PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 6, Pages 1965–1969
S 0002-9939(02)06668-6
Article electronically published on October 18, 2002

A FUNCTION SPACE $C_p(X)$ WITHOUT A CONDENSATION ONTO A σ-COMPACT SPACE

WITOLD MARCISZEWSKI

(Communicated by Alan Dow)

Abstract. Assuming that the minimal cardinality of a dominating family in ω^ω is equal to 2^ω, we construct a subset X of a real line $\mathbb{R}$ such that the space $C_p(X)$ of continuous real-valued functions on X does not admit any continuous bijection onto a σ-compact space. This gives a consistent answer to a question of Arhangel'skii.

1. Introduction

All spaces under consideration are completely regular. The space $C_p(X)$ is the space of all continuous real-valued functions on a space X, equipped with the pointwise convergence topology. A condensation is a continuous bijection. Recently, A. V. Arhangel'skii [Ar4] has shown that, for every σ-compact metrizable space X, the space $C_p(X)$ condenses onto a metrizable compactum. This result was generalized by H. Michalewski [Mi] for analytic spaces X, i.e., metrizable continuous images of the space of irrational numbers. The problem of the existence of such maps is related to an old question of S. Banach whether every separable Banach space can be condensed onto a metrizable compactum. This question was solved in the affirmative by E. G. Pytkeev [Py]. A. V. Arhangel'skii [Ar4, Problem 4] asked whether $C_p(X)$ condenses onto a σ-compact space for every separable metrizable space X (similar questions were asked in [Ar3]). In this note we give a negative answer to this question under some additional set-theoretic assumption.

1.1. Example. Assume that $\mathfrak{d} = 2^\omega$. Then there exists a subset X of the real line $\mathbb{R}$ such that there is no condensation of $C_p(X)$ onto a σ-compact space.

Let us recall that the above assumption $\mathfrak{d} = 2^\omega$, meaning that the minimal cardinality of a dominating family in ω^ω is equal to 2^ω, can be equivalently stated as follows. The space of irrational numbers cannot be covered by less than 2^ω many compact subsets. In fact, we use yet another equivalent formulation of this statement. The intersection of less than 2^ω G_δ-subsets of $\mathbb{R}$ containing the rationals $\mathbb{Q}$ has the cardinality 2^ω. The statement $\mathfrak{d} = 2^\omega$ follows from the Continuum

Received by the editors July 2, 2001 and, in revised form, December 4, 2001 and February 8, 2002.
2000 *Mathematics Subject Classification.* Primary 54C35, 54A10.
Key words and phrases. Function space, pointwise convergence topology, $C_p(X)$, condensation.
The author was supported in part by KBN grant 2 P03A 011 15.

Hypothesis or the Martin Axiom, and is independent of the usual axioms of the set theory (see [vD]).

The construction of Example 1.1 is given in Section 3. Section 2 contains some auxiliary results.

2. Auxiliary results

Let D be a dense subset of the space X. By $C_D(X)$ we denote the space $\{f|D : f \in C_p(X)\}$ equipped with the topology of the pointwise convergence on D, i.e., we consider $C_D(X)$ as a subspace of the product $\mathbb{R}^D$. The following two propositions are well known; we include their short proofs for the reader's convenience.

2.1. Proposition (cf. [Ar4, Proof of Thm. 1]). *Let X be a separable metrizable space. If K is a compact space which is a continuous image of a subset A of $C_p(X)$, then K is metrizable.*

Proof. Since X has a countable base, the space $C_p(X)$ has a countable network (see [Ar2]). Therefore A and K also have countable networks, hence K is metrizable. □

2.2. Proposition. *Every space X which is a countable union of metrizable compacta condenses onto a metrizable σ-compact space.*

Proof. Let $X = \bigcup_{n\in\omega} K_n$, where each K_n is a metrizable compact subspace of X. Since the union of two metrizable compacta is a metrizable compactum, we may assume that, for $n \in \omega$, $K_n \subset K_{n+1}$. For every $n \in \omega$, let $i_n : K_n \to [0,1]^\omega$ be a homeomorphic embedding and let $f_n : X \to [0,1]^\omega$ be a continuous extension of i_n. Then the map $F = \triangle_{n\in\omega} f_n : X \to ([0,1]^\omega)^\omega$ is continuous. This map is also injective since every pair of points of X belongs to some K_n and f_n is injective on K_n. The image $F(X)$, being a continuous image of a σ-compact space X, is a σ-compact subspace of a metrizable space $([0,1]^\omega)^\omega$. □

Propositions 2.1 and 2.2 imply the following:

2.3. Corollary. *Let X be a separable metrizable space. If $C_p(X)$ condenses onto a σ-compact space, then there exists a condensation of $C_p(X)$ onto a metrizable σ-compact space.*

For a subset D of a set X, we denote the projection of $\mathbb{R}^X$ onto $\mathbb{R}^D$ by π_D.

We will use the following standard factorization result (see [Ar1, Theorem 6]):

2.4. Proposition. *Let E be a countable dense subset of a space X and let $\psi : C_p(X) \to \mathbb{R}^\omega$ be a continuous map. Then there exists a countable set $D \subset X$ and a continuous map $\varphi : C_D(X) \to \mathbb{R}^\omega$ such that $E \subset D$ and $\psi = \varphi \circ (\pi_D|C_p(X))$.*

2.5. Proposition. *Let D be a dense subset of a metrizable space X. For every proper subspace Y of X containing D, $C_D(X)$ is a proper subspace of $C_D(Y)$.*

Proof. The inclusion $C_D(X) \subset C_D(Y)$ is obvious. Take $z \in X \setminus Y$. Let d be a metric on X. Define the continuous function $f : Y \to \mathbb{R}$ by $f(y) = 1/d(y,z)$. It is clear that $f|D \in C_D(Y) \setminus C_D(X)$. □

3. Construction of the example

We assume that $\mathfrak{d} = 2^\omega$. We consider the following family of maps:

$$\mathcal{F} = \{\varphi : B \to \mathbb{R}^\omega : B \text{ is a Borel subset of } \mathbb{R}^D, \text{ for some countable } D \subset \mathbb{R} \text{ with } \mathbb{Q} \subset D, \varphi \text{ is continuous}\}.$$

One can easily compute that the family $\mathcal{F}$ has the cardinality 2^ω, so we can enumerate $\mathcal{F}$ as $\{\varphi_\alpha : B_\alpha \to \mathbb{R}^\omega : \alpha < 2^\omega\}$ and we assume that B_α is a subset of $\mathbb{R}^{D_\alpha}$.

By transfinite induction we will construct, for every $\alpha < 2^\omega$, points $x_\alpha, y_\alpha \in \mathbb{R}$, G_δ-subsets A_α of $\mathbb{R}$ containing $\mathbb{Q}$, and continuous functions $f_\alpha, g_\alpha : A_\alpha \to \mathbb{R}$ satisfying the following conditions (where $X_\alpha = \mathbb{Q} \cup \{x_\beta : \beta < \alpha\}$):

(i) $x_\beta \neq x_\alpha$, for $\beta < \alpha$,
(ii) $(X_\alpha \cup \{x_\alpha\}) \cap \{y_\beta : \beta \leq \alpha\} = \emptyset$,
(iii) $x_\alpha \in \bigcap_{\beta \leq \alpha} A_\beta$,
(iv) if $D_\alpha \setminus X_\alpha \neq \emptyset$, then $y_\alpha \in D_\alpha$,
(v) if $D_\alpha \subset X_\alpha$ and $C_{D_\alpha}(X_\alpha) \setminus B_\alpha \neq \emptyset$, then $X_\alpha \subset A_\alpha$ and $f_\alpha|D_\alpha \notin B_\alpha$,
(vi) if $D_\alpha \subset X_\alpha$, $C_{D_\alpha}(X_\alpha) \subset B_\alpha$, and $\varphi_\alpha|C_{D_\alpha}(X_\alpha)$ is not injective, then $X_\alpha \subset A_\alpha$, $f_\alpha \neq g_\alpha$, and $\varphi_\alpha(f_\alpha|D_\alpha) = \varphi_\alpha(g_\alpha|D_\alpha)$.

Suppose that we have chosen x_β, y_β, A_β, and f_β, g_β for $\beta < \alpha < 2^\omega$. We will consider four cases related to conditions (iv)–(vi):

Case 1: $D_\alpha \setminus X_\alpha \neq \emptyset$.

Then we take $y_\alpha \in D_\alpha \setminus X_\alpha$. Put $A_\alpha = \mathbb{R}$ and $f_\alpha = g_\alpha \equiv 0$. Notice that by our assumption that $\mathfrak{d} = 2^\omega$, the intersection $\bigcap_{\beta \leq \alpha} A_\beta$ has the cardinality 2^ω, hence we can find $x_\alpha \in \mathbb{R}$ satisfying conditions (i)–(iii).

Case 2: $D_\alpha \subset X_\alpha$ and $C_{D_\alpha}(X_\alpha) \setminus B_\alpha \neq \emptyset$.

Let $f' : X_\alpha \to \mathbb{R}$ be a continuous function such that $f'|D_\alpha \notin B_\alpha$. By the Lavrentiev Theorem we can extend f' to a continuous function $f_\alpha : A_\alpha \to \mathbb{R}$ defined on a G_δ-set in $\mathbb{R}$ containing X_α. We put $g_\alpha = f_\alpha$. Next, we can choose $x_\alpha \in \mathbb{R}$ satisfying (i)–(iii). Finally, we take any $y_\alpha \in \mathbb{R}$ satisfying condition (ii).

Case 3: $D_\alpha \subset X_\alpha$, $C_{D_\alpha}(X_\alpha) \subset B_\alpha$, and $\varphi_\alpha|C_{D_\alpha}(X_\alpha)$ is not injective.

In this case we start with distinct $f', g' \in C_p(X_\alpha)$ such that $\varphi_\alpha(f'|D_\alpha) = \varphi_\alpha(g'|D_\alpha)$. As above, using the Lavrentiev Theorem, we can find a G_δ-set $A_\alpha \subset \mathbb{R}$ containing X_α and continuous extensions $f_\alpha, g_\alpha : A_\alpha \to \mathbb{R}$ of f' and g', respectively. We proceed with the choice of x_α and y_α as in Case 2.

Case 4: $D_\alpha \subset X_\alpha$, $C_{D_\alpha}(X_\alpha) \subset B_\alpha$, and $\varphi_\alpha|C_{D_\alpha}(X_\alpha)$ is injective.

This case is easy; we simply put $A_\alpha = \mathbb{R}$ and $f_\alpha = g_\alpha \equiv 0$. We choose any $x_\alpha, y_\alpha \in \mathbb{R}$ satisfying conditions (i)–(iii).

Take $X = \mathbb{Q} \cup \{x_\alpha : \alpha < 2^\omega\}$. We will show that there is no continuous bijection of $C_p(X)$ onto a σ-compact space.

Assume towards a contradiction that $\psi : C_p(X) \to S$ is a condensation onto a σ-compact space S. By Corollary 2.3 we may assume that S is metrizable, hence we may consider S as a subset of $\mathbb{R}^\omega$. Proposition 2.4 implies the existence of a countable subset $D \subset X$ containing $\mathbb{Q}$ and a continuous map $\varphi' : C_D(X) \to \mathbb{R}^\omega$ such that $\psi = \varphi' \circ (\pi_D|C_p(X))$. By the Lavrentiev Theorem we can find a G_δ-set $G \subset \mathbb{R}^D$ containing $C_D(X)$ and a continuous extension $\varphi'' : G \to \mathbb{R}^\omega$ of φ'. Take $B = (\varphi'')^{-1}(S)$ and $\varphi = \varphi''|B$. The set B is Borel and contains $C_D(X)$. We have $\varphi(B) = S$ and φ maps $C_D(X)$ injectively onto S. The map φ belongs to $\mathcal{F}$, so we have $\varphi = \varphi_\alpha$, $B = B_\alpha$, and $D = D_\alpha$ for some $\alpha < 2^\omega$. We will consider the same four cases that we encountered during the inductive construction:

Case 1: $D_\alpha \setminus X_\alpha \neq \emptyset$.

Then we have $y_\alpha \in D_\alpha = D$ by condition (iv), and by (ii) we have $y_\alpha \notin X$. Hence $D \setminus X \neq \emptyset$, a contradiction.

Case 2: $D_\alpha \subset X_\alpha$ and $C_{D_\alpha}(X_\alpha) \setminus B_\alpha \neq \emptyset$.

By condition (v) we have $X_\alpha \subset A_\alpha$. Condition (iii) implies that $x_\beta \in A_\alpha$ for $\beta \geq \alpha$. Therefore $X \subset A_\alpha$ and $f_\alpha | X \in C_p(X)$. By (v) we obtain that $f_\alpha | D \in C_D(X) \setminus B$, a contradiction.

Case 3: $D_\alpha \subset X_\alpha$, $C_{D_\alpha}(X_\alpha) \subset B_\alpha$, and $\varphi_\alpha | C_{D_\alpha}(X_\alpha)$ is not injective.

Similarly as in Case 2, condition (vi) implies that $X \subset A_\alpha$ and $f_\alpha | X, g_\alpha | X \in C_p(X)$. Then $f_\alpha | D, g_\alpha | D$ are distinct elements of $C_D(X)$ and $\varphi(f_\alpha | D) = \varphi(g_\alpha | D)$, a contradiction.

Case 4: $D_\alpha \subset X_\alpha$, $C_{D_\alpha}(X_\alpha) \subset B_\alpha$, and $\varphi_\alpha | C_{D_\alpha}(X_\alpha)$ is injective.

Condition (i) implies that X_α is a proper subset of X. Therefore, by Proposition 2.5, the space $C_D(X)$ is a proper subset of $C_D(X_\alpha)$. Since φ is injective on $C_D(X_\alpha)$ and $\varphi(C_D(X_\alpha)) \subset \varphi(B) = S$, we conclude that $S \setminus \varphi(C_D(X)) \neq \emptyset$, a contradiction.

Remark. By a slight modification of the above construction we can achieve that the space X from Example 1.1 will be zero-dimensional. We can also obtain that, for a given family $\mathcal{A}$ of separable metrizable spaces of the cardinality $|\mathcal{A}| = 2^\omega$ (e.g., the family of projective sets), the space $C_p(X)$ could not be condensed onto any space $A \in \mathcal{A}$.

As we mentioned in the Introduction, we consider only completely regular spaces. However, let us note that it is possible to modify the construction of our example X to ensure that the function space $C_p(X)$ cannot be condensed onto any Hausdorff σ-compact space. To achieve this one should replace in the definition of the family $\mathcal{F}$, described at the beginning of this section, the requirement that the maps $\varphi : B \to \mathbb{R}^\omega$ are continuous, by the condition that these maps are Borel. One should also use the following facts:

(1) Every Hausdorff space with a countable network can be condensed onto a Hausdorff space with a countable base; see [AP, Ch. 2, Problem 149].
(2) For every Hausdorff space X which is a countable union of metrizable compacta, there exists an injective Borel map of X onto a metrizable σ-compact space (cf. Proposition 2.2).
(3) Let X be a separable metrizable space and Y be a second-countable Hausdorff space. Then every continuous map $\psi : C_p(X) \to Y$ has the factorization property described in Proposition 2.4. This follows easily from the fact that every open set in $C_p(X)$ is a countable union of basic open sets (since $C_p(X)$ is hereditarily Lindelöf).
(4) Every Borel map $\varphi : A \to \mathbb{R}^\omega$, where $A \subset \mathbb{R}^\omega$, can be extended to a Borel map $\varphi' : B \to \mathbb{R}^\omega$, which is defined on a Borel set $B \subset \mathbb{R}^\omega$ containing A; see [Ku, §35.VI].

Acknowledgment

The author is grateful to Alexander V. Arhangel'skii and Sławomir Solecki for valuable suggestions.

References

[Ar1] A. V. Arhangel'skii, *Factorization theorems and function spaces: Stability and monolithicity*, Soviet Math. Dokl. **26** (1982), 177–181.

[Ar2] ———, *Topological Function Spaces*, Kluwer Academic Publishers, Dordrecht, 1992.

[Ar3] ———, *C_p-Theory*, Recent Progress in General Topology (M. Hušek and J. van Mill, eds.), Elsevier, 1992, pp. 1–56.

[Ar4] ———, *On condensations of C_p-spaces onto compacta*, Proc. Amer. Math. Soc. **128** (2000), 1881–1883. MR **2001c:**54002

[AP] A. V. Arhangel'skii and V. I. Ponomarev, *Fundamentals of General Topology in Problems and Exercises*, D. Reidel Publ. Co., Dordrecht-Boston, 1984. MR **56:**3781

[vD] E. van Douwen, *The integers and topology*, Handbook of Set-Theoretic Topology (K. Kunen and J. Vaughan, eds.), North-Holland, 1984, pp. 111–167. MR **87f:**54008

[En] R. Engelking, *General Topology*, PWN, Warszawa, 1977. MR **58:**18316b

[Ku] K. Kuratowski, *Topology* I, Academic Press and PWN, New York and London, 1966. MR **36:**840

[Mi] H. Michalewski, *Condensations of projective sets onto compacta*, Proc. Amer. Math. Soc., to appear.

[Py] E. G. Pytkeev, *Upper bounds of topologies*, Math. Notes **20** (1976), 831–837. MR **55:**1262

Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warszawa, Poland

E-mail address: `wmarcisz@mimuw.edu.pl`

Current address: Faculty of Sciences, Division of Mathematics and Computer Science, Vrije Universiteit, De Boelelaan 1081[a], 1081 HV Amsterdam, The Netherlands