

## A PRODUCT DECOMPOSITION OF INFINITE SYMMETRIC GROUPS

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ABSTRACT. We prove that for any infinite  $\kappa$ , the full symmetric group  $\text{Sym}(\kappa)$  is the product of at most 14 abelian subgroups. This is a strengthening of a recent result of M. Abért.

### 1. INTRODUCTION

For a cardinality  $\kappa$ , where  $\kappa$  can be finite or infinite, let  $f(\kappa)$  denote the minimum cardinality  $\lambda$  such that  $\text{Sym}(\kappa)$  is the product of  $\lambda$  abelian subgroups. For positive integers  $n$ , it is easy to see that  $f(n) \geq \log_3 n$ , and M. Abért [Ab] proved that  $f(n) \leq 3 \log_2 n$ . Abért also proved the surprising result that for all infinite  $\kappa$ ,  $f(\kappa)$  is bounded from above by an absolute constant; his construction shows that  $f(\kappa) \leq 161$ . The purpose of this note is to strengthen this result.

**Theorem 1.** *Let  $\kappa$  be any infinite cardinality. Then the full symmetric group  $\text{Sym}(\kappa)$  can be written as the product of 14 abelian subgroups. Namely, there are four abelian subgroups  $A, B, C, D \leq \text{Sym}(\kappa)$  such that*

$$\text{Sym}(\kappa) = ACABCABABDBABA.$$

For  $\kappa = \omega$ , Abért also proved that there are two abelian subgroups  $A, B \leq \text{Sym}(\omega)$  such that  $\text{Sym}(\omega) = (AB)^{144}A$  (i.e.,  $\text{Sym}(\omega)$  can be written as the product of 289 abelian subgroups, using only two different subgroups as terms in the product). Abért's construction that showed  $f(\kappa) \leq 161$  uses three different subgroups, and P. Komjáth [Ko] proved that one of these can be expressed as a short product of the other two, yielding  $\text{Sym}(\kappa) = (AB)^{96}A$ . In the case  $\kappa = 2^\omega$ , Abért and Keleti [AK] also demonstrated that  $\text{Sym}(2^\omega) = (AB)^{104}A$  for some abelian subgroups  $A, B$ . Moreover, combining the methods of [AK] and [Ko], one can show that  $\text{Sym}(\kappa) = (AB)^{40}A$  for any infinite  $\kappa$  (see Remark 15 in [AK]). In this direction, our best result is the following.

**Theorem 2.** *Let  $\kappa$  be any infinite cardinality. Then there exist abelian subgroups  $A, B \leq \text{Sym}(\kappa)$  such that  $\text{Sym}(\kappa) = (AB)^{16}A$ .*

The bound  $f(\kappa) \leq 14$  is small enough that we can start to speculate about the exact value of  $f(\kappa)$ . Ito [It] proved that if a group  $G$  is the product of two abelian

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subgroups, then  $G'$  is abelian. Since the derived subgroup of  $\text{Sym}(\kappa)$  is not abelian, Ito's theorem implies that  $f(\kappa) \geq 3$  for all infinite  $\kappa$ . We conjecture that  $f(\kappa) < 10$ . As a first step, it would be interesting to decide whether  $f(\kappa) = 3$  holds for some  $\kappa$ . We also conjecture that  $f(\kappa)$  is the same for all infinite  $\kappa$ .

Although the factorization of infinite groups into a product of two subgroups has an extensive literature (see the monograph [AFG] and its references), the study of products of more than two components seems to be more recent. Groups which can be written as the product of finitely many cyclic subgroups have been investigated by Lubotzky [Lu], Platonov and Rapinchuk [PR], and Tavgen [Ta]. Groups as products of abelian subgroups were studied by Abért, Pálffy, and Pyber [APP], who prove that for any field  $K$  and finite dimension  $n$ , the special linear group  $\text{SL}(n, K)$  is the product of 60 abelian subgroups.

## 2. THE GROUPS

In this section, we define the four groups  $A, B, C, D$  mentioned in Theorem 1, and introduce the notation used in the paper. As usual, we identify a cardinality  $\kappa$  with the set of ordinals less than  $\kappa$ . We use multiplicative notation for all groups.

We shall work most of the time with two abelian groups  $A$  and  $B$ , each having  $\kappa$  orbits of size  $\kappa$ , such that each orbit of  $A$  intersects each orbit of  $B$  in exactly one point. Therefore, to simplify notation, we identify the underlying set of  $\text{Sym}(\kappa)$  with  $\Omega := \kappa \times \kappa$ , which reflects this orbit structure.

For  $\alpha < \kappa$ , let  $A_\alpha$  be an abelian group that acts transitively (and so regularly) on the vertical line  $\Psi_\alpha := \{(\alpha, \beta) \in \Omega \mid \beta < \kappa\}$ , and define  $A := \prod_{\alpha < \kappa} A_\alpha$ . Similarly, let  $B_\alpha$  be an abelian group that acts transitively on the horizontal line  $\Phi_\alpha := \{(\beta, \alpha) \in \Omega \mid \beta < \kappa\}$ , and define  $B := \prod_{\alpha < \kappa} B_\alpha$ .

Let  $\Delta := \{(\alpha, \alpha) \in \Omega \mid \alpha < \kappa\}$  denote the diagonal in  $\Omega$ . The third group  $C$  is cyclic, and is generated by some  $c \in \text{Sym}(\Omega)$  that maps  $\Omega \setminus \Delta$  onto  $\Delta$  (and so maps  $\Delta$  onto  $\Omega \setminus \Delta$ ). The fourth group  $D$  is the product of cyclic groups. Let  $\Delta = \bigcup_{\alpha < \kappa} \Delta_\alpha$  be a partition of the diagonal into  $\kappa$  sets such that  $|\Delta_\alpha| \leq \omega$  for each  $\alpha < \kappa$ , for each positive integer  $n$  there are  $\kappa$  sets of size  $n$  among the  $\Delta_\alpha$ , and there are  $\kappa$  sets of size  $\omega$  among the  $\Delta_\alpha$ . Let  $D_\alpha$  be a transitive cyclic group on  $\Delta_\alpha$ , and let  $D := \prod_{\alpha < \kappa} D_\alpha$ . Note that  $D$  fixes  $\Omega \setminus \Delta$  pointwise.

Our basic tool is the following simple lemma.

**Lemma 3.** *Let  $h \in \text{Sym}(\Omega)$ , and let  $\Gamma$  be a subset of  $\Delta$  such that  $\Gamma^h \subseteq \Delta$ . Then there exist  $a \in A$  and  $b \in B$  such that the restrictions  $h|_\Gamma$  and  $ab|_\Gamma$  are equal.*

*Proof.* For  $(\alpha, \alpha) \in \Gamma$ , we define  $a_\alpha \in A_\alpha$  the following way. If  $(\alpha, \alpha)^h = (\beta, \beta)$ , then let  $a_\alpha$  be the unique element of  $A_\alpha$  such that  $(\alpha, \alpha)^{a_\alpha} = (\alpha, \beta)$ . Let  $a := \prod_{(\alpha, \alpha) \in \Gamma} a_\alpha$ .

After that, for  $(\beta, \beta) \in \Gamma^h$ , we define  $b_\beta \in B_\beta$ . If  $(\beta, \beta)^{h^{-1}} = (\alpha, \alpha)$ , then let  $b_\beta$  be the unique element of  $B_\beta$  such that  $(\alpha, \beta)^{b_\beta} = (\beta, \beta)$ . Let  $b := \prod_{(\beta, \beta) \in \Gamma^h} b_\beta$ .

Clearly,  $h|_\Gamma = ab|_\Gamma$ .  $\square$

## 3. PROOF OF THEOREM 1

Given  $g \in \text{Sym}(\Omega)$ , we shall write  $g$  as a product of elements of our groups in two steps. In the first step, we construct  $x \in ACABCA$  such that  $x^{-1}g$  has a significant number of fixed points, where we shall define the term “significant” in a precise

sense. In the second step, we write  $y := x^{-1}g$  as an element of  $ABABDBABA$ . Then  $g = xy$ , and  $xy \in (ACABCA)(ABABDBABA) = ACABCABABDBABA$ .

We start with the construction of  $x$ . Recursively for  $\alpha < \kappa$ , we define sets  $X_\alpha \subset \Psi_\alpha$  and an element  $\psi_\alpha \in \Psi_\alpha \setminus X_\alpha$  satisfying the following properties:

- (i) There exists  $t_\alpha \in A_\alpha$  such that  $X_\alpha = \{(\alpha, \beta)^{t_\alpha} \mid \beta \leq \alpha\}$  (i.e.,  $X_\alpha$  is a translate of the below-diagonal portion of  $\Psi_\alpha$ , including the diagonal);
- (ii)  $\psi_\alpha^g \notin \bigcup_{\beta < \alpha} X_\beta$ ; and
- (iii)  $\{\psi_\beta^g \mid \beta < \alpha\} \cap X_\alpha = \emptyset$ .

Such a set  $X_\alpha$  exists, since (iii) excludes less than  $\kappa$  elements of  $A_\alpha$  which cannot be used as  $t_\alpha$  in the definition of  $X_\alpha$ . Similarly, (ii) excludes less than  $\kappa$  elements of  $\Psi_\alpha$  which cannot be used as  $\psi_\alpha$ . Let  $X := \bigcup_{\alpha < \kappa} X_\alpha$ .

Let  $a_1$  be the unique element of  $A$  which maps  $\{\psi_\alpha \mid \alpha < \kappa\}$  onto  $\Delta$ . Then (ii) and (iii) ensure that  $X^{a_1} \subseteq \Omega \setminus \Delta$  and  $X^{g^{-1}a_1} \subseteq \Omega \setminus \Delta$ . Hence  $X^{g^{-1}a_1c} \subseteq \Delta$  and  $X^{a_1c} \subseteq \Delta$ . Applying Lemma 3 with  $\Gamma := X^{g^{-1}a_1c}$  and  $h := c^{-1}a_1^{-1}ga_1c$ , we obtain that there are  $a_2 \in A$  and  $b_2 \in B$  such that  $h|_\Gamma = a_2b_2|_\Gamma$ . Then for  $x := a_1ca_2b_2c^{-1}a_1^{-1} \in ACABCA$ , we have that  $(\gamma, \delta)^g = (\gamma, \delta)^x$  for all  $(\gamma, \delta) \in X^{g^{-1}}$ .

We record what we have proved so far.

**Lemma 4.** *There exist  $a_1, a_2 \in A$  and  $b_2 \in B$  such that the permutation  $x = a_1ca_2b_2c^{-1}a_1^{-1} \in ACABCA$  satisfies the property that  $x^{-1}g$  fixes  $X$  pointwise.  $\square$*

In the second step, we have to write  $y := x^{-1}g$  as a short product. The basic idea is similar to the one used in the construction of  $x$ : namely, we conjugate  $y$  such that the points moved by  $y$  are all mapped into  $\Delta$ , and write the appropriate permutation of  $\Delta$  as a short product. However, we need a strengthening of Lemma 3, since we also have to ensure that  $\Omega \setminus \Delta$  remains fixed pointwise.

**Lemma 5.** *Let  $h \in \text{Sym}(\Omega)$ , such that  $h$  fixes  $\Omega \setminus \Delta$  pointwise, and  $h$  also fixes  $\kappa$  elements of  $\Delta$  pointwise. Then there exist  $a \in A$ ,  $b \in B$ , and  $d \in D$  such that  $h = abdb^{-1}a^{-1} \in ABDBA$ .*

*Proof.* The assumption that  $h$  fixes  $\kappa$  elements of  $\Delta$  ensures that there exists some  $d \in D$  with the same cycle structure as  $h$ , and so there exists some  $r \in \text{Sym}(\Omega)$  which fixes  $\Omega \setminus \Delta$  pointwise, and conjugates  $h$  to  $d$ . By Lemma 3, there are  $a \in A$  and  $b \in B$  such that  $r|_\Delta = ab|_\Delta$ . This means that  $h = abdb^{-1}a^{-1}$ , since  $abdb^{-1}a^{-1}$  fixes  $\Omega \setminus \Delta$  pointwise.  $\square$

**Lemma 6.** (a) *There exist  $a_3 \in A$  and  $b_3 \in B$  such that  $b_3a_3$  moves all points of the closed upper triangle  $\{(\alpha, \beta) \in \Omega \mid \alpha \leq \beta\}$  into  $\Delta$ .*

(b) *There exist  $a_4 \in A$  and  $b_4 \in B$  such that  $a_4b_4$  moves all points of the closed lower triangle  $\{(\alpha, \beta) \in \Omega \mid \alpha \geq \beta\}$  into  $\Delta$ .*

(c) *In addition, there exists  $a_5 \in A$  such that  $a_5b_3a_3$  moves all points of  $\Omega \setminus X$  and  $\kappa$  points of  $X$  into  $\Delta$ .*

*Proof.* (a) Recursively for  $\alpha < \kappa$ , we define  $u_\alpha \in B_\alpha$  with the property that the projection of  $\{(\gamma, \alpha)^{u_\alpha} \mid \gamma \leq \alpha\}$  onto the first coordinate is disjoint from the projections of the sets  $\{(\gamma, \beta)^{u_\beta} \mid \gamma \leq \beta\}$ , for all  $\beta < \alpha$ . Formally, we define  $u_\alpha \in B_\alpha$  such that the sets  $\{\delta < \kappa \mid (\exists \gamma \leq \alpha) ((\gamma, \alpha)^{u_\alpha} = (\delta, \alpha))\}$  and

$$\bigcup_{\beta < \alpha} \{\delta < \kappa \mid (\exists \gamma \leq \beta) ((\gamma, \beta)^{u_\beta} = (\delta, \beta))\}$$

are disjoint. Such  $u_\alpha$  exists, since the previously defined  $u_\beta$  exclude less than  $\kappa$  elements of  $B_\alpha$  which cannot be used as  $u_\alpha$ . Let  $b_3 := \prod_{\alpha < \kappa} u_\alpha$ .

Since  $\{(\alpha, \beta) \in \Omega \mid \alpha \leq \beta\}^{b_3}$  intersects each vertical line  $\Psi_\alpha$  in at most one point, there exists  $a_3 \in A$  which maps this set into  $\Delta$ .

(b) The proof is the same as in part (a), reversing the roles of  $A$  and  $B$ .

(c) Let  $a_5 := \prod_{\alpha < \kappa} t_\alpha^{-1}$  for the permutations  $t_\alpha \in A_\alpha$  used in the definition of  $X_\alpha$ . Then  $a_5$  maps  $\Omega \setminus X$  onto the open upper triangle  $\{(\alpha, \beta) \in \Omega \mid \alpha < \beta\}$ , and  $X^{a_5} \supset \Delta$ . Therefore, by part (a),  $a_5 b_3 a_3$  maps all points of  $\Omega \setminus X$  and  $\kappa$  points of  $X$  into  $\Delta$ . □

**Lemma 7.** *Let  $a_3, a_5 \in A$  and  $b_3 \in B$  as defined in Lemma 6. Then there exist  $a_6 \in A, b_6 \in B$ , and  $d \in D$  such that  $y = a_5 b_3 a_3 a_6 b_6 d b_6^{-1} a_6^{-1} a_3^{-1} b_3^{-1} a_5^{-1} \in ABABDBABA$ .*

*Proof.* By Lemma 6(c), the conjugate  $h := a_3^{-1} b_3^{-1} a_5^{-1} y a_5 b_3 a_3$  of  $y$  fixes  $\Omega \setminus \Delta$  pointwise, and  $h$  also fixes  $\kappa$  elements of  $\Delta$  pointwise. Hence, by Lemma 5,  $h = a_6 b_6 d b_6^{-1} a_6^{-1}$  for some  $a_6 \in A, b_6 \in B$ , and  $d \in D$ . □

By Lemmas 4 and 7,  $g = xy \in ACABCABABDBABA$ . Since in this argument  $g$  was an arbitrary element of  $\text{Sym}(\Omega)$ , Theorem 1 follows.

#### 4. PROOF OF THEOREM 2

Given  $g \in \text{Sym}(\Omega)$ , we define  $x$  and  $y$  as in Section 3.

**Lemma 8.**  $x \in (AB)^4 A$ .

*Proof.* By Lemma 4,  $x = a_1 c a_2 b_2 c^{-1} a_1^{-1}$  for some  $a_1, a_2 \in A$  and  $b_2 \in B$ . Hence it is enough to express  $c$  as a short product of elements of  $A, B$ .

P. Komjáth [Ko] proved that for any  $\Gamma \subset \Omega$  satisfying  $|\Gamma| = \kappa$  and  $|\Omega \setminus \Gamma| = \kappa$ , there exists  $p \in ABABA$  or  $p \in BABAB$  such that  $\Gamma^p = \Delta$ . Since  $\Gamma := \Omega \setminus \Delta$  satisfies this property, we obtain that  $c$  can be chosen to be in  $ABABA \cup BABAB$ .

In fact, analyzing the proof in [Ko], it turns out that  $\Omega \setminus \Delta$  is in the class  $\mathcal{C}^+$  of sets defined in that paper, and so Lemmas 3, 4, 5 of [Ko] imply that  $\Omega \setminus \Delta$  can be mapped onto  $\Delta$  by some element of  $ABAB$  or  $BABA$ . Moreover, since  $\Omega \setminus \Delta$  is invariant for the operation exchanging the two coordinates of points, it can be mapped onto  $\Delta$  both by some element of  $ABAB$  and some element of  $BABA$ . Thus we can choose  $c \in ABAB$ , implying that  $x \in A(ABAB)AB(BABA)A = (AB)^4 A$ . □

**Lemma 9.** *Let  $h \in \text{Sym}(\Omega)$  such that  $h$  fixes  $\Omega \setminus \Delta$  pointwise. Then  $h \in (AB)^{10}$ .*

*Proof.* By a theorem of Ore [Or], there exist  $\bar{h}_1, \bar{h}_2 \in \text{Sym}(\Delta)$  such that  $h|_\Delta = [\bar{h}_1, \bar{h}_2]$ . We claim that there exists  $r_1 \in ABABA$  such that  $r_1$  fixes the open lower triangle  $\{(\alpha, \beta) \in \Omega \mid \alpha > \beta\}$  pointwise,  $r_1$  fixes  $\Delta$  setwise, and  $r_1|_\Delta = \bar{h}_1$ . We also claim that similarly there exists  $r_2 \in BABAB$  such that  $r_2$  fixes the open upper triangle  $\{(\alpha, \beta) \in \Omega \mid \alpha < \beta\}$  pointwise,  $r_2$  fixes  $\Delta$  setwise, and  $r_2|_\Delta = \bar{h}_2$ . These two claims imply that  $h = [r_1, r_2] \in ((ABABA)(BABAB))^2 = (AB)^{10}$ .

For  $i = 1, 2$ , let  $h_i \in \text{Sym}(\Omega)$  be the permutation that fixes  $\Omega \setminus \Delta$  pointwise, and  $h_i|_\Delta = \bar{h}_i$ .

Now we prove the existence of  $r_1$ . Let  $a_4 \in A$  and  $b_4 \in B$  be as defined in Lemma 6(b) such that  $a_4 b_4$  maps the closed lower triangle into  $\Delta$ . The conjugate  $p_1 := b_4^{-1} a_4^{-1} h_1 a_4 b_4$  of  $h_1$  fixes  $(\Omega \setminus \Delta)^{a_4 b_4}$  pointwise and, since  $a_4 b_4$  maps  $\Delta$  into a subset of  $\Delta$ ,  $p_1$  fixes  $\Delta$  setwise. By Lemma 3, there exist  $a_7 \in A$  and  $b_7 \in B$

such that  $p_1|_{\Delta} = a_7 b_7|_{\Delta}$ ; in particular,  $a_7 b_7$  fixes each element of  $\{(\alpha, \beta) \in \Omega \mid \alpha > \beta\}^{a_4 b_4} \subset \Delta$ . Hence  $r_1 := a_4 b_4 a_7 b_7 a_4^{-1} a_7^{-1} \in ABABA$  satisfies the required properties.

Similarly, using  $a_3 \in A$  and  $b_3 \in B$  from Lemma 6(a), we obtain that  $p_2 := a_3^{-1} b_3^{-1} h_2 b_3 a_3$  fixes  $\Delta$  setwise and fixes  $\{(\alpha, \beta) \in \Omega \mid \alpha > \beta\}^{b_3 a_3} \subset \Delta$  pointwise. By Lemma 3, there exist  $a_8 \in A$  and  $b_8 \in B$  such that  $p_2|_{\Delta} = a_8 b_8|_{\Delta}$ , and so  $r_2 := b_3 a_3 a_8 b_8 a_3^{-1} b_3^{-1} \in BABAB$  satisfies the required properties.  $\square$

**Lemma 10.**  $y \in (AB)^{12}A$ .

*Proof.* As we have seen in the proof of Lemma 7,  $y = a_5 b_3 a_3 h a_3^{-1} b_3^{-1} a_5^{-1}$  for some  $a_3, a_5 \in A$ ,  $b_3 \in B$ , and  $h \in \text{Sym}(\Omega)$  which fixes  $\Omega \setminus \Delta$  pointwise. Hence, by Lemma 9,  $h \in (AB)^{10}$  and so  $y \in ABA(AB)^{10}ABA = (AB)^{12}A$ .  $\square$

By Lemmas 8 and 10,  $g = xy \in (AB)^4 A (AB)^{12} A = (AB)^{16} A$ , and this implies Theorem 2.

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