VAN DER WAERDEN SPACES AND HINDMAN SPACES ARE NOT THE SAME

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Abstract. A Hausdorff topological space $X$ is van der Waerden if for every sequence $(x_n)_{n \in \omega}$ in $X$ there is a converging subsequence $(x_n)_{n \in A}$ where $A \subseteq \omega$ contains arithmetic progressions of all finite lengths. A Hausdorff topological space $X$ is Hindman if for every sequence $(x_n)_{n \in \omega}$ in $X$ there is an IP-converging subsequence $(x_n)_{n \in FS(B)}$ for some infinite $B \subseteq \omega$.

We show that the continuum hypothesis implies the existence of a van der Waerden space which is not Hindman.

1. Introduction

A Hausdorff topological space $X$ is van der Waerden if for every sequence $(x_n)_{n \in \omega}$ in $X$ there is a converging subsequence $(x_n)_{n \in A}$ where $A \subseteq \omega$ contains arithmetic progressions of all finite lengths. A Hausdorff topological space $X$ is Hindman if for every sequence $(x_n)_{n \in \omega}$ in $X$ there is an IP-converging subsequence $(x_n)_{n \in FS(B)}$ for some infinite $B \subseteq \omega$.

The term $FS(B)$ stands for the set of all finite sums (with no repetitions) over $B$ and IP-convergence to a point $x \in X$ means: for every neighborhood $U$ of $x$, there is some $n_0$ so that $\{x_n : n \in FS(B \setminus \{0, 1, \ldots, n_0 - 1\})\} \subseteq U$.

The classes of van der Waerden and of Hindman spaces were introduced in [2], [3] where it was shown that each class was productive and properly contained in the class of sequentially compact spaces, and that every Hausdorff space $X$ in which the closure of every countable set is compact and first countable is both van der Waerden and Hindman. The question was raised as to whether every Hausdorff space $X$ is van der Waerden if and only if it is Hindman. We answer this question in the negative using the Continuum Hypothesis.

1.1. Notation and combinatorial preliminaries. A set $A \subseteq \omega$ is an AP-set if it contains arithmetic progressions of all finite lengths. By van der Waerden’s theorem [4], if an AP-set $A$ is partitioned into finitely many parts, at least one of...
the parts is AP. Let \( \mathcal{I}_{AP} \) denote the collection of all subsets of \( \omega \) which are not AP. \( \mathcal{I}_{AP} \) is a proper ideal over \( \omega \) and a set \( A \subseteq \omega \) is AP if and only if \( A \notin \mathcal{I}_{AP} \).

A set \( A \subseteq \omega \) is an IP-set if there exists an infinite set \( B \subseteq \omega \) so that \( FS(B) \subseteq A \).

\( FS(B) = \{ \sum F : F \subseteq A, |F| < \aleph_0 \} \), where \( \sum F \) stands for \( \sum_{n \in F} n \). By Hindman’s theorem [1], if an IP-set \( A \) is partitioned into finitely many parts, at least one of the parts is IP. Let \( \mathcal{I}_{IP} \) denote the collection of all subsets of \( \omega \) which are not IP. \( \mathcal{I}_{IP} \) is a proper ideal over \( \omega \) and a set \( A \subseteq \omega \) is IP if and only if \( A \notin \mathcal{I}_{IP} \).

We shall need the following lemma which relates \( \mathcal{I}_{AP} \) to \( \mathcal{I}_{IP} \).

**Lemma 1.** Let \( A \) be an AP set and let \( f : \omega \to \omega \). There exists an AP set \( C \subseteq A \) such that either

1. \( |f[C]| = 1 \) or
2. \( f \) is finite-to-one on \( C \) and if \( \langle x_n \rangle_{n=0}^{\infty} \) enumerates \( f[C] \) in increasing order, then \( \lim_{n \to \infty} (x_{n+1} - x_n) = \infty \).

In particular, \( f[C] \in \mathcal{I}_{IP} \).

**Proof.** Suppose that for every AP set \( C \subseteq A \), \( |f[C]| > 1 \). We construct an AP set \( C \subseteq A \) for which conclusion (2) holds.

For each \( m \in \omega \), \( A \cap f^{-1}[[0,1,\ldots,m-1]] \) is not an AP set because it is the finite union of sets on which \( f \) is constant, and thus \( A \setminus f^{-1}[[0,1,\ldots,m-1]] \) is an AP set. (Here we are using the fact that when an AP set is partitioned into finitely many parts, one of these parts is an AP set.)

We inductively construct sets \( C_n \) for each \( n \in \mathbb{N} \) such that

(a) for each \( n \in \mathbb{N} \), \( C_n \) is a length \( n \) arithmetic progression and
(b) for all \( n, m \in \mathbb{N} \), all \( x \in C_m \), and all \( y \in C_n \), if \( m < n \), then \( f(y) \geq f(x) + n \) and if \( m = n \), then either \( f(x) = f(y) \) or \( f(x) - f(y) \geq n \).

Let \( C_1 \) be any singleton subset of \( A \). Let \( n \in \mathbb{N} \) and assume that we have chosen \( C_1, C_2, \ldots, C_n \). Let \( k = \max \{ \sum_{i=1}^{C_i} |f[C_i]| \} \) and choose \( i \in \{ 0, 1, \ldots, n \} \) such that \( (A \setminus f^{-1}[[0,1,\ldots,k+n]]) \cap f^{-1}[[n+1,\omega+i]] \) is an AP set. Let \( C_{n+1} \) be a length \( n+1 \) arithmetic progression contained in \( (A \setminus f^{-1}[[0,1,\ldots,k+n]]) \cap f^{-1}[[n+1,\omega+i]] \). Given \( m \leq n + 1 \), \( x \in C_m \), and \( y \in C_{n+1} \), if \( m \leq n \), then \( f(x) \leq k \) and \( f(y) \geq k + n + 1 \), while if \( m = n + 1 \), then either \( f(x) = f(y) \) or \( f(x) - f(y) \geq n + 1 \).

Let \( C = \bigcup_{n=1}^{\infty} C_n \).

\( \square \)

2. THE SPACE

**Lemma 2.** Assume CH. Then there exists a maximal almost disjoint family \( A \subseteq \mathcal{I}_{IP} \) so that for every AP-set \( B \subseteq \omega \) and every finite-to-one function \( f : B \to \omega \) there exists an AP-set \( C \subseteq B \) and \( A \in A \) so that \( f[C] \subseteq A \).

**Proof.** We construct from CH an almost disjoint family \( A = \{ A_\alpha : \alpha < \omega_1 \} \subseteq \mathcal{I}_{IP} \) by induction on \( \alpha \). The enumeration \( \{ A_\alpha : \alpha < \omega_1 \} \) may contain repetitions. Let \( \{ A_n : n < \omega \} \subseteq \mathcal{I}_{IP} \) be a collection of infinite and pairwise disjoint sets.

Fix a list \( \langle (f_\alpha, B_\alpha) : \omega \leq \alpha < \omega_1 \rangle \) of all pairs \( (f, B) \) in which \( B \subseteq \omega \) is an AP-set and \( f : B \to \omega \) is a finite-to-one function.

Suppose \( \omega \leq \alpha < \omega_1 \) and that \( A_\beta \) has been chosen for all \( \beta < \alpha \). Consider the pair \( (f_\alpha, B_\alpha) \). If there exists a finite set \( \{ \beta_0, \beta_1, \ldots, \beta_\ell \} \subseteq \alpha \) so that \( f_\alpha^{-1}[\bigcup_{i \leq \ell} A_{\beta_i}] \) is AP, let \( A_\alpha = A_0 \).
Otherwise, enumerate $\alpha$ as $\langle \beta_i : i < \omega \rangle$, and now for all $n < \omega$ the set $f^{-1}_\alpha(\bigcup_{i<n} A_{\beta_i})$ is not AP, hence $B_\alpha \setminus f^{-1}_\alpha(\bigcup_{i<n} A_{\beta_i})$ is AP. Let an arithmetic progression $D_n \subseteq B_\alpha \setminus f^{-1}_\alpha(\bigcup_{i<n} A_{\beta_i})$ of length $n$ be chosen for all $n$. Then $B' := \bigcup_{n<\omega} D_n$ is an AP-subset of $B_\alpha$, $f_\alpha[B']$ is infinite (because $f_\alpha$ is finite-to-one) and $|f_\alpha[B'] \cap A_{\beta}| < \aleph_0$ for all $\beta < \alpha$. By Lemma 5 we find an AP-set $B'' \subseteq B'$, so that $f_\alpha[B''] \in \mathcal{I}_{IP}$, and define $A_\alpha = f_\alpha[B'']$.

The family $A = \{A_\alpha : \alpha < \omega_1\}$ is clearly an almost disjoint family of (infinite) sets, and $A \subseteq \mathcal{I}_{IP}$.

Suppose now that $B \subseteq \omega$ is an AP-set and that $f : B \to \omega$ is finite-to-one. There is an index $\omega \leq \alpha < \omega_1$ for which $(B, f) = (B_\alpha, f_\alpha)$. At stage $\alpha$ of the construction of $\mathcal{A}$, either $f^{-1}[A_{\beta_0} \cup \cdots \cup A_{\beta_\ell}]$ was AP for some finite set $\{\beta_0, \ldots, \beta_\ell\} \subseteq \alpha$, hence $f^{-1}[A_\beta]$ was AP for some finite $\beta < \alpha$, or else $f^{-1}[A_\alpha]$ was AP. In either case, there is an AP-set $C \subseteq B$ and $A \in \mathcal{A}$ so that $f[C] \subseteq A$.

Finally, to verify that $\mathcal{A}$ is maximal let an infinite set $D \subseteq \omega$ be given and let $f : \omega \to D$ be the increasing enumeration of $D$. Since there is an AP-set $C \subseteq \omega$ and $A \in \mathcal{A}$ so that $f[C] \subseteq A$, it is clear that $D \cap A$ is infinite. \hfill \qed

**Theorem 3.** Suppose CH holds. Then there exists a compact, separable van der Waerden space which is not Hindman.

*Proof.* Let $\mathcal{A}$ be as stated in Lemma 2. For each $A \in \mathcal{A}$ let $p_A \notin \omega$ be a distinct point. Define a topology $\tau$ on $Y = \omega \cup \{p_A : A \in \mathcal{A}\}$ by requiring that $Z \in \tau$ if and only if for all $p_A \in Z$ the set $A \setminus Z$ is finite. Then for each $A \in \mathcal{A}$, $A \cup \{p_A\}$ is a compact neighborhood of $p_A$, so $\tau$ is a locally compact Hausdorff topology in which $\omega$ is a dense and discrete subspace. Let $X = Y \cup \{p\}$ be the one-point compactification of $\tau$.

It was shown in [3, Theorem 10] that when $\mathcal{A} \subseteq \mathcal{I}_{IP}$ is maximal almost disjoint, the space constructed in this way is sequentially compact but not Hindman. To keep this paper self-contained, we repeat the simple argument showing that $X$ is not Hindman. For each $n \in \omega$, let $x_n = n$ and suppose we have some infinite $B \subseteq \omega$ such that $(x_n)_{n \in FS(B)}$ IP-converges to $q \in X$. Then $q \notin \omega$. If $q = p_A$ for some $A \in \mathcal{A}$, then $A$ is an IP set. So $q = p$. By the maximality of $\mathcal{A}$, pick $A \in \mathcal{A}$ such that $A \cap B$ is infinite. But then $X \setminus (A \cup \{p_A\})$ is a neighborhood of $p$ and for no $n$ does one have $FS(B \setminus \{0, 1, \ldots, n-1\}) \subseteq X \setminus (A \cup \{p_A\})$.

We have yet to see that $X$ is van der Waerden. Suppose $f : \omega \to X$ is given. Let $g : f(\omega) \to \omega$ be 1-1. By Lemma 5 we can find an AP set $B \subseteq \omega$ so that $(g \circ f)|B$ is constant or finite-to-one, and hence $f[B]$ is constant or finite-to-one. In the former case, the sequence $(f(n))_{n \in B}$ is constant, and therefore converges. So assume that $f[B]$ is finite-to-one. Since either $f^{-1}[\omega] \cap B$ or $B \setminus f^{-1}[\omega]$ is AP, we may assume, by shrinking $B$ to some AP-subset, that either $f[B] \subseteq \omega$ or $f[B] \subseteq X \setminus (\omega \cup \{p\})$.

In the former case, there is some $A \in \mathcal{A}$ and AP-set $C \subseteq B$ so that $f[C] \subseteq A$. Since $f[B]$ is finite-to-one, $(f(n))_{n \in C}$ converges to $p_A$. In the latter case, we claim that the sequence $(f(n))_{n \in B}$ converges to $p$. To see this, let $Z$ be a compact subset of $Y$, so that $X \setminus Z$ is a basic neighborhood of $p$. Then $Z \setminus \omega$ is finite so, since $f[B]$ is finite-to-one, $(f(n))_{n \in B}$ is eventually in $X \setminus Z$. \hfill \qed

**References**


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