partitionings (2.8) and (2.9) of \mathbf{K} and \mathbf{P} we get

$$\left\langle \mathbf{P} \begin{bmatrix} c_\ell \\ K_{w^{(\ell)}}^{(2)} d_\ell \end{bmatrix}, \begin{bmatrix} c_j \\ K_{w^{(j)}}^{(2)} d_j \end{bmatrix} \right\rangle_{\mathbb{C}^n \oplus \mathcal{H}(K^{(2)})} = \begin{bmatrix} c_j^* & d_j^* \end{bmatrix} \mathbf{K}(w^{(j)}, w^{(\ell)}) \begin{bmatrix} c_\ell \\ d_\ell \end{bmatrix}.$$

By linearity and in view of (2.14),

$$\langle \mathbf{P} f, f \rangle_{\mathbb{C}^n \oplus \mathcal{H}(K^{(2)})} = \sum_{j,\ell=1}^r \begin{bmatrix} c_j^* & d_j^* \end{bmatrix} \mathbf{K}(w^{(j)}, w^{(\ell)}) \begin{bmatrix} c_\ell \\ d_\ell \end{bmatrix}.$$

Since the kernel $\mathbf{K}(z, w)$ is positive on Ω , the expression on the right-hand side of the last equality is nonnegative. Thus, $\langle \mathbf{P} f, f \rangle_{\mathbb{C}^n \oplus \mathcal{H}(K^{(2)})} \geq 0$ for every vector f of the form (2.14). Since the set of all such vectors is dense in $\mathbb{C}^n \oplus \mathcal{H}(K^{(2)})$, \mathbf{P} is positive semidefinite.

(4) \Rightarrow (1). If \mathbf{P} is positive semidefinite, then in particular, $\mathbf{M}_S \mathbf{M}_S^* \leq I$ and therefore, $S \in \mathcal{S}(K^{(1)}, K^{(2)})$. It remains to show that the interpolation condition (1.5) is valid. To this end let us consider the block operator

$$\widehat{\mathbf{P}} = \begin{bmatrix} I_{\mathcal{H}(K^{(1)})} & \mathbf{M}_F & \mathbf{M}_S^* \\ \mathbf{M}_F^* & \mathbf{M}_H^* \mathbf{M}_H & \mathbf{M}_H^* \\ \mathbf{M}_S & \mathbf{M}_H & I_{\mathcal{H}(K^{(2)})} \end{bmatrix} : \begin{bmatrix} \mathcal{H}(K^{(1)}) \\ \mathbb{C}^n \\ \mathcal{H}(K^{(2)}) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}(K^{(1)}) \\ \mathbb{C}^n \\ \mathcal{H}(K^{(2)}) \end{bmatrix}.$$

Here we use a somewhat sloppy notation: the domain and range of a multiplication operator \mathbf{M}_X depends on the X . Specifically, we have

$$\mathbf{M}_F : \mathbb{C}^n \rightarrow \mathcal{H}(K^{(1)}), \quad \mathbf{M}_H : \mathbb{C}^n \rightarrow \mathcal{H}(K^{(2)})$$

with therefore

$$\mathbf{M}_F^* : \mathcal{H}(K^{(1)}) \rightarrow \mathbb{C}^n, \quad \mathbf{M}_H^* : \mathcal{H}(K^{(2)}) \rightarrow \mathbb{C}^n,$$

but

$$\mathbf{M}_S : \mathcal{H}(K^{(1)}) \rightarrow \mathcal{H}(K^{(2)})$$

with therefore

$$\mathbf{M}_S^* : \mathcal{H}(K^{(2)}) \rightarrow \mathcal{H}(K^{(1)}).$$

Taking advantage of Remark 2.2, we obtain from (2.5) and (2.6) the representations

$$P = \mathbf{M}_H^* \mathbf{M}_H - \mathbf{M}_F^* \mathbf{M}_F, \quad \mathbf{M}_B = \mathbf{M}_H - \mathbf{M}_S \mathbf{M}_F,$$

which allow us to conclude that the operator \mathbf{P} given in (2.9) is the Schur complement of the (1, 1) block entry of $\widehat{\mathbf{P}}$:

$$\mathbf{P} = \begin{bmatrix} \mathbf{M}_H^* \mathbf{M}_H & \mathbf{M}_H^* \\ \mathbf{M}_H & I_{\mathcal{H}(K^{(2)})} \end{bmatrix} - \begin{bmatrix} \mathbf{M}_F^* \\ \mathbf{M}_S \end{bmatrix} [\mathbf{M}_F \quad \mathbf{M}_S^*].$$

Since \mathbf{P} is positive semidefinite, it follows that $\widehat{\mathbf{P}} \geq 0$. Therefore the Schur complement of the (3, 3) block entry of $\widehat{\mathbf{P}}$ is positive semidefinite:

$$\begin{aligned} & \begin{bmatrix} I_{\mathcal{H}(K^{(1)})} & \mathbf{M}_F \\ \mathbf{M}_F^* & \mathbf{M}_H^* \mathbf{M}_H \end{bmatrix} - \begin{bmatrix} \mathbf{M}_S^* \\ \mathbf{M}_H^* \end{bmatrix} [\mathbf{M}_S \quad \mathbf{M}_H] \\ &= \begin{bmatrix} I - \mathbf{M}_S^* \mathbf{M}_S & \mathbf{M}_F - \mathbf{M}_S^* \mathbf{M}_H \\ \mathbf{M}_F^* - \mathbf{M}_H^* \mathbf{M}_S & 0 \end{bmatrix} \geq 0. \end{aligned}$$

The last relation implies that $\mathbf{M}_F - \mathbf{M}_S^* \mathbf{M}_H = 0$, which is equivalent to (1.5). \square

3. EXAMPLE

In this section we apply the preceding analysis to a class $\mathcal{S}^{p \times q}$ of $\mathbb{C}^{p \times q}$ -valued functions S analytic in the unit ball $\mathbb{B}^d = \{z = (z_1, \dots, z_d) \in \mathbb{C}^d : \sum_1^d |z_j|^2 < 1\}$ of \mathbb{C}^d and such that

$$(3.1) \quad K_S(z, w) = \frac{I_p - S(z)S(w)^*}{1 - \langle z, w \rangle} \succeq 0 \quad (z, w \in \mathbb{B}^d).$$

The Nevanlinna–Pick problem for these functions (in the operator-valued version) has been recently considered in [6]. It was shown that every solution of the problem corresponds to a unitary extension of a partially defined isometric operator, which led to a parametrization of all solutions given in terms of a Redheffer linear fractional transformation. We shall pose a more general interpolation problem and, upon including it in the general scheme of Problem 1.1, shall get a different parametrization of all its solutions.

We shall use standard notations: points in \mathbb{C}^d will be denoted by $z = (z_1, \dots, z_d)$, where $z_j \in \mathbb{C}$ and $\langle z, w \rangle = \sum_{j=1}^d z_j \bar{w}_j$ will stand for the standard inner product in \mathbb{C}^d . For multiindices $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ we set

$$n_1 + n_2 + \dots + n_d = |\mathbf{n}|, \quad n_1! n_2! \dots n_d! = \mathbf{n}!, \quad z_1^{n_1} z_2^{n_2} \dots z_d^{n_d} = z^{\mathbf{n}}.$$

The kernel

$$(3.2) \quad \mathbf{k}(z, w) = \frac{1}{1 - \langle z, w \rangle}$$

is positive on \mathbb{B}^d . It can be shown (see, e.g., [5, Lemma 3.8]) that in the metric of $\mathcal{H}(\mathbf{k})$,

$$\langle z^{\mathbf{n}}, z^{\mathbf{m}} \rangle_{\mathcal{H}(\mathbf{k})} = \begin{cases} \frac{\mathbf{n}!}{|\mathbf{n}|!} & \text{if } \mathbf{n} = \mathbf{m}, \\ 0 & \text{otherwise,} \end{cases}$$

which leads to the following characterization of $\mathcal{H}(\mathbf{k})$:

$$(3.3) \quad \mathcal{H}(\mathbf{k}) = \left\{ f(z) = \sum_{\mathbf{n} \in \mathbb{N}^d} f_{\mathbf{n}} z^{\mathbf{n}}, \text{ with } \|f\|_{\mathcal{H}(\mathbf{k})}^2 = \sum_{\mathbf{n} \in \mathbb{N}^d} \frac{\mathbf{n}!}{|\mathbf{n}|!} |f_{\mathbf{n}}|^2 < \infty \right\}.$$

The space $\mathcal{H}(\mathbf{k}I_p)$ can be viewed as the tensor product Hilbert space $\mathcal{H}(\mathbf{k}) \otimes \mathbb{C}^{p \times 1}$ and we denote it $\mathcal{H}^p(\mathbf{k})$ for short. Similarly, we use the notation $\mathcal{H}^{p \times q}(\mathbf{k})$ for the space of $\mathbb{C}^{p \times q}$ -valued functions with entries in $\mathcal{H}(\mathbf{k})$. Note that the bilinear form defined in (2.4) takes in this context the form

$$(3.4) \quad [H, F]_{\mathcal{H}(\mathbf{k})} = \sum_{\mathbf{n} \in \mathbb{N}^d} \frac{\mathbf{n}!}{|\mathbf{n}|!} F_{\mathbf{n}}^* H_{\mathbf{n}},$$

and makes sense for every choice of $H \in \mathcal{H}^{p \times m}(\mathbf{k})$ and $F \in \mathcal{H}^{p \times \ell}(\mathbf{k})$.

The kernel K_S defined in (3.1) is a particular case of (1.5) corresponding to the particular choice of $K^{(1)} = \mathbf{k}I_q$ and $K^{(2)} = \mathbf{k}I_p$, and condition (3.1) means that S is a contractive multiplier from $\mathcal{H}^q(\mathbf{k})$ to $\mathcal{H}^p(\mathbf{k})$.

Let matrices $C_1 \in \mathbb{C}^{p \times m}$, $C_2 \in \mathbb{C}^{q \times m}$ and $A_1, \dots, A_d \in \mathbb{C}^{m \times m}$ be such that

1. The joint spectrum of A_1, \dots, A_d sits inside \mathbb{B}^d :

$$(3.5) \quad \sigma_{\text{joint}}(A_1, \dots, A_d) \subset \mathbb{B}^d.$$

2. For every two products (“words”) $W_{\mathbf{n}}(A_1, \dots, A_d)$ and $W'_{\mathbf{n}}(A_1, \dots, A_d)$ containing the same number n_j of a letter A_j (for all $j = 1, \dots, d$),

$$(3.6) \quad \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} W_{\mathbf{n}}(A_1, \dots, A_d) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} W'_{\mathbf{n}}(A_1, \dots, A_d)$$

(for instance, the last condition is clearly satisfied if $A_{\ell}A_j = A_jA_{\ell}$ ($\ell, j = 1, \dots, d$)). We set

$$(3.7) \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix} \quad \text{and} \quad Z(z) = [z_1 I_m \quad z_2 I_m \quad \dots \quad z_d I_m],$$

and we shall use the shorthand notation

$$(3.8) \quad C\mathbf{A}^{\mathbf{n}} = CA_1^{n_1} A_2^{n_2} \dots A_d^{n_d}$$

for a multiindex $\mathbf{n} = (n_1, n_2, \dots, n_d)$, when the order of multipliers is not essential. We assume furthermore that

3. The series $\sum_{\mathbf{n} \in \mathbb{N}^d} \frac{|\mathbf{n}|!}{\mathbf{n}!} (\mathbf{A}^{\mathbf{n}})^* C^* C \mathbf{A}^{\mathbf{n}}$ converges.

Note that the assumptions 1 and 3 provide that the function

$$(3.9) \quad G(z) = (I_m - Z(z)\mathbf{A})^{-1}$$

is analytic in $\overline{\mathbb{B}}_d$ and belongs to $\mathcal{H}^{m \times m}(\mathbf{k})$. Making use of notation (3.8) one can write

$$(3.10) \quad CG(z) = C \left(I_m - \sum_{j=1}^d z_j A_j \right)^{-1} = C \sum_{\mathbf{n} \in \mathbb{N}^d} \frac{|\mathbf{n}|!}{\mathbf{n}!} \mathbf{A}^{\mathbf{n}} z^{\mathbf{n}}.$$

The symbol $\mathbf{IP}(\mathbf{A}, C)$ will be used to denote the following interpolation problem:

Given \mathbf{A}, C and G as above, find necessary and sufficient conditions which insure the existence of a function $S \in \mathcal{S}^{p \times q}$ such that

$$(3.11) \quad \mathbf{M}_S^*(C_1 G(z)) = C_2 G(z)$$

and describe the set of all such functions.

Remark 3.1. It can be easily seen that the particular choice of

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} b_1 & \dots & b_m \\ c_1 & \dots & c_m \end{bmatrix}, \quad A_j = \begin{bmatrix} \bar{w}_1^{(j)} & & \\ & \ddots & \\ & & \bar{w}_m^{(j)} \end{bmatrix} \quad (j = 1, \dots, d)$$

in (3.11) for prescribed m points $w^{(\ell)} = (w_1^{(\ell)}, \dots, w_d^{(\ell)}) \in \mathbb{B}^d$ and vectors $b_\ell \in \mathbb{C}^p$, $c_\ell \in \mathbb{C}^q$, leads to left-sided Nevanlinna–Pick conditions $b_\ell^* S(w^{(\ell)}) = c_\ell^*$ ($\ell = 1, \dots, m$).

The **IP**(\mathbf{A}, C) can be included in the general scheme of Problem 1.1 upon setting

$$(3.12) \quad H(z) = C_1 G(z) \in \mathcal{H}^{p \times m}(\mathbf{k}) \quad \text{and} \quad F(z) = C_2 G(z) \in \mathcal{H}^{q \times m}(\mathbf{k}).$$

Then the function B defined in (2.6) takes the form $B(z) = (C_1 - S(z)C_2)G(z)$. By Theorem 2.4 we get that a $\mathbb{C}^{p \times q}$ -valued function S analytic in \mathbb{B}^d is a solution to the **IP**(\mathbf{A}, C) if and only if

$$(3.13) \quad \mathbf{K}(z, w) := \begin{bmatrix} P & G(w)^*(C_1^* - C_2^* S(w)^*) \\ (C_1 - S(z)C_2)G(z) & \frac{I_p - S(z)S(w)^*}{1 - \langle z, w \rangle} \end{bmatrix} \succeq 0 \quad (z, w \in \mathbb{B}^d),$$

where

$$(3.14) \quad P := [C_2 G, C_2 G]_{\mathcal{H}(\mathbf{k})} - [C_1 G, C_1 G]_{\mathcal{H}(\mathbf{k})}.$$

Making use of the signature matrix

$$(3.15) \quad J = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$$

we can represent P as

$$(3.16) \quad P = [JCG, CG]_{\mathcal{H}(\mathbf{k})}.$$

In contrast to the general case, inequality (3.13) admits a nice description of all its solutions. First we note that the Pick matrix P defined in (3.14) satisfies the generalized Stein equation

$$(3.17) \quad P - \sum_{j=1}^d A_j^* P A_j = C^* J C.$$

Indeed, due to conditions (3.6) one can solve (3.17) iteratively to get a (unique) solution P of (3.17) in the form of a uniformly converging series

$$P = \sum_{\mathbf{n} \in \mathbb{N}^d} \frac{|\mathbf{n}|!}{\mathbf{n}!} (\mathbf{A}^{\mathbf{n}})^* C^* J C \mathbf{A}^{\mathbf{n}}.$$

On the other hand, substituting the Taylor expansion (3.10) for G into the left-hand side in (3.16) and making use of (3.4) we come to the same expression for P :

$$\begin{aligned} [JCG, CG]_{\mathcal{H}(\mathbf{k})} &= \left[JC \sum_{\mathbf{n} \in \mathbb{N}^d} \frac{|\mathbf{n}|!}{\mathbf{n}!} \mathbf{A}^{\mathbf{n}} z^{\mathbf{n}}, C \sum_{\mathbf{n} \in \mathbb{N}^d} \frac{|\mathbf{n}|!}{\mathbf{n}!} \mathbf{A}^{\mathbf{n}} z^{\mathbf{n}} \right]_{\mathcal{H}(\mathbf{k})} \\ &= \sum_{\mathbf{n} \in \mathbb{N}^d} \frac{|\mathbf{n}|!}{\mathbf{n}!} (\mathbf{A}^{\mathbf{n}})^* C^* J C \mathbf{A}^{\mathbf{n}}. \end{aligned}$$

Let us assume that P is positive definite. Then the $\mathbb{C}^{(p+q) \times (md+p+q)}$ -valued function

$$(3.18) \quad \Theta(z) = \begin{bmatrix} 0 & I_{p+q} \\ +CG(z)P^{-1} [(z_1I_m - A_1^*)P^{\frac{1}{2}} & \dots & (z_dI_m - A_d^*)P^{\frac{1}{2}} & -C^*J] \end{bmatrix}$$

is analytic in \mathbb{B}^d and satisfies

$$(3.19) \quad \frac{J - \Theta(z)\mathbf{J}\Theta(w)^*}{1 - \langle z, w \rangle} = CG(z)P^{-1}G(w)^*C^*, \quad \text{where } \mathbf{J} = \begin{bmatrix} I_{md} & 0 \\ 0 & J \end{bmatrix},$$

for every choice of $z = (z_1, \dots, z_d)$ and $w = (w_1, \dots, w_d)$ in \mathbb{B}_d . Indeed, it follows readily from (3.19) that

$$(3.20) \quad J - \Theta(z)\mathbf{J}\Theta(w)^* = CG(z)P^{-1}T(z, w)P^{-1}G(w)^*C,$$

where

$$T(z, w) = \left(I - \sum_{j=1}^d \bar{w}_j A_j^* \right) P + P \left(I - \sum_{j=1}^d z_j A_j \right) - \sum_{j=1}^d (z_j I - A_j^*) P (\bar{w}_j I - A_j) - C^* J C.$$

Making use of (3.17), we get

$$T(z, w) = 2P - \sum_{j=1}^d (z_j \bar{w}_j P + A_j^* P A_j) - C^* J C = (1 - \langle z, w \rangle) P,$$

which together with (3.20) imply (3.19).

Still assuming that P is positive definite, we conclude that (3.13) is equivalent to

$$\frac{I_p - S(z)S(w)^*}{1 - \langle z, w \rangle} - (C_1 - S(z)C_2)G(z)P^{-1}G(w)^*(C_1^* - C_2^*S(w)^*) \succeq 0 \quad (z, w \in \mathbb{B}^d),$$

which in its turn, can be written as

$$[I_p \quad -S(z)] \left\{ \frac{J}{1 - \langle z, w \rangle} - CG(z)P^{-1}G(w)^*C \right\} \begin{bmatrix} I_p \\ -S(w)^* \end{bmatrix} \succeq 0 \quad (z, w \in \mathbb{B}^d).$$

Taking advantage of (3.19), we rewrite the last inequality as

$$(3.21) \quad [I_p \quad -S(z)] \frac{\Theta(z)\mathbf{J}\Theta(w)^*}{1 - \langle z, w \rangle} \begin{bmatrix} I_p \\ -S(w)^* \end{bmatrix} \succeq 0 \quad (z, w \in \mathbb{B}^d).$$

Theorem 3.2. *Let P be positive definite and let*

$$(3.22) \quad \Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} : \begin{bmatrix} \mathbb{C}^{md+p} \\ \mathbb{C}^q \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{C}^p \\ \mathbb{C}^q \end{bmatrix}$$

be the partition of the function Θ given by (3.18) into four blocks of the indicated sizes. Then the set of all solutions S of the $\mathbf{IP}(\mathbf{A}, C)$ are parametrized by the linear fractional transformation

$$(3.23) \quad S(z) = (\Theta_{11}(z)\mathcal{E}(z) + \Theta_{12}(z)) (\Theta_{21}(z)\mathcal{E}(z) + \Theta_{22}(z))^{-1},$$

when the parameter \mathcal{E} varies on the set $\mathcal{S}^{(md+p) \times q}$.

Proof. It follows from (3.19) and (3.22) that

$$-I_q - \Theta_{21}(z)\Theta_{21}(z)^* + \Theta_{22}(z)\Theta_{22}(z)^* \geq 0 \quad (z \in \mathbb{B}^d).$$

Therefore, $\Theta_{22}(z)$ is invertible at every point $z \in \mathbb{B}^d$ and $\|\Theta_{22}(z)^{-1}\Theta_{21}(z)\| < 1$. Then the function

$$\Theta_{21}(z)\mathcal{E}(z) + \Theta_{22}(z) = \Theta_{22}(z)(\Theta_{22}(z)^{-1}\Theta_{21}(z)\mathcal{E}(z) + I_q)$$

is invertible in \mathbb{B}^d for every $\mathcal{E} \in \mathcal{S}^{(md+p) \times q}$, which means that the transformation (3.2) is well defined on the set $\mathcal{S}^{(md+p) \times q}$.

According to the preceding analysis S is a solution of the $\mathbf{IP}(\mathbf{A}, C)$ if and only if it satisfies the inequality (3.21). Setting

$$(3.24) \quad \begin{bmatrix} u(z) & -v(z) \end{bmatrix} = \begin{bmatrix} I_p & -S(z) \end{bmatrix} \Theta(z)$$

where u and v are respectively, $\mathbb{C}^{p \times (md+p)}$ - and $\mathbb{C}^{p \times q}$ -valued, one can rewrite (3.21) as

$$\frac{u(z)u(w)^* - v(z)v(w)^*}{1 - \langle z, w \rangle} \succeq 0 \quad (z, w \in \mathbb{B}^d),$$

which is equivalent (see, e.g., [3, Theorem 3.1]) to the existence of a function $\mathcal{E} \in \mathcal{S}^{(md+p) \times q}$ such that

$$u(z)\mathcal{E}(z) = v(z) \quad (z \in \mathbb{B}^d).$$

By (3.24), we conclude that S is a solution of the $\mathbf{IP}(\mathbf{A}, C)$ if and only if

$$\begin{bmatrix} I_p & -S(z) \end{bmatrix} \Theta(z) = u(z) \begin{bmatrix} I_{md+p} & -\mathcal{E}(z) \end{bmatrix}$$

for some bounded analytic function u and a function $\mathcal{E} \in \mathcal{S}^{(md+p) \times q}$. The latter is equivalent to

$$\begin{bmatrix} I_p & -S(z) \end{bmatrix} \Theta(z) \begin{bmatrix} \mathcal{E}(z) \\ I_q \end{bmatrix} = 0,$$

which, being rewritten as

$$\Theta_{11}(z)\mathcal{E}(z) + \Theta_{12}(z) - S(z)(\Theta_{21}(z)\mathcal{E}(z) + \Theta_{22}(z)) = 0,$$

is evidently equivalent to (3.23). □

As a consequence of the last theorem we get that under the assumption $P > 0$, the $\mathbf{IP}(\mathbf{A}, C)$ has infinitely many solutions. Using the standard approximation argument it can be easily shown that if P is positive semidefinite, then there exists a solution to the $\mathbf{IP}(\mathbf{A}, C)$ (the questions about parametrization of all solutions and uniqueness criteria are more delicate and will be considered elsewhere). Therefore, the condition $P \geq 0$ is necessary and sufficient for the $\mathbf{IP}(\mathbf{A}, C)$ to be solvable (the necessity of this condition follows readily from (3.13)). This conclusion is not surprising in light of recent papers [1], [14], [12], [13], [6] where this result has been established (for a wide class of reproducing kernel Hilbert spaces) for the Nevanlinna–Pick problem.

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DEPARTMENT OF MATHEMATICS, THE COLLEGE OF WILLIAM AND MARY, WILLIAMSBURG, VIRGINIA 23187-8795

E-mail address: vladi@math.wm.edu