

## VAN DER WAERDEN SPACES AND HINDMAN SPACES ARE NOT THE SAME

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ABSTRACT. A Hausdorff topological space  $X$  is *van der Waerden* if for every sequence  $(x_n)_{n \in \omega}$  in  $X$  there is a converging subsequence  $(x_n)_{n \in A}$  where  $A \subseteq \omega$  contains arithmetic progressions of all finite lengths. A Hausdorff topological space  $X$  is *Hindman* if for every sequence  $(x_n)_{n \in \omega}$  in  $X$  there is an *IP-converging* subsequence  $(x_n)_{n \in FS(B)}$  for some infinite  $B \subseteq \omega$ .

We show that the continuum hypothesis implies the existence of a van der Waerden space which is not Hindman.

### 1. INTRODUCTION

A Hausdorff topological space  $X$  is *van der Waerden* if for every sequence  $(x_n)_{n \in \omega}$  in  $X$  there is a converging subsequence  $(x_n)_{n \in A}$  where  $A \subseteq \omega$  contains arithmetic progressions of all finite lengths. A Hausdorff topological space  $X$  is *Hindman* if for every sequence  $(x_n)_{n \in \omega}$  in  $X$  there is an *IP-converging* subsequence  $(x_n)_{n \in FS(B)}$  for some infinite  $B \subseteq \omega$ . The term  $FS(B)$  stands for the set of all *finite sums* (with no repetitions) over  $B$  and IP-convergence to a point  $x \in X$  means: for every neighborhood  $U$  of  $x$ , there is some  $n_0$  so that  $\{x_n : n \in FS(B \setminus \{0, 1, \dots, n_0 - 1\})\} \subseteq U$ .

The classes of van der Waerden and of Hindman spaces were introduced in [2], [3] where it was shown that each class was productive and properly contained in the class of sequentially compact spaces, and that every Hausdorff space  $X$  in which the closure of every countable set is compact and first countable is both van der Waerden and Hindman. The question was raised as to whether every Hausdorff space  $X$  is van der Waerden if and only if it is Hindman. We answer this question in the negative using the Continuum Hypothesis.

**1.1. Notation and combinatorial preliminaries.** A set  $A \subseteq \omega$  is an *AP-set* if it contains arithmetic progressions of all finite lengths. By van der Waerden's theorem [4], if an AP-set  $A$  is partitioned into finitely many parts, at least one of

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the parts is AP. Let  $\mathcal{I}_{AP}$  denote the collection of all subsets of  $\omega$  which are not AP.  $\mathcal{I}_{AP}$  is a proper ideal over  $\omega$  and a set  $A \subseteq \omega$  is AP if and only if  $A \notin \mathcal{I}_{AP}$ .

A set  $A \subseteq \omega$  is an *IP-set* if there exists an infinite set  $B \subseteq \omega$  so that  $FS(B) \subseteq A$ .  $FS(B) = \{\sum F : F \subseteq B, |F| < \aleph_0\}$ , where  $\sum F$  stands for  $\sum_{n \in F} n$ . By Hindman's theorem [1], if an IP-set  $A$  is partitioned into finitely many parts, at least one of the parts is IP. Let  $\mathcal{I}_{IP}$  denote the collection of all subsets of  $\omega$  which are not IP.  $\mathcal{I}_{IP}$  is a proper ideal over  $\omega$  and a set  $A \subseteq \omega$  is IP if and only if  $A \notin \mathcal{I}_{IP}$ .

We shall need the following lemma which relates  $\mathcal{I}_{AP}$  to  $\mathcal{I}_{IP}$ .

**Lemma 1.** *Let  $A$  be an AP set and let  $f : \omega \rightarrow \omega$ . There exists an AP set  $C \subseteq A$  such that either*

- (1)  $|f[C]| = 1$  or
- (2)  $f$  is finite-to-one on  $C$  and if  $\langle x_n \rangle_{n=0}^\infty$  enumerates  $f[C]$  in increasing order, then  $\lim_{n \rightarrow \infty} (x_{n+1} - x_n) = \infty$ .

In particular,  $f[C] \in \mathcal{I}_{IP}$ .

*Proof.* Suppose that for every AP set  $C \subseteq A$ ,  $|f[C]| > 1$ . We construct an AP set  $C \subseteq A$  for which conclusion (2) holds.

For each  $m \in \omega$ ,  $A \cap f^{-1}\{0, 1, \dots, m - 1\}$  is not an AP set because it is the finite union of sets on which  $f$  is constant, and thus  $A \setminus f^{-1}\{0, 1, \dots, m - 1\}$  is an AP set. (Here we are using the fact that when an AP set is partitioned into finitely many parts, one of these parts is an AP set.)

We inductively construct sets  $C_n$  for each  $n \in \mathbb{N}$  such that

- (a) for each  $n \in \mathbb{N}$ ,  $C_n$  is a length  $n$  arithmetic progression and
- (b) for all  $n, m \in \mathbb{N}$ , all  $x \in C_m$ , and all  $y \in C_n$ , if  $m < n$ , then  $f(y) \geq f(x) + n$  and if  $m = n$ , then either  $f(x) = f(y)$  or  $|f(x) - f(y)| \geq n$ .

Let  $C_1$  be any singleton subset of  $A$ . Let  $n \in \mathbb{N}$  and assume that we have chosen  $C_1, C_2, \dots, C_n$ . Let  $k = \max \bigcup_{i=1}^n f[C_i]$  and choose  $i \in \{0, 1, \dots, n\}$  such that  $(A \setminus f^{-1}\{0, 1, \dots, k + n\}) \cap f^{-1}[(n + 1)\omega + i]$  is an AP set. Let  $C_{n+1}$  be a length  $n + 1$  arithmetic progression contained in  $(A \setminus f^{-1}\{0, 1, \dots, k + n\}) \cap f^{-1}[(n + 1)\omega + i]$ . Given  $m \leq n + 1$ ,  $x \in C_m$ , and  $y \in C_{n+1}$ , if  $m \leq n$ , then  $f(x) \leq k$  and  $f(y) \geq k + n + 1$ , while if  $m = n + 1$ , then either  $f(x) = f(y)$  or  $|f(x) - f(y)| \geq n + 1$ .

Let  $C = \bigcup_{n=1}^\infty C_n$ . □

## 2. THE SPACE

**Lemma 2.** *Assume CH. Then there exists a maximal almost disjoint family  $\mathcal{A} \subseteq \mathcal{I}_{IP}$  so that for every AP-set  $B \subseteq \omega$  and every finite-to-one function  $f : B \rightarrow \omega$  there exists an AP-set  $C \subseteq B$  and  $A \in \mathcal{A}$  so that  $f[C] \subseteq A$ .*

*Proof.* We construct from CH an almost disjoint family  $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\} \subseteq \mathcal{I}_{IP}$  by induction on  $\alpha$ . The enumeration  $\{A_\alpha : \alpha < \omega_1\}$  may contain repetitions. Let  $\{A_n : n < \omega\} \subseteq \mathcal{I}_{IP}$  be a collection of infinite and pairwise disjoint sets.

Fix a list  $\langle (f_\alpha, B_\alpha) : \omega \leq \alpha < \omega_1 \rangle$  of all pairs  $(f, B)$  in which  $B \subseteq \omega$  is an AP-set and  $f : B \rightarrow \omega$  is a finite-to-one function.

Suppose  $\omega \leq \alpha < \omega_1$  and that  $A_\beta$  has been chosen for all  $\beta < \alpha$ . Consider the pair  $(f_\alpha, B_\alpha)$ . If there exists a finite set  $\{\beta_0, \beta_1, \dots, \beta_\ell\} \subseteq \alpha$  so that  $f_\alpha^{-1}[\bigcup_{i \leq \ell} A_{\beta_i}]$  is AP, let  $A_\alpha = A_0$ .

Otherwise, enumerate  $\alpha$  as  $\langle \beta_i : i < \omega \rangle$ , and now for all  $n < \omega$  the set  $f_\alpha^{-1}[\bigcup_{i < n} A_{\beta_i}]$  is not AP, hence  $B_\alpha \setminus f_\alpha^{-1}[\bigcup_{i < n} A_{\beta_i}]$  is AP. Let an arithmetic progression  $D_n \subseteq B_\alpha \setminus f_\alpha^{-1}[\bigcup_{i < n} A_{\beta_i}]$  of length  $n$  be chosen for all  $n$ . Then  $B' := \bigcup_{n \in \omega} D_n$  is an AP-subset of  $B_\alpha$ ,  $f_\alpha[B']$  is infinite (because  $f_\alpha$  is finite-to-one) and  $|f_\alpha[B'] \cap A_\beta| < \aleph_0$  for all  $\beta < \alpha$ . By Lemma 1 find an AP-set  $B'' \subseteq B'$ , so that  $f_\alpha[B''] \in \mathcal{I}_{IP}$ , and define  $A_\alpha = f_\alpha[B'']$ .

The family  $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$  is clearly an almost disjoint family of (infinite) sets, and  $\mathcal{A} \subseteq \mathcal{I}_{IP}$ .

Suppose now that  $B \subseteq \omega$  is an AP-set and that  $f : B \rightarrow \omega$  is finite-to-one. There is an index  $\omega \leq \alpha < \omega_1$  for which  $(B, f) = (B_\alpha, f_\alpha)$ . At stage  $\alpha$  of the construction of  $\mathcal{A}$ , either  $f^{-1}[A_{\beta_0} \cup \dots \cup A_{\beta_\ell}]$  was AP for some finite set  $\{\beta_0, \dots, \beta_\ell\} \subseteq \alpha$ , hence  $f^{-1}[A_\beta]$  was AP for some single  $\beta < \alpha$ , or else  $f^{-1}[A_\alpha]$  was AP. In either case, there is an AP-set  $C \subseteq B$  and  $A \in \mathcal{A}$  so that  $f[C] \subseteq A$ .

Finally, to verify that  $\mathcal{A}$  is maximal let an infinite set  $D \subseteq \omega$  be given and let  $f : \omega \rightarrow D$  be the increasing enumeration of  $D$ . Since there is an AP-set  $C \subseteq \omega$  and  $A \in \mathcal{A}$  so that  $f[C] \subseteq A$ , it is clear that  $D \cap A$  is infinite.  $\square$

**Theorem 3.** *Suppose CH holds. Then there exists a compact, separable van der Waerden space which is not Hindman.*

*Proof.* Let  $\mathcal{A}$  be as stated in Lemma 2. For each  $A \in \mathcal{A}$  let  $p_A \notin \omega$  be a distinct point. Define a topology  $\tau$  on  $Y = \omega \cup \{p_A : A \in \mathcal{A}\}$  by requiring that  $Z \in \tau$  if and only if for all  $p_A \in Z$  the set  $A \setminus Z$  is finite. Then for each  $A \in \mathcal{A}$ ,  $A \cup \{p_A\}$  is a compact neighborhood of  $p_A$ , so  $\tau$  is a locally compact Hausdorff topology in which  $\omega$  is a dense and discrete subspace. Let  $X = Y \cup \{p\}$  be the one-point compactification of  $\tau$ .

It was shown in [3, Theorem 10] that when  $\mathcal{A} \subseteq \mathcal{I}_{IP}$  is maximal almost disjoint, the space constructed in this way is sequentially compact but not Hindman. To keep this paper self-contained, we repeat the simple argument showing that  $X$  is not Hindman. For each  $n \in \omega$ , let  $x_n = n$  and suppose we have some infinite  $B \subseteq \omega$  such that  $(x_n)_{n \in FS(B)}$  IP-converges to  $q \in X$ . Then  $q \notin \omega$ . If  $q = p_A$  for some  $A \in \mathcal{A}$ , then  $A$  is an IP set. So  $q = p$ . By the maximality of  $\mathcal{A}$ , pick  $A \in \mathcal{A}$  such that  $A \cap B$  is infinite. But then  $X \setminus (A \cup \{p_A\})$  is a neighborhood of  $p$  and for no  $n$  does one have  $FS(B \setminus \{0, 1, \dots, n-1\}) \subseteq X \setminus (A \cup \{p_A\})$ .

We have yet to see that  $X$  is van der Waerden. Suppose  $f : \omega \rightarrow X$  is given. Let  $g : f[\omega] \rightarrow \omega$  be 1-1. By Lemma 1 we can find an AP set  $B \subseteq \omega$  so that  $(g \circ f) \upharpoonright B$  is constant or finite-to-one, and hence  $f \upharpoonright B$  is constant or finite-to-one. In the former case, the sequence  $(f(n))_{n \in B}$  is constant, and therefore converges. So assume that  $f \upharpoonright B$  is finite-to-one. Since either  $f^{-1}[\omega] \cap B$  or  $B \setminus f^{-1}[\omega]$  is AP, we may assume, by shrinking  $B$  to some AP-subset, that either  $f[B] \subseteq \omega$  or  $f[B] \subseteq X \setminus (\omega \cup \{p\})$ .

In the former case, there is some  $A \in \mathcal{A}$  and AP-set  $C \subseteq B$  so that  $f[C] \subseteq A$ . Since  $f \upharpoonright B$  is finite-to-one,  $(f(n))_{n \in C}$  converges to  $p_A$ . In the latter case, we claim that the sequence  $(f(n))_{n \in B}$  converges to  $p$ . To see this, let  $Z$  be a compact subset of  $Y$ , so that  $X \setminus Z$  is a basic neighborhood of  $p$ . Then  $Z \setminus \omega$  is finite so, since  $f \upharpoonright B$  is finite-to-one,  $(f(n))_{n \in B}$  is eventually in  $X \setminus Z$ .  $\square$

REFERENCES

[1] N. Hindman. Finite sums from sequences within cells of a partition of  $\mathbb{N}$ . *J. Comb. Theory (Series A)*, 17:1–11, 1974. MR 50:2067

- [2] M. Kojman. Van der Waerden spaces. *Proc. Amer. Math. Soc.*, 130:631–635, 2002. MR **2002i**:54018
- [3] M. Kojman. Hindman spaces. *Proc. Amer. Math. Soc.*, 130:1597–1602, 2002.
- [4] B. L. van der Waerden. Beweis eine Baudetschen Vermutung *Nieuw Arch. Wisk.*, 15:212–216, 1927.

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