PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 6, Pages 1687–1694
S 0002-9939(02)06741-2
Article electronically published on October 1, 2002

ANDRÉ-QUILLEN HOMOLOGY VIA FUNCTOR HOMOLOGY

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(Communicated by Paul Goerss)

ABSTRACT. We obtain André-Quillen homology for commutative algebras using relative homological algebra in the category of functors on finite pointed sets.

1. Introduction

Let Γ be the small category of finite pointed sets. For any $n \geq 0$, let $[n]$ be the set $\{0, 1, ..., n\}$ with basepoint 0. We assume that the objects of Γ are the sets $[n]$. A left Γ-module is a covariant functor $\Gamma \to \mathsf{Vect}$ to the category of vector spaces over a field K. For a left Γ-module F we put

$$\pi_0(F) := \mathsf{Coker}(d_0 - d_1 + d_2 : F([2]) \to F([1])),$$

where d_1 is induced by the folding map $[2] \to [1]$, $1, 2 \mapsto 1$ while d_0 and d_2 are induced by the projection maps $[2] \to [1]$ given respectively by $1 \mapsto 1, 2 \mapsto 0$ and $1 \mapsto 0, 2 \mapsto 1$. The category Γ-*mod* of left Γ-modules is an abelian category with enough projective and injective objects. Therefore one can form the left derived functors of the functor $\pi_0 : \Gamma\text{-}mod \to \mathsf{Vect}$, which we will denote by π_*. Thanks to [5] and [6] we know that $\pi_* F$ is isomorphic to the homotopy of the spectrum corresponding to the Γ-space F according to Segal (see [9] and [1]).

Let A be a commutative algebra over a ground field K and let M be an A-module. There exists a functor $\mathcal{L}(A, M) : \Gamma \to \mathsf{Vect}$, which assigns $M \otimes A^{\otimes n}$ to $[n]$ (see [3] or section 3). Here all tensor products are taken over K. It was proved in [7] that $\pi_*(\mathcal{L}(A, M))$ is isomorphic to a brave new algebra version of André-Quillen homology $\mathsf{H}^\Gamma_*(A, M)$ constructed by Alan Robinson and Sarah Whitehouse [10]. The main result of this paper shows that a similar isomorphism also exists for André-Quillen homology if one takes an appropriate relative derived functor of the same functor $\pi_0 : \Gamma\text{-}mod \to \mathsf{Vect}$.

2. A class of proper exact sequences

Thanks to the Yoneda lemma, Γ^n, $n \geq 0$, are projective generators of the category Γ-*mod*. Here

$$\Gamma^n := K[\mathsf{Hom}_\Gamma([n], -)],$$

Received by the editors October 13, 2001 and, in revised form, January 17, 2002.
2000 *Mathematics Subject Classification.* Primary 13D03, 18G25.
The author was partially supported by the grant INTAS-99-00817 and by the TMR network K-theory and algebraic groups, ERB FMRX CT-97-0107.

and $K[S]$ denotes the free vector space spanned by a set S. For left Γ-modules F and T one defines the pointwise tensor product $F \otimes T$ to be the left Γ-module given by $(F \otimes T)([n]) = F([n]) \otimes T([n])$. Since $\Gamma^n \otimes \Gamma^m \cong \Gamma^{n+m}$ one sees that the tensor product of two projective left Γ-modules is still projective. We also have $\Gamma^n \cong (\Gamma^1)^{\otimes n}$.

A *partition* $\lambda = (\lambda_1, \cdots, \lambda_k)$ is a sequence of natural numbers $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$. The sum of partition is given by $s(\lambda) := \lambda_1 + \cdots + \lambda_k$, while the group $\Sigma(\lambda)$ is a product of the corresponding symmetric groups

$$\Sigma(\lambda) := \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_k},$$

which is identified with the Young subgroup of $\Sigma_{s(\lambda)}$. Let us observe that $\Sigma_n = Aut_\Gamma([n])$ and therefore Σ_n acts on $\Gamma^n \cong (\Gamma^1)^{\otimes n}$. For a partition λ with $s(\lambda) = n$ we let $\Gamma(\lambda)$ be the coinvariants of Γ^n under the action of $\Sigma(\lambda) \subset \Sigma_n$.

For a vector space V we let $S^*(V)$, $\Lambda^*(V)$ and $D^*(V)$ be respectively the symmetric, exterior and divided power algebra generated by V. Let us recall that $S^n(V) = (V^{\otimes n})/\Sigma_n$ is the space of coinvariants of $V^{\otimes n}$ under the action of the symmetric group Σ_n, while $D^n(V) = (V^{\otimes n})^{\Sigma_n}$ is the space of invariants. Moreover for a partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ we put

$$S^\lambda := S^{\lambda_1} \otimes \cdots \otimes S^{\lambda_k}.$$

We similarly define Λ^λ and D^λ. It follows from the definition that

$$\Gamma(\lambda) \cong S^\lambda \circ \Gamma^1.$$

In particular $\Gamma(1, \cdots, 1) \cong \Gamma^n$ and $\Gamma(n) \cong S^n \circ \Gamma^1$.

Let

$$0 \to T_1 \to T \to T_2 \to 0$$

be an exact sequence of left Γ-modules. It is called a *$\mathcal{Y}$-exact sequence* if for any partition λ with $s(\lambda) = n$ the induced map

$$T([n])^{\Sigma(\lambda)} \to T_2([n])^{\Sigma(\lambda)}$$

is surjective. Here and elsewhere, M^G denotes the subspace of G-fixed elements of a G-module M. For a $\mathcal{Y}$-exact sequence $0 \to T_1 \to T \to T_2 \to 0$ the sequence

$$0 \to T_1([n])^{\Sigma(\lambda)} \to T([n])^{\Sigma(\lambda)} \to T_2([n])^{\Sigma(\lambda)} \to 0$$

is also exact. Following to Section XII.4 of [4] we introduce the relative notions. An epimorphism $f : F \to T$ is called a *$\mathcal{Y}$-epimorphism* if

$$0 \to \mathsf{Ker}(f) \to F \to T \to 0$$

is a $\mathcal{Y}$-exact sequence. Similarly, a monomorphism $f : F \to T$ is called a *$\mathcal{Y}$-monomorphism* if

$$0 \to F \to T \to \mathsf{Coker}(f) \to 0$$

is a $\mathcal{Y}$-exact sequence. A morphism $f : F \to T$ is called a *$\mathcal{Y}$-morphism* if $F \to \mathsf{Im}(f)$ is a $\mathcal{Y}$-epimorphism and $\mathsf{Im}(f) \to T$ is a $\mathcal{Y}$-monomorphism. A left Γ-module Z is called *$\mathcal{Y}$-projective* if for any $\mathcal{Y}$-epimorphism $f : F \to T$ and any morphism $g : Z \to T$ there exists a morphism $h : Z \to F$ such that $g = fh$.

Lemma 2.1. i) *If a short exact sequence is isomorphic to a $\mathcal{Y}$-exact sequence, then it is also a $\mathcal{Y}$-exact sequence.*

ii) *A split short exact sequence is $\mathcal{Y}$-exact.*

iii) *A composition of two $\mathcal{Y}$-epimorphisms is still a $\mathcal{Y}$-epimorphism.*

iv) *If f and g are two composable epimorphisms and fg is a $\mathcal{Y}$-epimorphism, then f is also a $\mathcal{Y}$-epimorphism.*
v) *A composition of two $\mathcal{Y}$-monomorphisms is still a $\mathcal{Y}$-monomorphism.*
vi) *If f and g are two composable monomorphisms and fg is a $\mathcal{Y}$-monomorphism, then g is also a $\mathcal{Y}$-monomorphism.*

Proof. The properties i)- iv) are clear. Let $f : B \to C$ and $g : A \to B$ be monomorphisms. One can form the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{g} & B & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle f} & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{fg} & C & \longrightarrow & Z & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Y & \xrightarrow{1_Y} & Y & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Assume f and g are are $\mathcal{Y}$-monomorphisms; then for any partition λ with $s(\lambda) = n$ one has a commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A([n])^{\Sigma(\lambda)} & \longrightarrow & B([n])^{\Sigma(\lambda)} & \longrightarrow & X([n])^{\Sigma(\lambda)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_A} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A([n])^{\Sigma(\lambda)} & \longrightarrow & C([n])^{\Sigma(\lambda)} & \xrightarrow{h} & Z([n])^{\Sigma(\lambda)} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Y([n])^{\Sigma(\lambda)} & \xrightarrow{1_Y} & Y([n])^{\Sigma(\lambda)} & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

The diagram chasing shows that h is an epimorphism and therefore fg is a $\mathcal{Y}$-monomorphism and v) is proved. Assume now that fg is a $\mathcal{Y}$-monomorphism.

Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A([n])^{\Sigma(\lambda)} & \longrightarrow & B([n])^{\Sigma(\lambda)} & \xrightarrow{l} & X([n])^{\Sigma(\lambda)} & & \\
 & & \downarrow{\scriptstyle 1_A} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A([n])^{\Sigma(\lambda)} & \longrightarrow & C([n])^{\Sigma(\lambda)} & \longrightarrow & Z([n])^{\Sigma(\lambda)} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Y([n])^{\Sigma(\lambda)} & \xrightarrow{1_Y} & Y([n])^{\Sigma(\lambda)} & &
\end{array}$$

The diagram chasing shows that l is an epimorphism and therefore f is a $\mathcal{Y}$-monomorphism and therefore we get vi). □

As an immediate corollary we obtain that the class of all $\mathcal{Y}$-exact sequences is proper in the sense of Mac Lane [4]. We now show that there are enough $\mathcal{Y}$-projective objects.

Lemma 2.2. i) *For any partition λ the left Γ-module $\Gamma(\lambda)$ is a $\mathcal{Y}$-projective object.*

ii) *A morphism $f : F \to T$ of left Γ-modules is a $\mathcal{Y}$-epimorphism iff for any partition λ the induced morphism*

$$\mathsf{Hom}_{\Gamma-mod}(\Gamma(\lambda), F) \to \mathsf{Hom}_{\Gamma-mod}(\Gamma(\lambda), T)$$

is an epimorphism.

iii) *For any left Γ-module F there is a $\mathcal{Y}$-projective object Z and a $\mathcal{Y}$-epimorphism $f : Z \to F$.*

iv) *Any projective $\mathcal{Y}$-module is a direct summand of the sum of objects of the form $\Gamma(\lambda)$.*

v) *The tensor product of two $\mathcal{Y}$-projective left Γ-modules is still $\mathcal{Y}$-projective.*

Proof. Let λ be a partition with $s(\lambda) = n$. By definition $\Gamma(\lambda) = H_0(\Sigma(\lambda), \Gamma^n)$. Hence for any left Γ-module F one has

$$\mathsf{Hom}_{\Gamma-mod}(\Gamma(\lambda), F) \cong H^0(\Sigma(\lambda), \mathsf{Hom}_{\Gamma-mod}(\Gamma^n, F) \cong F(n)^{\Sigma(\lambda)}.$$

The assertions i) and ii) are immediate consequences of this isomorphism. To prove iii) we set

$$X(\lambda) := \mathsf{Hom}_{\Gamma-mod}(\Gamma(\lambda), F).$$

Moreover, for each $x \in X(\lambda)$ we let $f_x : \Gamma(\lambda) \to F$ be the corresponding morphism. Take $Z = \bigoplus_\lambda \bigoplus_{x \in X(\lambda)} \Gamma(\lambda)$. Then the collection f_x, $x \in X(\lambda)$, yields the morphism $f : Z \to F$. We have to show that it is a $\mathcal{Y}$-epimorphism. Let $g : \Gamma(\lambda) \to F$ be a morphism of left Γ-modules. By ii) we need to lift g to Z. By our construction $g \in X(\lambda)$ and therefore the inclusion $\Gamma(\lambda) \to Z$ corresponding to the summand $g \in X(\lambda)$ is an expected lifting and iii) is proved. The proof of iii) shows that one can assume P to be a sum of Γ^λ and iv) follows. To prove the last statement one observes that, for any partitions λ and μ, one has

$$\Gamma(\lambda) \otimes \Gamma(\mu) \cong (\Gamma^{s(\lambda)} \otimes \Gamma^{s(\mu)})^{\Sigma(\lambda)\times\Sigma(\mu)} = (\Gamma^{s(\lambda)+s(\mu)})^{\Sigma(\lambda)\times\Sigma(\mu)}$$

and therefore $\Gamma(\lambda) \otimes \Gamma(\mu)$ is $\mathcal{Y}$-projective. □

3. Definition of André-Quillen homology and the functor $\mathcal{L}$

The definition of André-Quillen homology is based on the framework of homotopical algebra [8] and it is given as follows. We let $C_*(V_*)$ be the chain complex associated to a simplicial vector space V_*. Let A be a commutative algebra over a ground field K and let M be an A-module. *A simplicial resolution of* A is an augmented simplicial object $P_* \to A$ in the category of commutative algebras, which is a weak equivalence (in other words $C_*(P_*) \to A$ is a weak equivalence). A simplicial resolution is called *free* if P_n is a polynomial algebra over K for all $n \geq 0$. Any commutative algebra possesses a free simplicial resolution which is unique up to homotopy. Then the André-Quillen homology is defined by

$$\mathsf{D}_*(A, M) := H_*(C_*(\Omega^1_{P_*} \otimes_{P_*} M)),$$

where Ω^1 is the Kähler 1-differential and $P_* \to A$ is a free simplicial resolution. In dimension 0 we have $\mathsf{D}_0(A, M) \cong \Omega^1_A \otimes_A M$.

As we mentioned above the functor $\mathcal{L}(A, M) : \Gamma \to \mathsf{Vect}$ is given on objects by $[n] \mapsto M \otimes A^{\otimes n}$. For a pointed map $f : [n] \to [m]$, the action of f on $\mathcal{L}(A, M)$ is given by

$$f_*(a_0 \otimes \cdots \otimes a_n) := b_0 \otimes \cdots \otimes b_m,$$

where $b_j = \prod_{f(i)=j} a_i,\ j = 0, \cdots, n.$

Example 3.1. Let $M = A = K[t]$. In this case one has an isomorphism

$$\mathcal{L}(K[t], K[t]) \cong S^* \circ \Gamma^1.$$

To see this isomorphism, one observes that Γ^1 assigns the free vector space on a set $[n]$ to $[n]$ and therefore both functors in question assign the ring $K[t_0, \cdots, t_n]$ to $[n]$. An important consequence of this isomorphism is the fact that the functor $\mathcal{L}(K[t], K[t])$ is $\mathcal{Y}$-projective.

Lemma 3.2. *For any commutative algebra A and any A-module M, one has a natural isomorphism $\pi_0(\mathcal{L}(A, M)) \cong \Omega^1_A \otimes_A M$.*

Proof. By the definition we have $\pi_0(\mathcal{L}(A, M)) = \mathsf{Coker}(b : M \otimes A^{\otimes 2} \to M \otimes A)$, where $b(m \otimes a \otimes b) = ma \otimes b - m \otimes ab + mb \otimes a$. Since

$$adb \otimes m \mapsto (ma \otimes b)\, mod\, \mathsf{Im}(b)$$

yields the isomorphism $\Omega^1_A \otimes_A M \to \mathsf{Coker}(b)$, the result follows. □

Lemma 3.3. i) *Let A be a commutative algebra and let*

$$0 \to M_1 \to M \to M_2 \to 0$$

be a short exact sequence of A-modules. Then

$$0 \to \mathcal{L}(A, M_1) \to \mathcal{L}(A, M) \to \mathcal{L}(A, M_2) \to 0$$

is a $\mathcal{Y}$-exact sequence.

ii) *Let $f : B \to A$ be a surjective homomorphism of commutative algebras. Then for any A-module M the induced morphism of left Γ-modules*

$$\mathcal{L}(B, M) \to \mathcal{L}(A, M)$$

is a $\mathcal{Y}$-epimorphism.

Proof. One observes that for any partition λ with $s(\lambda)=n$ one has

$$(\mathcal{L}(A,M)([n]))^{\Sigma(\lambda)} = (M\otimes A^{\otimes n})^{\Sigma(\lambda)} \cong M\otimes D^{\lambda}(A).$$

Since we are over a field the tensor product is exact and we obtain i). By the same reason f has a linear section, which also yields a linear section of $D^{\lambda}(B)\to D^{\lambda}(A)$, because D^{λ} is a functor defined on the category of vector spaces. □

4. Relative derived functors

By Lemma 2.2 the class of $\mathcal{Y}$-exact sequences has enough projective objects. Thanks to [4] this allows us to construct the relative derived functors. Let us recall that an augmented chain complex $X_*\to F$ is called a *$\mathcal{Y}$-resolution* of F if it is exact (that is, $H_i(X_*)=0$ for $i>0$ and $H_0(X_*)\cong F$) and all boundary maps $X_{n+1}\to X_n$ are $\mathcal{Y}$-morphisms, $n\geq 0$. It follows from Lemma 2.2 that $X_*\to F$ is a $\mathcal{Y}$-resolution iff for any partition λ the augmented complex

$$\mathsf{Hom}_{\Gamma-mod}(\Gamma(\lambda),X_*)\to \mathsf{Hom}_{\Gamma-mod}(\Gamma(\lambda),F)$$

is exact. A $\mathcal{Y}$-resolution $Z_*\to F$ is called a *$\mathcal{Y}$-projective resolution* if for all $n\geq 0$ the left Γ-module Z_n is a $\mathcal{Y}$-projective object. We define $\pi_*^{\mathcal{Y}}(F)$ using relative derived functors of the functor $\pi_0:\Gamma\text{-}mod\to \mathsf{Vect}$. In other words we put

$$\pi_n^{\mathcal{Y}}(F):=H_n(\pi_0(Z_*)),\ n\geq 0,$$

where $Z_*\to F$ is a $\mathcal{Y}$-projective resolution. By [4] this gives the well-defined functors $\pi_n^{\mathcal{Y}}:\Gamma\text{-}mod\to\mathsf{Vect},\ \ n\geq 0$.

Lemma 4.1. *If K is a field of characteristic zero, then $\pi_*(F)\cong\pi_*^{\mathcal{Y}}(F)$.*

Proof. In this case all exact sequences are $\mathcal{Y}$-exact, because for any finite group G, the functor $M\mapsto M^G$ is exact. □

Lemma 4.2. *For left Γ-modules F,T one has an isomorphism*

$$\pi_*^{\mathcal{Y}}(F\otimes T)\cong \pi_*^{\mathcal{Y}}(F)\otimes T([0])\oplus F([0])\otimes\pi_*^{Y}(T).$$

Proof. The result in dimension 0 is known (see Lemma 4.2 of [5]). Let $Z_*\to F$ and $R_*\to T$ be $\mathcal{Y}$-projective resolutions. By Lemma 2.2 $Z_*\otimes R_*\to F\otimes T$ is also a $\mathcal{Y}$-projective resolution. Thus

$$\begin{aligned}\pi_*^{\mathcal{Y}}(F\otimes T) &= H_*(\pi_0(Z_*\otimes R_*))\\ &\cong H_*(\pi_0^{\mathcal{Y}}(Z_*)\otimes R_*([0])\oplus Z_*([0])\otimes\pi_0^{Y}(R_*))\\ &\cong \pi_*^{\mathcal{Y}}(F)\otimes T([0])\oplus F([0])\otimes \pi_*^{Y}(T),\end{aligned}$$

where the last isomorphism follows from the Eilenberg-Zilber theorem and Künneth theorem. □

Lemma 4.3. *Let $\epsilon:X_*\to A$ be a simplicial resolution in the category of commutative algebras and let M be an A-module. Then the associated chain complex of the simplicial Γ-module $C_*(\mathcal{L}(X_*,M))\to\mathcal{L}(A,M)$ is a $\mathcal{Y}$-resolution.*

Proof. Since ϵ is a weak equivalence of simplicial algebras it is a homotopy equivalence in the category of simplicial vector spaces. Thus $M\otimes D^{\lambda}(X_*)\to M\otimes D^{\lambda}(A_*)$ is also a homotopy equivalence, for any partition λ. It follows that

$$\mathcal{L}(X_*,M)([n])^{\Sigma(\lambda)}\to \mathcal{L}(A,M)([n])^{\Sigma(\lambda)}$$

is also a homotopy equivalence of simplicial vector spaces. □

The following is our main result.

Theorem 4.4. *For any commutative ring A and any A-module M, there is a canonical isomorphism*

$$\mathsf{D}_i(A, M) \cong \pi_i^{\mathcal{Y}}(\mathcal{L}(A, M)),\ \ i \geq 0,$$

between the André-Quillen homology and relative derived functors of π_0 applied on the functor $\mathcal{L}(A, M)$.

Proof. Thanks to Lemma 3.2 the result is true for $i = 0$. First consider the case when $M = A = K[t]$. In this case the André-Quillen homology vanishes in positive dimensions by definition. On the other hand $\mathcal{L}(K[t], K[t])$ is $\mathcal{Y}$-projective thanks to Example 3.1 and therefore $\pi_i^{\mathcal{Y}}(\mathcal{L}(A, M))$ vanishes for all $i > 0$. One can use Lemma 4.2 to conclude that $\pi_i^{\mathcal{Y}}(\mathcal{L}(A, A))$ vanishes for all $i > 0$ provided A is a polynomial algebra. For the next step, we prove that the result is true if A is a polynomial algebra and M is any A-module. We have to prove that $\pi_i^{\mathcal{Y}}(\mathcal{L}(A, M))$ also vanishes for $i > 0$. We already proved this fact if $M = A$. By additivity the functor $\pi_i^{\mathcal{Y}}(\mathcal{L}(A, -))$ vanishes on free A-modules. By Lemma 3.3 the functor $\pi_*^{\mathcal{Y}}(\mathcal{L}(A, -))$ assigns the long exact sequence to a short exact sequence of A-modules. Therefore we can consider such an exact sequence associated to a short exact sequence of A-modules

$$0 \to N \to F \to M \to 0$$

with free F. Since the result is true if $i = 0$, one obtains by induction on i that $\pi_i^{\mathcal{Y}}(\mathcal{L}(A, M)) = 0$ provided $i > 0$. Now consider the general case. Let $P_* \to A$ be a free simplicial resolution in the category of commutative algebras. Then we have

$$\Omega^1_{P_*} \otimes_{P_*} M \cong \pi_0^{\mathcal{Y}}(\mathcal{L}(P_*, M)).$$

Thanks to Lemma 4.3 $C_*(\mathcal{L}(P_*, M)) \to \mathcal{L}(A, M)$ is a $\mathcal{Y}$-resolution consisting of $\pi_*^{\mathcal{Y}}$-acyclic objects and the result follows. □

The main theorem together with the main result of [7] yields:

Corollary 4.5. *If $Char(K) = 0$, then for any commutative algebra A and any A-module M one has a natural isomorphism*

$$\mathsf{D}_*(A, M) \cong \mathsf{H}_*^{\Gamma}(A, M).$$

This fact was also proved in [10] based on the combinatorical and homotopical analysis of the space of fully grown trees.

Remarks. i) We let $t : \Gamma^{op} \to \mathsf{Vect}$ be the functor which assigns the vector space of all maps $f : [n] \to K,\ f(0) = 0$ to $[n]$. Then $t \otimes_\Gamma F \cong \pi_0(F)$ (see Proposition 2.2 of [5]). Hence $\pi_*^{\mathcal{Y}}$ can also be defined as the relative derived functors of the functor $t \otimes_\Gamma (-) : \Gamma\text{-}mod \to \mathsf{Vect}$. More generally one can take any functor $T : \Gamma^{op} \to \mathsf{Vect}$ and define $\mathsf{Tor}_*^{\mathcal{Y}}(T, F)$ as the value of the relative derived functors (with respect to $\mathcal{Y}$-exact sequences) of the functor $T \otimes_\Gamma (-) : \Gamma\text{-}mod \to \mathsf{Vect}$. Then our result claims that

$$\mathsf{D}_*(A, M) \cong \mathsf{Tor}_*^{\mathcal{Y}}(t, \mathcal{L}(A, M)).$$

Based on Proposition 1.15 of [5] the argument given above shows that

$$\mathsf{D}_*^{\{n\}}(A, M) \cong \mathsf{Tor}_*^{\mathcal{Y}}(\Lambda^n \circ t, \mathcal{L}(A, M)),$$

where $\mathsf{D}_*^{\{n\}}(A,M)$ are defined using Kähler n-differentials:

$$\mathsf{D}_*^{\{n\}}(A,M) := H_*(C_*(\Omega^n_{P_*} \otimes_{P_*} M))$$

and for $n = 1$ one recovers the main theorem.

ii) All results remains true if K is any commutative ring and A and M are projective as K-modules.

Acknowledgments

This work was written during my visit to the Max-Planck-Institut für Mathematik at Bonn.

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