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RESCALINGS OF FREE PRODUCTS OF II_1–FACTORS

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Abstract. We introduce the notation $\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_r)$ for von Neumann algebra II_1–factors where r is allowed to be negative. This notation is defined by rescalings of free products of II_1–factors, and is proved to be consistent with known results and natural operations. We also give two statements which we prove are equivalent to isomorphism of free group factors.

Introduction

The rescaling $\mathcal{M}_t$ of a II_1–factor $\mathcal{M}$ by a positive number t was introduced by Murray and von Neumann [5]. In [4], we showed that if $\mathcal{Q}(1), \ldots, \mathcal{Q}(n)$ are II_1–factors ($n \in \{2, 3, \ldots\}$) and if $0 < t < \sqrt{1 - 1/n}$, then

$$\big(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n)\big)_t \cong \mathcal{Q}(1)_t * \cdots * \mathcal{Q}(n)_t * L(\mathbf{F}_r), \tag{1}$$

where $r = (n-1)(t^{-2} - 1)$. Here $L(\mathbf{F}_r)$, $r > 1$, is an interpolated free group factor ([2], [6]). For $\sqrt{1 - 1/n} \le t < 1$, we proved a similar formula, where $L(\mathbf{F}_r)$ was replaced by a hyperfinite von Neumann algebra with specified tracial state having free dimension ([1]) equal to $r = (n-1)(t^{-2} - 1) \le 1$. If one tries to use the formula (1) when $t > 1$, one obtains $L(\mathbf{F}_r)$ with $r < 0$.

In this note we introduce the notation

$$\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_r) \qquad (n \in \mathbf{N},\ 1 - n < r \le \infty). \tag{2}$$

If $r > 1$, then $L(\mathbf{F}_r)$ in (2) is an interpolated free group factor, while if $r \le 1$, then (2) defines a II_1 factor which is the rescaling by t of $\mathcal{Q}(1)_{1/t} * \cdots * \mathcal{Q}(n)_{1/t}$ if $n = 2$ or of $\mathcal{Q}(1)_{1/t} * L(\mathbf{F}_2)$ if $n = 1$ for an appropriate $t > 1$. We will prove that this notation is consistent with known results and natural operations involving free products. The notation (2) provides an elegent means of describing rescalings of free products of II_1–factors, and is used in [3].

Finally, we show that if the free group factors are isomorphic to each other, then $\mathcal{Q}(1) * \mathcal{Q}(2) \cong \mathcal{Q}(1) * \mathcal{Q}(2) * L(\mathbf{F}_\infty)$ for all II_1–factors $\mathcal{Q}(1)$ and $\mathcal{Q}(2)$ and we give one additional equivalent condition. It is conceivable that these conditions may be used to prove nonisomorphism of free group factors.

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RESCALINGS

Recall that the interpolated free group factors rescale as follows:

$$L(\mathbf{F}_r)_t \cong L(\mathbf{F}_{1+t^{-2}(r-1)}) \qquad (1 < r \leq \infty,\ 0 < t < \infty), \tag{3}$$

(see [2], [6]).

Lemma 1. *Let* $n \in \mathbf{N}$*, let* $\mathcal{Q}(1), \dots, \mathcal{Q}(n)$ *be II*$_1$*–factors, let* $1 < r \leq \infty$*, and let*

$$\mathcal{M} = \mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_r).$$

Then for every $0 < t < \sqrt{1 + (r-1)/n}$*,*

$$\mathcal{M}_t \cong \mathcal{Q}(1)_t * \cdots * \mathcal{Q}(n)_t * L(\mathbf{F}_{t^{-2}r+(n-1)(t^{-2}-1)}).$$

Proof. If $t \leq 1$, then this follows from [4]; see (1) and (3) above. Suppose $t > 1$. Note that t is taken so that $t^{-2}r + (n-1)(t^{-2}-1) > 1$. Applying (1) and (3), we have

$$\Big(\mathcal{Q}(1)_t * \cdots * \mathcal{Q}(n)_t * L(\mathbf{F}_{t^{-2}r+(n-1)(t^{-2}-1)})\Big)_{\frac{1}{t}} \cong \mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_r).$$

□

Proposition 2. *Let* $n \in \mathbf{N}$*, let* $\mathcal{Q}(1), \dots, \mathcal{Q}(n)$ *be II*$_1$*–factors, and let*

$$1 - n < r \leq 1.$$

Then there is a II$_1$*–factor* $\mathcal{M}$*, unique up to isomorphism, such that*

$$\mathcal{M}_t \cong \mathcal{Q}(1)_t * \cdots * \mathcal{Q}(n)_t * L(\mathbf{F}_{t^{-2}r+(n-1)(t^{-2}-1)}),$$

whenever $0 < t < \sqrt{1 + (r-1)/n}$*.*

Proof. Let $0 < s < t < \sqrt{1 + (r-1)/n}$ and let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be II$_1$–factors such that

$$\mathcal{M}_s \cong \mathcal{Q}(1)_s * \cdots * \mathcal{Q}(n)_s * L(\mathbf{F}_{s^{-2}r+(n-1)(s^{-2}-1)}),$$

$$\widetilde{\mathcal{M}}_t \cong \mathcal{Q}(1)_t * \cdots * \mathcal{Q}(n)_t * L(\mathbf{F}_{t^{-2}r+(n-1)(t^{-2}-1)}).$$

Then using Lemma 1 we have

$$\widetilde{\mathcal{M}}_s = (\widetilde{\mathcal{M}}_t)_{\frac{s}{t}} \cong \mathcal{Q}(1)_s * \cdots * \mathcal{Q}(n)_s * L(\mathbf{F}_{s^{-2}r+(n-1)(s^{-2}-1)}) \cong \mathcal{M}_s.$$

□

Definition 3. We denote the unique factor $\mathcal{M}$ in Proposition 2 by

$$\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_r).$$

Proposition 4. *Let* $n \in \mathbf{N}$*, let* $\mathcal{Q}(1), \mathcal{Q}(2), \dots, \mathcal{Q}(n)$ *be II*$_1$*–factors, and let* $1 - n < r \leq \infty$*.*

(i) *If* $0 < t < \infty$*, then*

$$\big(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_r)\big)_t \cong \mathcal{Q}(1)_t * \cdots * \mathcal{Q}(n)_t * L(\mathbf{F}_{t^{-2}r+(n-1)(t^{-2}-1)}).$$

(ii) *If* σ *is a permutation of* $\{1, 2, \dots, n\}$*, then*

$$\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_r) \cong \mathcal{Q}(\sigma(1)) * \cdots * \mathcal{Q}(\sigma(n)) * L(\mathbf{F}_r).$$

(iii) *If* $\mathcal{Q}(1) = L(\mathbf{F}_s)$ *with* $1 < s \leq \infty$*, then*

$$\mathcal{Q}(1) * \cdots \mathcal{Q}(n) * L(\mathbf{F}_r) \cong \begin{cases} \mathcal{Q}(2) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_{r+s}) & \text{if } n \geq 2, \\ L(\mathbf{F}_{r+s}) & \text{if } n = 1. \end{cases}$$

(iv) *If* $n \geq 2$ *and if* $r > 2 - n$, *then*

$$\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_r) \cong \mathcal{Q}(1) * \big(\mathcal{Q}(2) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_r)\big).$$

(v) *If* $\mathcal{Q}(1) = \mathcal{N}(1) * \mathcal{N}(2)$ *where* $\mathcal{N}(1)$ *and* $\mathcal{N}(2)$ *are* II_1*–factors, then*

$$\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_r) \cong \mathcal{N}(1) * \mathcal{N}(2) * \mathcal{Q}(2) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_r).$$

(vi) *If* $0 < \tilde{r} \leq \infty$, *then*

$$\big(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_r)\big) * L(\mathbf{F}_{\tilde{r}}) \cong \mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_{r+\tilde{r}}).$$

(vii) *If* $\tilde{n} \in \mathbf{N}$, *if* $\widetilde{\mathcal{Q}}(1), \ldots, \widetilde{\mathcal{Q}}(\tilde{n})$ *are* II_1*–factors, and if* $1 - \tilde{n} < \tilde{r} \leq \infty$, *then*

$$\begin{aligned}&\big(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_r)\big) * \big(\widetilde{\mathcal{Q}}(1) * \cdots * \widetilde{\mathcal{Q}}(\tilde{n}) * L(\mathbf{F}_{\tilde{r}})\big) \\ &\quad \cong \mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * \widetilde{\mathcal{Q}}(1) * \cdots * \widetilde{\mathcal{Q}}(\tilde{n}) * L(\mathbf{F}_{r+\tilde{r}}).\end{aligned}$$

(viii) *If* $n \geq 2$, *then*

$$\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_0) \cong \mathcal{Q}(1) * \cdots * \mathcal{Q}(n).$$

(ix) *If* $\mathcal{N}$ *is a* II_1*–factor and if* $\mathcal{A}$ *is a von Neumann algebra with specified normal faithful tracial state, where* $\mathcal{A} \neq \mathbf{C}$ *and* $\mathcal{A}$ *is either finite dimensional, hyperfinite, an interpolated free group factor or a (possibly countably infinite) direct sum of these, then*

$$\mathcal{N} * \mathcal{A} \cong \mathcal{N} * L(\mathbf{F}_r)$$

where r *is the quantity computed in* [1] *and (perhaps misleadingly) called the free dimension of* $\mathcal{A}$*; (in the revised notation of* [3, §1], r *is such that* $\mathcal{A}$ *has a generating set of free dimension* r*).*

Proof. For (i), if $r > 1$, then this is Lemma 1. If $r \leq 1$ but $0 < t < \sqrt{1 + (r-1)/n}$, then this is Definition 3. Suppose $r \leq 1$ and $\sqrt{1 + (r-1)/n} \leq t < \infty$. Let $\lambda > 0$ be such that $\lambda t < \sqrt{1 + (r-1)/n}$. Then applying Definition 3 twice gives

$$\begin{aligned}&\Big(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_r)\Big)_{\lambda t} \\ &\quad \cong \mathcal{Q}(1)_{\lambda t} * \cdots * \mathcal{Q}(n)_{\lambda t} * L(\mathbf{F}_{\lambda^{-2}t^{-2}r+(n-1)(\lambda^{-2}t^{-2}-1)}) \\ &\quad \cong \Big(\mathcal{Q}(1)_t * \cdots * \mathcal{Q}(n)_t * L(\mathbf{F}_{t^{-2}r+(n-1)(t^{-2}-1)})\Big)_{\lambda}.\end{aligned}$$

Now the proofs of (ii)–(viii) are obtained by rescaling both sides of the desired isomorphisms by the same $t > 0$ which is small enough and applying (i) and perhaps equation (1). For example, to prove (vii) let

$$0 < t < \min\Big(\frac{1}{\sqrt{2}}, \sqrt{1 + \frac{r-1}{n}}, \sqrt{1 + \frac{\tilde{r}-1}{\tilde{n}}}, \sqrt{1 + \frac{r+\tilde{r}-1}{n+\tilde{n}}}\Big)$$

and use (i) three times to get

$$\begin{aligned}&\Big(\big(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_r)\big) * \big(\widetilde{\mathcal{Q}}(1) * \cdots * \widetilde{\mathcal{Q}}(\tilde{n}) * L(\mathbf{F}_{\tilde{r}})\big)\Big)_t \\ &\quad \cong \Big(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L(\mathbf{F}_r)\Big)_t * \Big(\widetilde{\mathcal{Q}}(1) * \cdots * \widetilde{\mathcal{Q}}(\tilde{n}) * L(\mathbf{F}_{\tilde{r}})\Big)_t * L(\mathbf{F}_{t^{-2}-1}) \\ &\quad \cong \mathcal{Q}(1)_t * \cdots * \mathcal{Q}(n)_t * \widetilde{\mathcal{Q}}(1)_t * \cdots * \widetilde{\mathcal{Q}}(\tilde{n})_t * L(\mathbf{F}_{t^{-2}(r+\tilde{r})+(n+\tilde{n}-1)(t^{-2}-1)}) \\ &\quad \cong \Big(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * \widetilde{\mathcal{Q}}(1) * \cdots * \widetilde{\mathcal{Q}}(\tilde{n}) * L(\mathbf{F}_{r+\tilde{r}})\Big)_t.\end{aligned}$$

For (ix), if $k \in \mathbf{N}$ is large enough, then by [1], $M_k(\mathbf{C}) * \mathcal{A}$ is the interpolated free group factor $L(F_{r+1-k^{-2}})$. By [1, Thm. 1.2],

$$\begin{aligned}(\mathcal{N} * \mathcal{A})_{\frac{1}{k}} &\cong \big((\mathcal{N}_{\frac{1}{k}} \otimes M_k(\mathbf{C})) * \mathcal{A}\big)_{\frac{1}{k}} \cong \mathcal{N}_{\frac{1}{k}} * \big(M_k(\mathbf{C}) * \mathcal{A}\big)_{\frac{1}{k}} \\ &\cong \mathcal{N}_{\frac{1}{k}} * L(\mathbf{F}_{r+1-k^{-2}})_{\frac{1}{k}} \cong \mathcal{N}_{\frac{1}{k}} * L(\mathbf{F}_{k^2 r}) \cong \big(\mathcal{N} * L(\mathbf{F}_r)\big)_{\frac{1}{k}}. \qquad \square\end{aligned}$$

Formula (1) can now be extended to all values of t.

Theorem 5. *Let $n \in \{2, 3, \ldots\}$, let $\mathcal{Q}(1), \ldots, \mathcal{Q}(n)$ be II_1–factors, and let $0 < t < \infty$. Then*

$$\big(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n)\big)_t = \mathcal{Q}(1)_t * \cdots * \mathcal{Q}(n)_t * L(\mathbf{F}_{(n-1)(t^{-2}-1)}).$$

Proof. Use part (viii) followed by part (i) of Proposition 4. ☐

We know from [6] (see also [2]) that the interpolated free group factors $(L(\mathbf{F}_t))_{1<t\leq\infty}$ are either all isomorphic to each other or all mutually nonisomorphic.

Theorem 6. *The following are equivalent:*

(a) *$L(F_s) \cong L(F_t)$ for some, and then for all, $1 < s < t \leq \infty$;*

(b) *for every II_1–factor $\mathcal{Q}$ and every $r > 0$,*

$$\mathcal{Q} * L(\mathbf{F}_r) \cong \mathcal{Q} * L(\mathbf{F}_\infty);$$

(c) *for all II_1–factors $\mathcal{Q}(1)$ and $\mathcal{Q}(2)$,*

$$\mathcal{Q}(1) * \mathcal{Q}(2) \cong \mathcal{Q}(1) * \mathcal{Q}(2) * L(\mathbf{F}_\infty).$$

Proof. For (a) $\Longrightarrow$ (b), if $0 < t < \sqrt{r}$, then by part (i) of Proposition 4,

$$\big(\mathcal{Q} * L(\mathbf{F}_r)\big)_t \cong \mathcal{Q}_t * L(\mathbf{F}_{t^{-2}r}) \cong \mathcal{Q}_t * L(\mathbf{F}_\infty) \cong \big(\mathcal{Q} * L(\mathbf{F}_\infty)\big)_t,$$

while (b) $\Longrightarrow$ (a) can be seen by choosing $\mathcal{Q} = L(\mathbf{F}_2)$. For (a) $\Longrightarrow$ (c), if $0 < t < 1/\sqrt{2}$, then using Lemma 1,

$$\begin{aligned}\big(\mathcal{Q}(1) * \mathcal{Q}(2)\big)_t &\cong \mathcal{Q}(1)_t * \mathcal{Q}(2)_t * L(\mathbf{F}_{t^{-2}-1}) \\ &\cong \mathcal{Q}(1)_t * \mathcal{Q}(2)_t * L(\mathbf{F}_\infty) \cong \big(\mathcal{Q}(1) * \mathcal{Q}(2) * L(\mathbf{F}_\infty)\big)_t.\end{aligned}$$

Taking $\mathcal{Q}(1) \cong \mathcal{Q}(2) \cong L(\mathbf{F}_2)$ shows (c) $\Longrightarrow$ (a). ☐

References

[1] K. Dykema, *Free products of hyperfinite von Neumann algebras and free dimension,* Duke Math. J. **69** (1993), 97-119. MR **93m:**46071

[2] ———, *Interpolated free group factors,* Pacific J. Math. **163** (1994), 123-135. MR **95c:**46103

[3] ———, *Free subproducts and free scaled products of II_1–factors,* J. Funct. Anal. (to appear).

[4] K. Dykema, F. Rădulescu, *Compressions of free products of von Neumann algebras,* Math. Ann. **316** (2000), 61-82. MR **2001f:**46100

[5] F.J. Murray and J. von Neumann, *Rings of operators. IV,* Ann. of Math. **44** (1943), 716-808. MR **5:**101a

[6] F. Rădulescu, *Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index,* Invent. Math. **115** (1994), 347-389. MR **95c:**46102

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