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# MASS POINTS OF MEASURES ON THE UNIT CIRCLE AND REFLECTION COEFFICIENTS

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ABSTRACT. Measures on the unit circle and orthogonal polynomials are completely determined by their reflection coefficients through the Szegő recurrences. We find the conditions on the reflection coefficients which provide the lack of a mass point at $\zeta = 1$. We show that the result is sharp in a sense.

## 1. Introduction

Let $\mathcal{P}$ be the set of all probability measures $\mu$ on the unit circle $\mathbb{T} = \{|\zeta| = 1\}$ with infinite support. The latter is defined as the smallest *closed* set with the complement having $\mu$-measure zero. The polynomials $\phi_n(z) = \kappa_n z^n + \ldots$, orthonormal on the unit circle with respect to $\mu$ are uniquely determined by the requirement that $\kappa_n > 0$ and

$$\int_{\mathbb{T}} \phi_n(\zeta)\overline{\phi_m(\zeta)}\, d\mu = \delta_{n,m}, \qquad n, m = 0, 1, \ldots,\ \zeta \in \mathbb{T}.$$

The monic orthogonal polynomials $\Phi_n$ are $\Phi_n(z) = \kappa_n^{-1}\phi_n(z) = z^n + \ldots$, and the values $a_n = a_n(\mu) \stackrel{\text{def}}{=} \Phi_n(0)$ are known as the *reflection coefficients*.

Let us recall that the orthogonal polynomials (both monic and orthonormal) as well as the measure itself are completely determined by their reflection coefficients through the Szegő recurrences

$$\Phi_n(z) = z\Phi_{n-1}(z) + a_n\Phi^*_{n-1}(z), \quad n \in \mathbb{N} \stackrel{\text{def}}{=} \{1, 2, \ldots\},\ \Phi_0 = 1 \tag{1}$$

(cf. [5, formula (11.4.7), p. 293]), and the connection between the reflection coefficients and the leading coefficients $\kappa_n$ is given by

$$\kappa_n^2 = \prod_{k=1}^{n} \left(1 - |a_n|^2\right)^{-1}, \quad n \in \mathbb{N},\ \kappa_0 = 1 \tag{2}$$

(cf. [1, formula (8.6), p. 156]). Here the reversed $^*$-polynomial of a polynomial $p_n$ of degree $n$ is defined by $p_n^*(z) \stackrel{\text{def}}{=} z^n\,\overline{p_n(1/\bar{z})}$. Moreover, each sequence $a_n$ of points from the open unit disk $\mathbb{D}$ comes up as a sequence of reflection coefficients for a

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certain uniquely determined probability measure $\mu$. Hence, we have some sort of parametrization of the set $\mathcal{P}$ with free parameters from $\mathbb{D} \times \mathbb{D} \times \dots$.

The problem we study in the present paper is the relation between the existence of a mass point at $\zeta = 1$ and behavior of the reflection coefficients. The argument here relies upon the equivalence

$$\mu\{\zeta\} > 0, \quad \zeta \in \mathbb{T} \iff \sum_{n=0}^{\infty} |\phi_n(\zeta)|^2 < \infty \tag{3}$$

(cf. [4, pp. 45–46], [2, § 20]). Note that within the Szegő class of measures, which can be defined by the condition $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, the sequence $\{\kappa_n\}$ is increasing and bounded, and hence

$$\sum_{n=0}^{\infty} |\phi_n(\zeta)|^2 < \infty \iff \sum_{n=0}^{\infty} |\Phi_n(\zeta)|^2 < \infty.$$

The starting point for us is the following result due to P. Nevai [3, Theorem 2.5].

**Theorem.** *If the reflection coefficients $a_n$ are real for all $n$ and they are nonnegative for all but finitely many values of $n$, then the corresponding measure $\mu$ has no mass point at 1.*

An example below shows that the first assumption of this theorem (which is actually the subject of our investigation) cannot be discarded.

**Example 1.** Put

$$a_1 = \frac{-1+i}{2}, \qquad a_2 = \frac{-1-i}{2}, \qquad a_k = \frac{1}{k}, \quad k \ge 3,$$

which gives rise to some measure $\nu$ in the Szegő class. We have by (1) $\Phi_1(z) = z + a_1$, $\Phi_2(z) = z^2 + (a_1 + a_2\overline{a}_1)z + a_2$, so that

$$\Phi_1(1) = 1 + a_1 = \frac{1+i}{2}, \qquad \Phi_2(1) = 1 + a_1 + a_2 + a_2\overline{a}_1 = \frac{i}{2},$$

which is a pure imaginary number. Now keeping in mind that $\Phi_m^*(1) = \overline{\Phi_m(1)}$, write (1) at $z = 1$:

$$\Phi_n(1) = \Phi_{n-1}(1) + a_n\overline{\Phi_{n-1}(1)}, \quad n \in \mathbb{N}. \tag{4}$$

After separating the real and imaginary parts in (4) and iterating up, we come to

$$\Re\Phi_n(1) = \Re\Phi_2(1) \prod_{k=3}^{n} (1 + a_k) = 0,$$
$$\Im\,\Phi_n(1) = \Im\Phi_2(1) \prod_{k=3}^{n} (1 - a_k) = \frac{1}{2} \prod_{k=3}^{n} \left(1 - \frac{1}{k}\right) = \frac{1}{n}, \qquad n \ge 3.$$

It follows that $\Phi_n(1)$ is square summable and hence $\nu\{1\} > 0$.

On the other hand, the following modification of Nevai's theorem holds.

**Theorem 1.** *Let $\Re a_n \ge 0$ for all $n \in \mathbb{N}$ and*

$$\sum_{n=1}^{\infty} \prod_{k=1}^{n} (1 - |\Im a_n|)^2 = \infty. \tag{5}$$

*Then the corresponding measure $\mu$ has no mass point at 1.*

We start out with the following simple lemma.

**Lemma 1.** *If* $\Re a_n \geq 0$ *for all* $n \in \mathbb{N}$*, then* $|\Im \Phi_n(1)| \leq \Re \Phi_n(1)$.

*Proof.* We proceed by induction. It is clear that

$$\Re \Phi_1(1) = 1 + \Re a_1 \geq 1 > |a_1| = |\Im \Phi_1(1)| .$$

Suppose that the statement is true for $k = 1, 2, \dots, n-1$. Put

$$a_n = \alpha_n + i\beta_n, \qquad \Phi_n(1) = \Re \Phi_n(1) + i\Im \Phi_n(1) = u_n + iv_n$$

and write (4) as a system of two linear recurrences

$$\begin{aligned} u_n &= u_{n-1} + \alpha_n u_{n-1} + \beta_n v_{n-1}, \\ v_n &= v_{n-1} + \beta_n u_{n-1} - \alpha_n v_{n-1}. \end{aligned} \tag{6}$$

We want to show that $u_n \pm v_n \geq 0$. We have

$$\begin{aligned} u_n + v_n &= u_{n-1}(1 + \alpha_n + \beta_n) + v_{n-1}(1 - \alpha_n + \beta_n) \\ &\geq u_{n-1}(1 + \alpha_n + \beta_n) - |v_{n-1}|\,|1 - \alpha_n + \beta_n|. \end{aligned}$$

But $|1 - \alpha_n + \beta_n| \leq 1 + \alpha_n + \beta_n$ since

$$\begin{aligned} 1 + \alpha_n + \beta_n + 1 - \alpha_n + \beta_n &= 2 + 2\beta_n \geq 0, \\ 1 + \alpha_n + \beta_n - 1 + \alpha_n - \beta_n &= 2\alpha_n \geq 0 \end{aligned}$$

by the assumption of Lemma 1. Hence, $u_n + v_n \geq 0$.

The same reasoning applied to orthogonal polynomials generated by the sequence $\{\overline{a}_n\}$ leads to the second inequality $u_n - v_n \geq 0$. □

*Proof of Theorem* 1. It is well known that all zeros of $\Phi_n$ lie inside $\mathbb{D}$ [1, p. 9], so that $\Phi_n(1) \neq 0$. By Lemma 1 this implies $u_n = \Re \Phi_n(1) > 0$ for all $n \in \mathbb{N}$.

For $a_n = \alpha_n + i\beta_n$ put $\omega_n \stackrel{\text{def}}{=} \prod_{k=1}^{n}(1 + \alpha_k) \geq 1$ and divide the first equation in (6) by $\omega_n$:

$$\frac{u_n}{\omega_n} = \frac{u_{n-1}}{\omega_{n-1}} + \frac{\beta_n}{1 + \alpha_n}\,\frac{v_{n-1}}{\omega_{n-1}}.$$

Next, by Lemma 1

$$\frac{u_n}{\omega_n} \geq \frac{u_{n-1}}{\omega_{n-1}} - \frac{|\beta_n|}{1 + \alpha_n}\,\frac{|v_{n-1}|}{\omega_{n-1}} \geq (1 - |\beta_n|)\,\frac{u_{n-1}}{\omega_{n-1}}.$$

Finally,

$$u_m \geq \frac{u_m}{\omega_m} \geq \frac{u_{n-1}}{\omega_{n-1}} \prod_{k=n}^{m}(1 - |\beta_k|), \qquad m > n.$$

The latter inequality along with (5) yields

$$\sum_{n=0}^{\infty} |\Phi_n(1)|^2 \geq \sum_{n=0}^{\infty} u_n^2 = \infty.$$

The result follows immediately from (2) and (3). □

Note that (5) holds as long as $\sum_{n=1}^{\infty} |\Im a_n| < \infty$.

**Corollary.** *If the reflection coefficients* $a_n$ *satisfy* $\Re a_n \geq 0$ *for all* $n \in \mathbb{N}$ *and they are nonnegative for all but finitely many values of* $n$*, then the corresponding measure* $\mu$ *has no mass point at* 1.

It turns out that the first assumption in Theorem 1 is sharp in a sense.

**Theorem 2.** *For every $\epsilon > 0$ there is a measure $\mu$ in the Szegő class with the reflection coefficients $a_n$ such that $\Re a_n \ge -\epsilon$ for all $n \in \mathbb{N}$, $a_n \ge 0$ for all sufficiently large $n$ and $\mu\{1\} > 0$.*

*Proof.* We follow the line of reasoning from the example above, but in a more sophisticated way. For $a_n = \alpha_n + i\beta_n$ put

$$\alpha_k = -\alpha, \quad \beta_k = (-1)^k \alpha_k, \quad k = 1, 2, \dots, 2p; \qquad a_k = \frac{1}{k}, \quad k \ge 2p+1.$$

We will show that the parameters $\alpha$ and $p$ can be found from the conditions

$$0 < \alpha < \epsilon, \qquad u_{2p} = \Re\Phi_{2p}(1) = 0. \tag{7}$$

To that end let us go back to (4) and write (6) in the matrix form (the matrix product is taken from right to left)

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = (I + V_n) \begin{bmatrix} u_{n-1} \\ v_{n-1} \end{bmatrix}, \qquad \begin{bmatrix} u_n \\ v_n \end{bmatrix} = \overset{\curvearrowleft}{\prod_{1\le k\le n}} (I + V_k) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where $I$ is the $2 \times 2$ identity matrix and

$$V_k = \begin{pmatrix} \alpha_k & \beta_k \\ \beta_k & -\alpha_k \end{pmatrix}, \qquad k = 1, 2, \dots, n.$$

To meet $u_{2p} = 0$ we need to choose $p$ and $\alpha = \alpha_p$ in such a way that

$$\overset{\curvearrowleft}{\prod_{1\le k\le 2p}} (I + V_k) = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}.$$

In our case for $1 \le m \le p$

$$V_{2m} = V_+ = \begin{pmatrix} -\alpha & -\alpha \\ -\alpha & \alpha \end{pmatrix}, \qquad V_{2m-1} = V_- = \begin{pmatrix} -\alpha & \alpha \\ \alpha & \alpha \end{pmatrix}$$

and

$$\overset{\curvearrowleft}{\prod_{1\le k\le 2p}} (I + V_k) = \left[(I + V_+)(I + V_-)\right]^p.$$

Note also that

$$V_- = JV_+J, \quad I + V_- = J(I + V_+)J, \qquad J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that our matrix product is just the power of a single matrix

$$\overset{\curvearrowleft}{\prod_{1\le k\le 2p}} (I + V_k) = \left[(I + V_+)J\right]^{2p} = U_\alpha^{2p}, \qquad U_\alpha \overset{\text{def}}{=} \begin{pmatrix} -1+\alpha & -\alpha \\ \alpha & 1+\alpha \end{pmatrix}.$$

It is a matter of undergraduate linear algebra to find the eigenvalues of $U_\alpha$,

$$\lambda_1 = \alpha + \sqrt{1-\alpha^2} > 1, \qquad \lambda_2 = \alpha - \sqrt{1-\alpha^2} < 0$$

(we assume that $\alpha < 1/2$ and, hence, $\det U_\alpha = 2\alpha^2 - 1 < 0$), and to reduce $U_\alpha$ to diagonal form by means of a nonsingular transformation

$$T = \begin{pmatrix} 1 & 1 \\ y_1 & y_2 \end{pmatrix}, \qquad T^{-1} = \frac{1}{y_2 - y_1} \begin{pmatrix} y_2 & -1 \\ -y_1 & 1 \end{pmatrix}$$

with

$$y_1 = \frac{1-\lambda_2}{1-\lambda_1} = \frac{1-\alpha+\sqrt{1-\alpha^2}}{1-\alpha-\sqrt{1-\alpha^2}}\,,$$
$$y_2 = \frac{1-\lambda_1}{1-\lambda_2} = \frac{1-\alpha-\sqrt{1-\alpha^2}}{1-\alpha+\sqrt{1-\alpha^2}} = y_1^{-1}.$$

Finally, we arrive at the following expression for the object we are interested in:

$$U_\alpha^{2p} = T\begin{pmatrix} \lambda_1^{2p} & 0 \\ 0 & \lambda_2^{2p}\end{pmatrix} T^{-1} = \frac{1}{y_2-y_1}\begin{pmatrix} \lambda_1^{2p}y_2 - \lambda_2^{2p}y_1 & * \\ * & * \end{pmatrix}.$$

It remains only to choose $p$ and $\alpha$ to satisfy $\lambda_1^{2p}y_2 = \lambda_2^{2p}y_1$ or, equivalently

$$\left(\frac{\lambda_2}{\lambda_1}\right)^{2p} = \frac{y_2}{y_1}, \qquad \left(\frac{\sqrt{1-\alpha^2}-\alpha}{\sqrt{1-\alpha^2}+\alpha}\right)^p = \frac{\sqrt{1-\alpha^2}-1+\alpha}{\sqrt{1-\alpha^2}+1-\alpha} \tag{8}$$

(note that both $y_1$ and $y_2$ are negative numbers).

Consider a transcendental equation

$$\left(\frac{1-\tan x}{1+\tan x}\right)^p = \tan\frac{x}{2}\,,$$

which has a unique solution $x = x_p$ on the interval $(0, \pi/4)$. It is clear that $x_p \to 0$ as $p \to \infty$. Put $\alpha_p = \sin x_p$ and pick $p$ to have $0 < \alpha_p < \epsilon$. A routine calculation shows that

$$\left(\frac{\sqrt{1-\alpha_p^2}-\alpha_p}{\sqrt{1-\alpha_p^2}+\alpha_p}\right)^p = \left(\frac{1-\tan x_p}{1+\tan x_p}\right)^p,$$
$$\frac{\sqrt{1-\alpha_p^2}-1+\alpha_p}{\sqrt{1-\alpha_p^2}+1-\alpha_p} = \frac{\cos x_p - 1 + \sin x_p}{\cos x_p + 1 - \sin x_p} = \tan\frac{x_p}{2}\,,$$

so that both (8) and (7) hold.

Once this is done, the rest is clear in view of Example 1. Indeed, $\Phi_{2p}(1) \neq 0$ now implies $\Im\Phi_{2p}(1) \neq 0$, and for $n \geq 2p+1$ we have

$$\Re\Phi_n(1) = \Re\Phi_{2p}(1)\prod_{k=2p+1}^{n}(1+a_k) = 0,$$
$$\Im\Phi_n(1) = \Im\Phi_{2p}(1)\prod_{k=2p+1}^{n}\left(1-\frac{1}{k}\right) = \frac{2p}{n}\Im\Phi_{2p}(1),$$

so that $\{\Phi_n(1)\}$ is a square summable sequence and $\mu\{1\} > 0$. □

*Remark.* Theorem 2 shows that for every $\epsilon > 0$ there is a measure $\mu$ in the Szegő class with the reflection coefficients $a_n$ such that $|\Im a_n| \leq \epsilon$ for all $n \in \mathbb{N}$, $a_n \geq 0$ for sufficiently large $n$ and $\mu\{1\} > 0$. Therefore, the first assumption in Nevai's theorem is also sharp in the same sense.

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