VAN DER WAERDEN SPACES AND HINDMAN SPACES ARE NOT THE SAME

MENACHEM KOJMAN AND SAHARON SHELAH

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Abstract. A Hausdorff topological space $X$ is van der Waerden if for every sequence $(x_n)_{n \in \omega}$ in $X$ there is a converging subsequence $(x_n)_{n \in A}$ where $A \subseteq \omega$ contains arithmetic progressions of all finite lengths. A Hausdorff topological space $X$ is Hindman if for every sequence $(x_n)_{n \in \omega}$ in $X$ there is an IP-converging subsequence $(x_n)_{n \in FS(B)}$ for some infinite $B \subseteq \omega$.

We show that the continuum hypothesis implies the existence of a van der Waerden space which is not Hindman.

1. Introduction

A Hausdorff topological space $X$ is van der Waerden if for every sequence $(x_n)_{n \in \omega}$ in $X$ there is a converging subsequence $(x_n)_{n \in A}$ where $A \subseteq \omega$ contains arithmetic progressions of all finite lengths. A Hausdorff topological space $X$ is Hindman if for every sequence $(x_n)_{n \in \omega}$ in $X$ there is an IP-converging subsequence $(x_n)_{n \in FS(B)}$ for some infinite $B \subseteq \omega$. The term $FS(B)$ stands for the set of all finite sums (with no repetitions) over $B$ and IP-convergence to a point $x \in X$ means: for every neighborhood $U$ of $x$, there is some $n_0$ so that $\{x_n : n \in FS(B \setminus \{0, 1, \ldots, n_0 - 1\})\} \subseteq U$.

The classes of van der Waerden and of Hindman spaces were introduced in [2], [3] where it was shown that each class was productive and properly contained in the class of sequentially compact spaces, and that every Hausdorff space $X$ in which the closure of every countable set is compact and first countable is both van der Waerden and Hindman. The question was raised as to whether every Hausdorff space $X$ is van der Waerden if and only if it is Hindman. We answer this question in the negative using the Continuum Hypothesis.

1.1. Notation and combinatorial preliminaries. A set $A \subseteq \omega$ is an AP-set if it contains arithmetic progressions of all finite lengths. By van der Waerden’s theorem [4], if an AP-set $A$ is partitioned into finitely many parts, at least one of...
Lemma 1. Let $A$ be an AP set and let $f : \omega \to \omega$. There exists an AP set $C \subseteq A$ such that either

1. $|f[C]| = 1$ or
2. $f$ is finite-to-one on $C$ and if $\langle x_n \rangle_{n=0}^\infty$ enumerates $f[C]$ in increasing order, then $\lim_{n \to \infty} (x_{n+1} - x_n) = \infty$.

In particular, $f[C] \in \mathcal{I}_{IP}$.

Proof. Suppose that for every AP set $C \subseteq A$, $|f[C]| > 1$. We construct an AP set $C \subseteq A$ for which conclusion (2) holds.

For each $m \in \omega$, $A \cap f^{-1}([0,1,\ldots,m-1])$ is not an AP set because it is the finite union of sets on which $f$ is constant, and thus $A \setminus f^{-1}([0,1,\ldots,m-1])$ is an AP set. (Here we are using the fact that when an AP set is partitioned into finitely many parts, one of these parts is an AP set.)

We inductively construct sets $C_n$ for each $n \in \mathbb{N}$ such that

(a) for each $n \in \mathbb{N}$, $C_n$ is a length $n$ arithmetic progression and
(b) for all $n, m \in \mathbb{N}$, all $x \in C_m$, and all $y \in C_n$, if $m < n$, then $f(y) \geq f(x) + n$ and if $m = n$, then either $f(x) = f(y)$ or $|f(x) - f(y)| \geq n$.

Let $C_1$ be any singletons subset of $A$. Let $n \in \mathbb{N}$ and assume that we have chosen $C_1, C_2, \ldots, C_n$. Let $k = \max\{\sum_{i=1}^{n} f[C_i] \}$ and choose $i \in \{0,1,\ldots,n\}$ such that $(A \setminus f^{-1}([0,1,\ldots,k+n])) \cap f^{-1}([n+1]\omega + i)$ is an AP set. Let $C_{n+1}$ be a length $n+1$ arithmetic progression contained in $(A \setminus f^{-1}([0,1,\ldots,k+n])) \cap f^{-1}([n+1]\omega + i)$. Given $m \leq n+1$, $x \in C_m$, and $y \in C_{n+1}$, if $m \leq n$, then $f(x) \leq k$ and $f(y) \geq k + n + 1$, while if $m = n + 1$, then either $f(x) = f(y)$ or $|f(x) - f(y)| \geq n + 1$.

Let $C = \bigcup_{n=1}^\infty C_n$. \hfill \Box

2. The space

Lemma 2. Assume CH. Then there exists a maximal almost disjoint family $A \subseteq \mathcal{I}_{IP}$ so that for every AP-set $B \subseteq \omega$ and every finite-to-one function $f : B \to \omega$ there exists an AP-set $C \subseteq B$ and $A \in A$ so that $f[C] \subseteq A$.

Proof. We construct from CH an almost disjoint family $A = \{A_\alpha : \omega_1 < \omega \} \subseteq \mathcal{I}_{IP}$ by induction on $\alpha$. The enumeration $\{A_\alpha : \alpha < \omega_1\}$ may contain repetitions. Let $\{A_n : n < \omega_1\} \subseteq \mathcal{I}_{IP}$ be a collection of infinite and pairwise disjoint sets.

Fix a list $\langle f_\alpha, B_\alpha \rangle : \omega \leq \alpha < \omega_1$ of all pairs $(f,B)$ in which $B \subseteq \omega$ is an AP-set and $f : B \to \omega$ is a finite-to-one function.

Suppose $\omega \leq \alpha < \omega_1$ and that $A_\beta$ has been chosen for all $\beta < \alpha$. Consider the pair $(f_\alpha, B_\alpha)$. If there exists a finite set $\{\beta_0, \beta_1, \ldots, \beta_k\} \subseteq \alpha$ so that $f_\alpha^{-1}(\bigcup_{\beta \leq \ell} A_\beta]$ is AP, let $A_\alpha = A_\beta$.
Otherwise, enumerate $\alpha$ as $\langle \beta_i : i < \omega \rangle$, and now for all $n < \omega$ the set $f^{-1}_{\omega} \cup A_{\beta_i}$ is not AP, hence $B_n \setminus f^{-1}_{\omega} \cup A_{\beta_i}$ is AP. Let an arithmetic progression $D_n \subseteq B_n \setminus f^{-1}_{\omega} \cup A_{\beta_i}$ of length $n$ be chosen for all $n$. Then $B' := \bigcup_{n \in \omega} D_n$ is an AP-subset of $B_n$, $f_{\alpha}[B']$ is infinite (because $f_{\alpha}$ is finite-to-one) and $|f_{\alpha}[B'] \cap A_{\beta}| < \aleph_0$ for all $\beta < \alpha$. By Lemma 4 find an AP-set $B'' \subseteq B'$, so that $f_{\alpha}[B''] \subseteq I_{1P}$, and define $A_{\alpha} = f_{\alpha}[B'']$.

The family $A = \{A_{\alpha} : \alpha < \omega_1 \}$ is clearly an almost disjoint family of (infinite) sets, and $A \subseteq I_{1P}$.

Suppose now that $B \subseteq \omega$ is an AP-set and that $f : B \to \omega$ is finite-to-one. There is an index $\omega \leq \alpha < \omega_1$ for which $(B, f) = (B_n, f_{\alpha})$. At stage $\alpha$ of the construction of $A$, either $f^{-1}_{\omega}[A_{\beta_0} \cup \cdots \cup A_{\beta_{\ell}}]$ was AP for some finite set $\{\beta_0, \ldots, \beta_{\ell} \} \subseteq \alpha$, hence $f^{-1}_{\omega}[A_{\beta}]$ was AP for some single $\beta < \alpha$, or else $f^{-1}_{\omega}[A_{\alpha}]$ was AP. In either case, there is an AP-set $C \subseteq B$ and $A \in A$ so that $f[C] \subseteq A$.

Finally, to verify that $A$ is maximal let an infinite set $D \subseteq \omega$ be given and let $f : \omega \to D$ be the increasing enumeration of $D$. Since there is an AP-set $C \subseteq \omega$ and $A \in A$ so that $f[C] \subseteq A$, it is clear that $D \cap A$ is infinite. \(\square\)

**Theorem 3.** Suppose CH holds. Then there exists a compact, separable van der Waerden space which is not Hindman.

**Proof.** Let $A$ be as stated in Lemma 2. For each $A \in A$ let $p_A \notin \omega$ be a distinct point. Define a topology $\tau$ on $Y = \omega \cup \{p_A : A \in A\}$ by requiring that $Z \in \tau$ if and only if for all $p_A \in Z$ the set $A \setminus Z$ is finite. Then for each $A \in A$, $A \cup \{p_A\}$ is a compact neighborhood of $p_A$, so $\tau$ is a locally compact Hausdorff topology in which $\omega$ is a dense and discrete subspace. Let $X = Y \cup \{p\}$ be the one-point compactification of $\tau$.

It was shown in [3, Theorem 10] that when $A \subseteq I_{1P}$ is maximal almost disjoint, the space constructed in this way is sequentially compact but not Hindman. To keep this paper self-contained, we repeat the simple argument showing that $X$ is not Hindman. For each $n \in \omega$, let $x_n = n$ and suppose we have some infinite $B \subseteq \omega$ such that $(x_n)_{n \in F(S(B))}$ IP-converges to $q \in X$. Then $q \notin \omega$. If $q = p_A$ for some $A \in A$, then $A$ is an IP set. So $q = p$. By the maximality of $A$, pick $A \in A$ such that $A \cap B$ is infinite. But then $X \setminus (A \cup \{p_A\})$ is a neighborhood of $p$ and for no $n$ does one have $F(S(B) \setminus \{0, 1, \ldots, n - 1\}) \subseteq X \setminus (A \cup \{p_A\})$.

We have yet to see that $X$ is van der Waerden. Suppose $f : \omega \to X$ is given. Let $g : f[\omega] \to \omega$ be 1-1. By Lemma 1 we can find an AP set $B \subseteq \omega$ so that $(g \circ f)[B]$ is constant or finite-to-one, and hence $f[B]$ is constant or finite-to-one. In the former case, the sequence $(f(n))_{n \in B}$ is constant, and therefore converges. So assume that $f[B]$ is finite-to-one. Since either $f^{-1}[\omega] \cap B$ or $B \setminus f^{-1}[\omega]$ is AP, we may assume, by shrinking $B$ to some AP-subset, that either $f[B] \subseteq \omega$ or $f[B] \subseteq X \setminus (\omega \cup \{p\})$.

In the former case, there is some $A \in A$ and AP-set $C \subseteq B$ so that $f[C] \subseteq A$. Since $f[B]$ is finite-to-one, $(f(n))_{n \in C}$ converges to $p_A$. In the latter case, we claim that the sequence $(f(n))_{n \in B}$ converges to $p$. To see this, let $Z$ be a compact subset of $Y$, so that $X \setminus Z$ is a basic neighborhood of $p$. Then $X \setminus Z$ is finite so, since $f[B]$ is finite-to-one, $(f(n))_{n \in B}$ is eventually in $X \setminus Z$. \(\square\)

**References**


Department of Mathematics, Ben Gurion University of the Negev, Beer Sheva, Israel

E-mail address: kojman@cs.bgu.ac.il

Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel

E-mail address: shelah@ma.huji.ac.il