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VAN DER WAERDEN SPACES AND HINDMAN SPACES ARE NOT THE SAME

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Abstract. A Hausdorff topological space X is *van der Waerden* if for every sequence $(x_n)_{n\in\omega}$ in X there is a converging subsequence $(x_n)_{n\in A}$ where $A \subseteq \omega$ contains arithmetic progressions of all finite lengths. A Hausdorff topological space X is *Hindman* if for every sequence $(x_n)_{n\in\omega}$ in X there is an *IP-converging* subsequence $(x_n)_{n\in FS(B)}$ for some infinite $B \subseteq \omega$.

We show that the continuum hypothesis implies the existence of a van der Waerden space which is not Hindman.

1. Introduction

A Hausdorff topological space X is *van der Waerden* if for every sequence $(x_n)_{n\in\omega}$ in X there is a converging subsequence $(x_n)_{n\in A}$ where $A \subseteq \omega$ contains arithmetic progressions of all finite lengths. A Hausdorff topological space X is *Hindman* if for every sequence $(x_n)_{n\in\omega}$ in X there is an *IP-converging* subsequence $(x_n)_{n\in FS(B)}$ for some infinite $B \subseteq \omega$. The term $FS(B)$ stands for the set of all *finite sums* (with no repetitions) over B and IP-convergence to a point $x \in X$ means: for every neighborhood U of x, there is some n_0 so that $\{x_n : n \in FS(B \setminus \{0, 1, \ldots, n_0 - 1\})\} \subseteq U$.

The classes of van der Waerden and of Hindman spaces were introduced in [2], [3] where it was shown that each class was productive and properly contained in the class of sequentially compact spaces, and that every Hausdorff space X in which the closure of every countable set is compact and first countable is both van der Waerden and Hindman. The question was raised as to whether every Hausdorff space X is van der Waerden if and only if it is Hindman. We answer this question in the negative using the Continuum Hypothesis.

1.1. **Notation and combinatorial preliminaries.** A set $A \subseteq \omega$ is an *AP-set* if it contains arithmetic progressions of all finite lengths. By van der Waerden's theorem [4], if an AP-set A is partitioned into finitely many parts, at least one of

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the parts is AP. Let $\mathcal{I}_{AP}$ denote the collection of all subsets of ω which are not AP. $\mathcal{I}_{AP}$ is a proper ideal over ω and a set $A \subseteq \omega$ is AP if and only if $A \notin \mathcal{I}_{AP}$.

A set $A \subseteq \omega$ is an *IP-set* if there exists an infinite set $B \subseteq \omega$ so that $FS(B) \subseteq A$. $FS(B) = \{\sum F : F \subseteq A, |F| < \aleph_0\}$, where $\sum F$ stands for $\sum_{n \in F} n$. By Hindman's theorem [1], if an IP-set A is partitioned into finitely many parts, at least one of the parts is IP. Let $\mathcal{I}_{IP}$ denote the collection of all subsets of ω which are not IP. $\mathcal{I}_{IP}$ is a proper ideal over ω and a set $A \subseteq \omega$ is IP if and only if $A \notin \mathcal{I}_{IP}$.

We shall need the following lemma which relates $\mathcal{I}_{AP}$ to $\mathcal{I}_{IP}$.

Lemma 1. *Let A be an AP set and let $f : \omega \to \omega$. There exists an AP set $C \subseteq A$ such that either*

(1) $|f[C]| = 1$ *or*
(2) *f is finite-to-one on C and if $\langle x_n \rangle_{n=0}^{\infty}$ enumerates $f[C]$ in increasing order, then* $\lim_{n\to\infty}(x_{n+1} - x_n) = \infty$.

In particular, $f[C] \in \mathcal{I}_{IP}$.

Proof. Suppose that for every AP set $C \subseteq A$, $|f[C]| > 1$. We construct an AP set $C \subseteq A$ for which conclusion (2) holds.

For each $m \in \omega$, $A \cap f^{-1}[\{0, 1, \ldots, m-1\}]$ is not an AP set because it is the finite union of sets on which f is constant, and thus $A \setminus f^{-1}[\{0, 1, \ldots, m-1\}]$ is an AP set. (Here we are using the fact that when an AP set is partitioned into finitely many parts, one of these parts is an AP set.)

We inductively construct sets C_n for each $n \in \mathbb{N}$ such that

(a) for each $n \in \mathbb{N}$, C_n is a length n arithmetic progression and
(b) for all $n, m \in \mathbb{N}$, all $x \in C_m$, and all $y \in C_n$, if $m < n$, then $f(y) \geq f(x) + n$ and if $m = n$, then either $f(x) = f(y)$ or $|f(x) - f(y)| \geq n$.

Let C_1 be any singleton subset of A. Let $n \in \mathbb{N}$ and assume that we have chosen $C_1, C_2, \ldots, C_n$. Let $k = \max \bigcup_{i=1}^{n} f[C_i]$ and choose $i \in \{0, 1, \ldots, n\}$ such that $(A \setminus f^{-1}[\{0, 1, \ldots, k+n\}]) \cap f^{-1}[(n+1)\omega + i]$ is an AP set. Let C_{n+1} be a length $n+1$ arithmetic progression contained in $(A \setminus f^{-1}[\{0, 1, \ldots, k+n\}]) \cap f^{-1}[(n+1)\omega + i]$. Given $m \leq n+1$, $x \in C_m$, and $y \in C_{n+1}$, if $m \leq n$, then $f(x) \leq k$ and $f(y) \geq k + n + 1$, while if $m = n+1$, then either $f(x) = f(y)$ or $|f(x) - f(y)| \geq n+1$.

Let $C = \bigcup_{n=1}^{\infty} C_n$. □

2. The space

Lemma 2. *Assume CH. Then there exists a maximal almost disjoint family $\mathcal{A} \subseteq \mathcal{I}_{IP}$ so that for every AP-set $B \subseteq \omega$ and every finite-to-one function $f : B \to \omega$ there exists an AP-set $C \subseteq B$ and $A \in \mathcal{A}$ so that $f[C] \subseteq A$.*

Proof. We construct from CH an almost disjoint family $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\} \subseteq \mathcal{I}_{IP}$ by induction on α. The enumeration $\{A_\alpha : \alpha < \omega_1\}$ may contain repetitions. Let $\{A_n : n < \omega\} \subseteq \mathcal{I}_{IP}$ be a collection of infinite and pairwise disjoint sets.

Fix a list $\langle (f_\alpha, B_\alpha) : \omega \leq \alpha < \omega_1 \rangle$ of all pairs (f, B) in which $B \subseteq \omega$ is an AP-set and $f : B \to \omega$ is a finite-to-one function.

Suppose $\omega \leq \alpha < \omega_1$ and that A_β has been chosen for all $\beta < \alpha$. Consider the pair (f_α, B_α). If there exists a finite set $\{\beta_0, \beta_1, \ldots, \beta_\ell\} \subseteq \alpha$ so that $f_\alpha^{-1}[\bigcup_{i \leq \ell} A_{\beta_i}]$ is AP, let $A_\alpha = A_0$.

Otherwise, enumerate α as $\langle \beta_i : i < \omega \rangle$, and now for all $n < \omega$ the set $f_\alpha^{-1}[\bigcup_{i<n} A_{\beta_i}]$ is not AP, hence $B_\alpha \setminus f_\alpha^{-1}[\bigcup_{i<n} A_{\beta_i}]$ is AP. Let an arithmetic progression $D_n \subseteq B_\alpha \setminus f_\alpha^{-1}[\bigcup_{i<n} A_{\beta_i}]$ of length n be chosen for all n. Then $B' := \bigcup_{n\in\omega} D_n$ is an AP-subset of B_α, $f_\alpha[B']$ is infinite (because f_α is finite-to-one) and $|f_\alpha[B'] \cap A_\beta| < \aleph_0$ for all $\beta < \alpha$. By Lemma 1 find an AP-set $B'' \subseteq B'$, so that $f_\alpha[B''] \in \mathcal{I}_{IP}$, and define $A_\alpha = f_\alpha[B'']$.

The family $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ is clearly an almost disjoint family of (infinite) sets, and $\mathcal{A} \subseteq \mathcal{I}_{IP}$.

Suppose now that $B \subseteq \omega$ is an AP-set and that $f : B \to \omega$ is finite-to-one. There is an index $\omega \leq \alpha < \omega_1$ for which $(B, f) = (B_\alpha, f_\alpha)$. At stage α of the construction of $\mathcal{A}$, either $f^{-1}[A_{\beta_0} \cup \cdots \cup A_{\beta_\ell}]$ was AP for some finite set $\{\beta_0, \ldots, \beta_\ell\} \subseteq \alpha$, hence $f^{-1}[A_\beta]$ was AP for some single $\beta < \alpha$, or else $f^{-1}[A_\alpha]$ was AP. In either case, there is an AP-set $C \subseteq B$ and $A \in \mathcal{A}$ so that $f[C] \subseteq A$.

Finally, to verify that $\mathcal{A}$ is maximal let an infinite set $D \subseteq \omega$ be given and let $f : \omega \to D$ be the increasing enumeration of D. Since there is an AP-set $C \subseteq \omega$ and $A \in \mathcal{A}$ so that $f[C] \subseteq A$, it is clear that $D \cap A$ is infinite. □

Theorem 3. *Suppose CH holds. Then there exists a compact, separable van der Waerden space which is not Hindman.*

Proof. Let $\mathcal{A}$ be as stated in Lemma 2. For each $A \in \mathcal{A}$ let $p_A \notin \omega$ be a distinct point. Define a topology τ on $Y = \omega \cup \{p_A : A \in \mathcal{A}\}$ by requiring that $Z \in \tau$ if and only if for all $p_A \in Z$ the set $A \setminus Z$ is finite. Then for each $A \in \mathcal{A}$, $A \cup \{p_A\}$ is a compact neighborhood of p_A, so τ is a locally compact Hausdorff topology in which ω is a dense and discrete subspace. Let $X = Y \cup \{p\}$ be the one-point compactification of τ.

It was shown in [3, Theorem 10] that when $\mathcal{A} \subseteq \mathcal{I}_{IP}$ is maximal almost disjoint, the space constructed in this way is sequentially compact but not Hindman. To keep this paper self-contained, we repeat the simple argument showing that X is not Hindman. For each $n \in \omega$, let $x_n = n$ and suppose we have some infinite $B \subseteq \omega$ such that $(x_n)_{n\in FS(B)}$ IP-converges to $q \in X$. Then $q \notin \omega$. If $q = p_A$ for some $A \in \mathcal{A}$, then A is an IP set. So $q = p$. By the maximality of $\mathcal{A}$, pick $A \in \mathcal{A}$ such that $A \cap B$ is infinite. But then $X \setminus (A \cup \{p_A\})$ is a neighborhood of p and for no n does one have $FS(B \setminus \{0, 1, \ldots, n-1\}) \subseteq X \setminus (A \cup \{p_A\})$.

We have yet to see that X is van der Waerden. Suppose $f : \omega \to X$ is given. Let $g : f[\omega] \to \omega$ be 1-1. By Lemma 1 we can find an AP set $B \subseteq \omega$ so that $(g \circ f)\restriction B$ is constant or finite-to-one, and hence $f\restriction B$ is constant or finite-to-one. In the former case, the sequence $(f(n))_{n\in B}$ is constant, and therefore converges. So assume that $f\restriction B$ is finite-to-one. Since either $f^{-1}[\omega] \cap B$ or $B \setminus f^{-1}[\omega]$ is AP, we may assume, by shrinking B to some AP-subset, that either $f[B] \subseteq \omega$ or $f[B] \subseteq X \setminus (\omega \cup \{p\})$.

In the former case, there is some $A \in \mathcal{A}$ and AP-set $C \subseteq B$ so that $f[C] \subseteq A$. Since $f\restriction B$ is finite-to-one, $(f(n))_{n\in C}$ converges to p_A. In the latter case, we claim that the sequence $(f(n))_{n\in B}$ converges to p. To see this, let Z be a compact subset of Y, so that $X \setminus Z$ is a basic neighborhood of p. Then $Z \setminus \omega$ is finite so, since $f\restriction B$ is finite-to-one, $(f(n))_{n\in B}$ is eventually in $X \setminus Z$. □

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