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ASYMPTOTIC LIMIT FOR CONDENSATE SOLUTIONS IN THE ABELIAN CHERN-SIMONS HIGGS MODEL

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ABSTRACT. In this paper we show that the maximal solutions $(\phi^\epsilon, A^\epsilon)$ in the Abelian Chern-Simons Higgs model on a 't Hooft type periodic domain converges to (ϕ_*, A_*) and ϕ_* is a harmonic map. We also study asymptotic behaviors of the energy density.

1. INTRODUCTION

Let Ω be a basic lattice cell in $\mathbb{R}^2$ generated by two independent vectors $\mathbf{a}^1$ and $\mathbf{a}^2$, namely,

$$\Omega = \{x \in \mathbb{R}^2 \mid x = s_1\mathbf{a}^1 + s_2\mathbf{a}^2,\ \ 0 < s_1, s_2 < 1\}.$$

The (static) Abelian Chern-Simons Higgs functional on Ω is given by [4, 5]

$$E_\epsilon(\phi, A) = \int_\Omega |D_A\phi|^2 + \frac{\epsilon^2}{4}\frac{|F_A|^2}{|\phi|^2} + \frac{1}{\epsilon^2}|\phi|^2(1-|\phi|^2)^2. \tag{1.1}$$

Here $\epsilon > 0$, $\phi : \Omega \to \mathbb{C}$ is the complex Higgs field, $A : \Omega \to \mathbb{R}^2$ is the coupled gauge potential, $D_A\phi = \nabla\phi - iA\phi$ is the covariant derivative, and $F_A = \mathbf{curl}A$ is the magnetic field.

We say that $(\phi, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ is *gauge equivalent* to $(\psi, B) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$, if there exists a function $\chi \in H^2(\Omega, \mathbb{R})$ such that

$$(\psi, B) = (e^{i\chi}\phi, A + \nabla\chi).$$

It is easy to check that the functional $E_\epsilon(\phi, A)$ is invariant under the gauge transformation

$$(\phi, A) \to (e^{i\chi}\phi, A + \nabla\chi).$$

Following [2], we specify suitable boundary conditions to study the vortex condensate solutions. In fact, in view of gauge invariance we impose the following 't Hooft boundary condition on Ω:

$$\begin{aligned} \exp(i\xi_j(x+\mathbf{a}^j))\phi(x+\mathbf{a}^j) &= \exp(i\xi_j(x))\phi(x), \\ (A_k + \partial_k\xi_j)(x+\mathbf{a}^j) &= (A_k + \partial_k\xi_j)(x), \quad k = 1, 2, \\ x \in \Gamma^1 \cup \Gamma^2 - \Gamma^j, &\qquad j = 1, 2, \end{aligned} \tag{1.2}$$

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where ξ_1 and ξ_2 are real-valued smooth functions defined in a neighborhood of $\Gamma^2 \cup \{\mathbf{a}^1 + \Gamma^2\}$, $\Gamma^1 \cup \{\mathbf{a}^2 + \Gamma^1\}$, respectively. Here

$$\Gamma^j = \{x \in \mathbf{R}^2 | x = s\mathbf{a}^j,\ \ 0 < s < 1\}, \quad j = 1, 2.$$

Let us denote $\xi_j(s_1, s_2) = \xi_j(s_1\mathbf{a}^1 + s_2\mathbf{a}^2)$ for simplicity. Since ϕ is a single-valued complex function, its phase change around Ω can only be a multiple of 2π, and hence the boundary condition (1.2) implies that

$$\begin{aligned} &\xi_1(1,1^-) - \xi_1(1,0^+) + \xi_1(0,0^+) - \xi_1(0,1^-) \\ &\quad + \xi_2(0^+,1) - \xi_2(1^-,1) + \xi_2(1^-,0) - \xi_2(0^+,0) + 2\pi N = 0 \end{aligned} \tag{1.3}$$

for some integer $N > 0$. As a consequence of (1.2) and (1.3) we are led to the quantization of the magnetic flux and the electric charge:

$$\begin{aligned} \Phi &= \int_\Omega F_A dx = \int_{\partial\Omega} A_j dx_j = 2\pi N, \\ Q &= \int_\Omega \rho dx = \epsilon\Phi = 2\pi\epsilon N. \end{aligned}$$

We can also rewrite the energy E as

$$E(\phi, A) = \int_\Omega \Big\{ \ |D_1\phi + iD_2\phi|^2 + \frac{1}{4}\Big(\frac{\epsilon}{|\phi|}F_A + \frac{2}{\epsilon}|\phi|(|\phi|^2 - 1)\Big)^2 \ \Big\}dx + \int_\Omega F_A dx.$$

Hence we obtain the Bogomol'nyi type energy lower bound

$$E(\phi, A) \geq \Phi = 2\pi N.$$

The lower bound is achieved if and only if the configuration (ϕ, A) satisfies the self-duality equations

$$D_1\phi + iD_2\phi \ = \ 0, \tag{1.4}$$

$$F_A + \frac{2}{\epsilon^2}|\phi|^2(|\phi|^2 - 1) \ = \ 0. \tag{1.5}$$

The equations (1.4) and (1.5) are gauge invariant and their solutions are often called the condensate solutions. To examine them further, we use the classical Jaffe-Taubes arguments [6]. In fact, equation (1.4) implies that ϕ is holomorphic up to a nonvanishing multiple factor and has exactly N zeros allowing multiplicities. Thus we may assume that ϕ takes the form

$$\phi(z) = \exp\Big(\ \frac{1}{2}u(z) + i\sum_{j=1}^k n_j \arg(z - p_j)\ \Big), \tag{1.6}$$

where $z = x_1 + ix_2$, and the points $p_1, \cdots, p_k$, called the vortex points, are the distinct zeros of ϕ with multiplicities $n_1, \cdots, n_k$, respectively. We observe that the arbitrary choice on the imaginary part of ϕ merely reflects the gauge invariance of (1.4) and (1.5). Now equation (1.5) can be reduced to

$$\begin{aligned} &\Delta u = \frac{4}{\epsilon^2}e^u(e^u - 1) + 4\pi\sum_{j=1}^k n_j\delta_{p_j} \quad \text{on } \Omega, \\ &u : \text{ doubly periodic.} \end{aligned} \tag{1.7}$$

Here δ_p denotes the Dirac measure concentrated on the point p. Conversely, once we find a solution u of (1.7), we may recover A from (1.4) by the formula

$$A_1 + iA_2 = -2i\overline{\partial}\ln\phi, \tag{1.8}$$

where $\overline{\partial} = (\partial_1 + i\partial_2)/2$.

Recently there have been various studies on equations (1.4) and (1.5) by analyzing (1.7). See [3] and the references therein. In this paper we are interested in the maximal solutions of (1.4) and (1.5) on Ω which takes the form (1.6) and (1.8). We first recall several results [2, 3, 7] about the existence of maximal solutions and their asymptotic behavior as $\epsilon \to 0$:

Theorem 1.1. *Let $p_1, \cdots, p_k$ be given points in Ω and $n_1, \cdots, n_k$ positive integers with $n_1 + \cdots + n_k = N$. Then, there is a critical value $\epsilon_c < \sqrt{|\Omega|/4\pi N}$ so that for $0 < \epsilon \le \epsilon_c$ the self-duality equations $(1.4)_\epsilon$ and $(1.5)_\epsilon$ with the boundary conditions (1.2) and (1.3) admits a maximal solution $(\phi^\epsilon, A^\epsilon)$ of the form (1.6) and (1.8). The solution $(\phi^\epsilon, A^\epsilon)$ is maximal in the sense that $|\phi^\epsilon|$ has the largest possible value among all the solutions to $(1.4)_\epsilon$ and $(1.5)_\epsilon$ with the same zeros and multiplicities. The energy, magnetic flux, and electric charge are quantized and given by*

$$E = 2\pi N, \quad \Phi = 2\pi N, \quad Q = 2\pi\epsilon N.$$

If $\epsilon > \epsilon_c$, then equations $(1.4)_\epsilon$ and $(1.5)_\epsilon$ have no solution realizing the zeros $p_1, \cdots, p_k$ with respective multiplicities $n_1, \cdots, n_k$.

Furthermore, the following hold as $\epsilon \to 0$:

(i) $|\phi^\epsilon| < 1$, and $|\phi^\epsilon| \to 1$ a.e. in Ω and in $W^{1,p}(\Omega)$ for $p \in (1, 2)$.

(ii)

$$F_{A^\epsilon} = \frac{2}{\epsilon^2}|\phi^\epsilon|^2(1 - |\phi^\epsilon|^2) \to 2\pi\sum_{j=1}^{k} n_j\delta_{p_j} \tag{1.9}$$

weakly in the sense of measure.

(iii) For every compact subset K of $\Omega\backslash\{p_1, \cdots, p_k\}$,

$$1 - |\phi^\epsilon(x)|^2 \le C_K\epsilon^2, \qquad \forall x \in K, \tag{1.10}$$

and

$$\frac{1}{\epsilon^2}(1 - |\phi^\epsilon|^2) \to \pi\sum_{j=1}^{k} 2n_j(2n_j + 1)\delta_{p_j} \tag{1.11}$$

weakly in the sense of measure.

The purpose of the present paper is to improve the above results concerning the asymptotic behavior of the maximal solutions. We now state the main result.

Theorem 1.2. *Let $(\phi^\epsilon, A^\epsilon)$ be the maximal solution of* (1.4) *and* (1.5) *corresponding to ϵ as in Theorem 1.1. Let $\Omega' = \Omega\backslash\{p_1, \cdots, p_k\}$. Then we have*

$$(\phi^\epsilon, A^\epsilon) \to (\phi_*, A_*) \qquad in \quad C^{1,\alpha}_{loc}(\Omega', \mathbb{C}) \times C^{0,\alpha}_{loc}(\Omega', \mathbb{R}^2),$$

where (ϕ_*, A_*) belongs to $W^{2,p}_{loc}(\Omega', \mathbb{C}) \times W^{1,p}_{loc}(\Omega', \mathbb{R}^2)$ for all $p > 1$ and satisfies

$$\begin{aligned} \Delta\phi_* + \phi_*|\nabla\phi_*|^2 &= 0, \\ |\phi_*| &= 1, \\ \mathbf{deg}(\phi_*, p_j) &= n_j, \\ A_* &= -i\overline{\phi_*}\nabla\phi_* \end{aligned} \tag{1.12}$$

on Ω'. In fact,

$$\phi_*(z) = \prod_{j=1}^{k} \frac{(z-p_j)^{n_j}}{|z-p_j|^{n_j}} \tag{1.13}$$

on Ω'.

Moreover, if we denote the energy density by

$$e_\epsilon(\phi, A) = |D_A\phi|^2 + \frac{\epsilon^2}{4}\frac{|F_A|^2}{|\phi|^2} + \frac{1}{\epsilon^2}|\phi|^2(1-|\phi|^2)^2,$$

then

$$e_\epsilon(\phi^\epsilon, A^\epsilon) \to 2\pi \sum_{j=1}^{k} n_j \delta_{p_j}, \tag{1.14}$$

weakly in the sense of measure.

Here $\mathbf{deg}(\phi_*, p_j)$ denotes the degree of ϕ_* restricted to a small circle centered at p_j with no vortex points other than p_j inside the circle. The equation (1.12) indicates that the map ϕ_* is a harmonic map, which often appears as a limit function in some asymptotic problems such as in the Ginzburg-Landau theory [1]. The equation (1.14) implies that the energy corresponding to maximal solutions tends to be concentrated at the vortex points as $\epsilon \to 0$.

2. Proof of Theorem 1.2

Proof of (1.12) *and* (1.13). Let $\epsilon_n \to 0$ be given. Let $x_0 \in \Omega'$ and choose $r < \min |x_0 - p_j|/2$. Set

$$\phi^{\epsilon_n}(z) = \exp(u_{\epsilon_n}(z)/2 + i\theta_{\epsilon_n}(z)/2).$$

Then it follows from (1.4) that

$$2A^{\epsilon_n} = \mathbf{curl} u_{\epsilon_n} + \nabla\theta_{\epsilon_n}. \tag{2.1}$$

Here $\mathbf{curl} u = (\partial_2 u, -\partial_1 u)$. Since ϕ^{ϵ_n} has no zeros in $B(x_0, 2r)$, θ_{ϵ_n} is harmonic in $B(x_0, 2r)$. Therefore, $\mathbf{div} A^{\epsilon_n} = 0$ and it follows from (1.7) that

$$\Delta u_{\epsilon_n} = \frac{4}{\epsilon_n^2} e^{u_{\epsilon_n}}(e^{u_{\epsilon_n}} - 1)$$

on $B(x_0, 2r)$. We observe from (1.10) that $1 - e^{2u_{\epsilon_n}} \le C\epsilon_n^2$ on $B(x_0, 2r)$. Hence (u_ϵ) is uniformly bounded in $W^{2,p}(B(x_0, r), \mathbb{R})$ for all $p > 1$. Consequently, $(\phi^{\epsilon_n}, A^\epsilon)$ is uniformly bounded in $W^{2,p}(B(x_0, r), \mathbb{C}) \times W^{1,p}(B(x_0, r), \mathbb{R}^2)$, and there exist a configuration $(\phi_*, A_*) \in W^{2,p}(B(x_0, r), \mathbb{C}) \times W^{1,p}(B(x_0, r), \mathbb{R}^2)$ such that passing to a subsequence, if necessary, we have

$$\begin{aligned} \phi^{\epsilon_n} \to \phi_* \quad & \text{in} \quad C^{1,\alpha}(B(x_0, r), \mathbb{C}), \\ A^{\epsilon_n} \to A_* \quad & \text{in} \quad C^{0,\alpha}(B(x_0, r), \mathbb{R}^2), \end{aligned}$$

with $\mathbf{div} A_* = 0$.

Let us denote $(\phi^{\epsilon_n}, A^{\epsilon_n})$ by (ϕ^n, A^n) for simplicity. Consider the identity

$$\begin{aligned} \overline{\phi^n} D_{A^n}\phi^n \cdot \nabla|\phi^n|^2 &= \overline{\phi^n}^2 D_{A^n}\phi^n \cdot D_{A^n}\phi^n + |\phi^n|^2 |D_{A^n}\phi^n|^2 \\ &= |\phi^n|^2 |D_{A^n}\phi^n|^2, \end{aligned}$$

where the last equality follows from (1.4). Letting $n \to \infty$ and keeping $|\phi_*| = 1$ in mind, we conclude that

$$|D_{A^n}\phi^n|^2 \to |D_{A_*}\phi_*| = 0 \quad \text{on} \;\; B(x_0, r). \tag{2.2}$$

Hence we obtain

$$A_* = -i\overline{\phi_*}\nabla\phi_*. \tag{2.3}$$

Since $\mathbf{div}A_* = 0$, the substitution of (2.3) into the equation $D^2_{A_*}\phi_* = 0$ gives

$$\Delta\phi_* + \phi_*|\nabla\phi_*|^2 = 0 \tag{2.4}$$

on $B(x_0, r)$.

Let $\eta = \min_{i\neq j}|p_i - p_j|/4$. We may assume by (1.10) that $|\phi^n| \geq 1/2$ on $\Omega\setminus\bigcup_{j=1}^k B(p_j, \eta)$. Hence $\mathbf{deg}(p_j, \phi^n)$, the degree of $\phi^n/|\phi^n|$ considered as a map from $\partial B(p_j, 2\eta)$ into S^1, is well-defined as

$$\mathbf{deg}(p_j, \phi^n) = \frac{1}{2\pi}\int_{\partial B(p_j,2\eta)} \frac{1}{|\phi^n|^2}\mathrm{Im}\Big(\overline{\phi^n}\frac{\partial\phi^n}{\partial\tau}\Big)d\sigma, \tag{2.5}$$

where τ is the unit tangent vector field to $\partial B(p_j, 2\eta)$. Since ϕ^n has a unique zero p_j with multiplicity n_j on $B(p_j, 2\eta)$, we have $\mathbf{deg}(p_j, \phi^n) = n_j$. Since

$$\phi^n \to \phi_* \quad \text{in} \;\; C^{1,\alpha}(\Omega\setminus\bigcup_{j=1}^k B(p_j, \eta), \mathbb{C}),$$

it follows that

$$n_j = \mathbf{deg}(p_j, \phi^n) \to \mathbf{deg}(p_j, \phi_*).$$

Now by (1.6) we can write

$$\phi^n(z) = (z - p_1)^{n_1}\cdots(z - p_k)^{n_k}e^{v_n(z)}$$

for some real-valued function v_n, and we may assume by the above argument that

$$\phi^n \to \phi_* = (z - p_1)^{n_1}\cdots(z - p_k)^{n_k}e^{v_*}$$

on Ω'. Since $|\phi_*| = 1$ on Ω', we conclude that

$$e^{v_*} = 1/|z - p_1|^{n_1}\cdots|z - p_k|^{n_k},$$

which proves (1.13).

Since $\epsilon_n \to 0$ was arbitrary, the convergence holds for the whole sequence. This completes the proof of (1.12) and (1.13). □

Proof of (1.14). By use of (1.5), we can rewrite the energy density as

$$e_\epsilon(\phi, A) = |D_A\phi|^2 + \frac{2}{\epsilon^2}|\phi|^2(1 - |\phi|^2)^2.$$

Let $\epsilon_n \to 0$ be given. Write $e_n(\phi^n, A^n) = e_{\epsilon_n}(\phi^{\epsilon_n}, A^{\epsilon_n})$ for brevity. Since $|\phi^n| < 1$, it follows from (1.9) and (2.2) that for any subset $K \subset\subset \Omega'$,

$$\int_K e_n(\phi^n, A^n) \to 0$$

as $n \to \infty$. Since $\|e_n(\phi^n, A^n)\|_{L^1(\Omega)} = 2\pi N$, after passing to a subsequence if necessary, we see that

$$e_n(\phi^n, A^n) \to \sum_{j=1}^{k} \alpha_j \delta_{p_j}$$

weakly in the sense of measure.

Let $B_j = B(p_j, 2\eta)$ where η is given as above. Then by (1.4),

$$\begin{aligned} \int_{B_j} |D_{A^n}\phi^n|^2 &= \int_{B_j} |D_1\phi^n + iD_2\phi^n|^2 + 2\mathrm{Re}(iD_1\phi^n \overline{D_2\phi^n}) \\ &= \int_{B_j} 2\mathrm{Re}(i\partial_1\phi^n \partial_2\overline{\phi^n}) + \int_{B_j} 2\mathrm{Re}(-\overline{\phi^n}A_2\partial_1\phi^n + \phi^n A_1 \partial_2\overline{\phi^n}) \\ &= (I) + (II). \end{aligned}$$

We observe that

$$\begin{aligned} (I) &= \int_{\partial B_j} \mathrm{Im}\Big(\overline{\phi^n}\frac{\partial\phi^n}{\partial\tau}\Big) \\ &= 2\pi\mathbf{deg}(p_j, \phi^n) + \int_{\partial B_j} (|\phi^n|^2 - 1)\frac{1}{|\phi^n|^2}\mathrm{Im}\Big(\overline{\phi^n}\frac{\partial\phi^n}{\partial\tau}\Big) \\ &= 2\pi n_j + o(1). \end{aligned}$$

The last equality follows from (1.10).

On the other hand, a short computation yields that

$$\begin{aligned} (II) &= -\int_{\partial B_j} |\phi^n|^2 A^n \cdot \tau + \int_{B_j} |\phi^n|^2 F_{A^n} \\ &= \int_{\partial B_j} (1 - |\phi^n|^2) A^n \cdot \tau - \int_{B_j} (1 - |\phi^n|^2) F_{A^n} \\ &= o(1) - \int_{B_j} \frac{2}{\epsilon_n^2}|\phi^n|^2(1 - |\phi^n|^2)^2 \end{aligned}$$

as $n \to \infty$, where the last equality comes from (1.5) and (1.10). We thus obtain

$$\int_{B_j} |D_{A^n}\phi^n|^2 + \frac{2}{\epsilon_n^2}|\phi^n|^2(1 - |\phi^n|^2)^2 = 2\pi n_j + o(1),$$

which implies that $\alpha_j = 2\pi n_j$. Since $\epsilon_n \to 0$ was arbitrary, we are led to (1.14). $\square$

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