TWO $F_{\sigma\delta}$ IDEALS

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Abstract. We find two $F_{\sigma\delta}$ ideals on $\mathbb{N}$ neither of which is $F_{\sigma}$ whose quotient Boolean algebras are homogeneous but nonisomorphic. This solves a problem of Just and Krawczyk (1984).

We consider Boolean algebras of the form $\mathcal{P}(\mathbb{N})/\mathcal{I}$, where $\mathcal{I}$ is an ideal on $\mathbb{N}$ containing the ideal $\text{Fin}$ of finite sets. In [3] Just and Krawczyk formulated several conditions on the ideals $\mathcal{I}, \mathcal{J}$ that guarantee their quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ to be isomorphic. By identifying sets of integers with their characteristic functions, we equip $\mathcal{P}(\mathbb{N})$ with the Cantor-space topology. We can therefore assign topological complexity to the ideals of sets of integers. In particular, we have $F_{\sigma}, F_{\sigma\delta}, \text{Borel}$, and so on, ideals on $\mathbb{N}$.

Just and Krawczyk have proved that the Continuum Hypothesis implies that
1. all quotients over $F_{\sigma}$ ideals are pairwise isomorphic, and
2. the quotient over the ideal of asymptotic density zero sets, $Z_0 = \{ A \subseteq \mathbb{N} : \limsup_{n \to \infty} |A \cap n|/n = 0 \}$, is isomorphic to the quotient over the ideal of logarithmic density zero sets, $Z_{\log} = \{ A \subseteq \mathbb{N} : \limsup_{n \to \infty} (\sum_{i \in A \cap n} 1/i)/(\sum_{i < n} 1/i) = 0 \}$.

They have also introduced a class of $E\!U$-ideals that contains both $Z_0$ and $Z_{\log}$ and proved that under CH all quotients over these ideals are homogeneous and pairwise isomorphic. (A Boolean algebra $\mathcal{B}$ is homogeneous if it is isomorphic to $\mathcal{B}_A = \{ B \in \mathcal{B} : B \leq A \}$, for every $A \in \mathcal{B} \setminus \{ 0_{\mathcal{B}} \}$.) Motivated by this result, Just and Krawczyk posed the following problem.

Problem 1 ([3 Problem C]). Is it true that if $\mathcal{I}, \mathcal{J}$ are $F_{\sigma\delta}$ and not $F_{\sigma}$ and both $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ are homogeneous, then $\mathcal{P}(\mathbb{N})/\mathcal{I} \approx \mathcal{P}(\mathbb{N})/\mathcal{J}$?

We will prove that this problem has a negative answer. We will also prove that there is an $F_{\sigma\delta}$ ideal whose quotient is not isomorphic to a quotient over any $P$-ideal. (Recall that $\mathcal{I}$ is a $P$-ideal if for every sequence $A_n (n \in \mathbb{N})$ in $\mathcal{I}$ there is an $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all $n$.)
Sequential topology. If $B$ is a σ-complete Boolean algebra, one can define a topology on $B$ as follows. A sequence $A_n$ ($n \in \mathbb{N}$) algebraically converges to $A$ if:

$$\lim_{m \to \infty} \bigwedge_{n=m}^{\infty} A_n = \bigwedge_{n=1}^{\infty} A_n.$$ 

A subset of $B$ is closed if it is closed under taking algebraic limits of sequences in it. Open sets are complements of closed sets. See [4] and [1] for more on this topology on complete Boolean algebras.

It is known that quotients over analytic ideals (or more generally, over ideals that have the property of Baire) are never σ-complete (see [3]). If $B$ is a (not necessarily σ-complete) Boolean algebra, define a topology $\tau$ on $B$ as follows. A sequence $A_n$ ($n \in \mathbb{N}$) algebraically converges to $A$ if:

1. For all $m$, $B_m = \bigwedge_{n=m}^{\infty} A_n$ exists.
2. For all $m$, $C_m = \bigvee_{n=m}^{\infty} A_n$ exists.
3. Both $\bigwedge_{m=1}^{\infty} C_m$ and $\bigvee_{m=1}^{\infty} B_m$ exist and are equal to $A$.

A subset of $B$ is $\tau$-closed if it is closed under taking algebraic limits of sequences in it. $\tau$-open sets are the complements of $\tau$-closed sets.

**Proposition 2.** If $\mathcal{I}$ is an analytic $P$-ideal, then there is a complete metric on $\mathcal{P}(\mathbb{N})/\mathcal{I}$ that induces $\tau$.

**Proof.** Let $\phi$ be a lower semicontinuous submeasure such that $\mathcal{I} = \{ A \subseteq \mathbb{N} : \limsup_{n \to \infty} \phi(A \setminus n) = 0 \}$, as guaranteed by [5]. Define a metric $d_\phi$ on $\mathcal{P}(\mathbb{N})/\mathcal{I}$ by

$$d_\phi([A]_\mathcal{I}, [B]_\mathcal{I}) = \limsup_n \phi((A \Delta B) \setminus n).$$

This metric is complete (see [2] Lemma 1.3.3). It is easily checked that a sequence is $d_\phi$-convergent if and only if it is $\tau$-convergent. □

**Theorem 3.** There are two ideals $\mathcal{I}$ and $\mathcal{J}$ such that

1. both $\mathcal{I}$ and $\mathcal{J}$ are $F_{\sigma, \delta}$ and neither $\mathcal{I}$ nor $\mathcal{J}$ is $F_\sigma$,
2. both quotient algebras over $\mathcal{I}$ and $\mathcal{J}$ are homogeneous,
3. these quotient algebras are not isomorphic.

**Proof.** We will take $\mathcal{I}$ and $\mathcal{J}$ to be the following ideals on $\mathbb{Q} \cap [0, 1]$:

- $\text{NWD}(\mathbb{Q}) = \{ A \subseteq \mathbb{Q} \cap [0, 1] : A \text{ is nowhere dense} \}$,
- $\text{NULL}(\mathbb{Q}) = \{ A \subseteq \mathbb{Q} \cap [0, 1] : \overline{A} \text{ is of Lebesgue measure 0} \}$.

(The closure $\overline{A}$ is taken in $\mathbb{R}$.) To see that $\text{NWD}(\mathbb{Q})$ is $F_{\sigma, \delta}$, enumerate the basis of $\mathbb{Q}$ as $\{ U_n \}$, $\mathbb{Q}$ as $\{ q_n \}$, and the basis of $\mathbb{Q} \cap U_m$ as $\{ V_{mn} \}$. The set

$$K_m = \{ A \subseteq \mathbb{Q} : (\exists n) A \cap V_{mn} \subseteq \{ q_1, \ldots, a_n \} \}$$

is hereditary and $F_{\sigma}$, and $A \in K_m$ if and only if $A \cap U_m$ is nowhere dense. Therefore, $\text{NWD}(\mathbb{Q}) = \bigcap_m K_m$.

To see that $\text{NULL}(\mathbb{Q})$ is $F_{\sigma, \delta}$, for each $n$ enumerate all finite unions of rational intervals of measure $\leq 1/n$ and proceed as above, using the compactness of $[0, 1]$.

Neither of these ideals is $F_\sigma$.

Ideals $\mathcal{I}$ and $\mathcal{J}$ are Rudin–Keisler isomorphic if there are $A \in \mathcal{I}$, $B \in \mathcal{J}$, and a bijection $h$ between $\mathbb{N} \setminus B$ and $\mathbb{N} \setminus A$ such that for all $X \subseteq \mathbb{N} \setminus A$ we have

$$X \in \mathcal{I} \iff h^{-1}(X) \in \mathcal{J}.$$
Claim 1. The quotient $\mathcal{P}(\mathbb{Q})/\text{NWD}(\mathbb{Q})$ is homogeneous.

Proof. Let $A$ be a positive set. Then the interior $B$ of $\overline{A}$ is nonempty, hence $B \cap A$ is dense in itself. Thus $B \cap A$ is homeomorphic to $\mathbb{Q}$, and $A \setminus B$ is nowhere dense. The homeomorphism is a Rudin–Keisler isomorphism between $\text{NWD}(\mathbb{Q})$ and $\text{NWD}(\mathbb{Q}) \upharpoonright A$, and it induces an isomorphism between $\mathcal{P}(\mathbb{Q})/\text{NWD}(\mathbb{Q})$ and $\mathcal{P}(A)/\text{NWD}(\mathbb{Q}) \upharpoonright A$. □

Claim 2. The quotient $\mathcal{P}(\mathbb{Q})/\text{NULL}(\mathbb{Q})$ is homogeneous.

Proof. As in the proof of Claim 1, we need to prove that for every positive $A$ the ideals $\text{NULL}(\mathbb{Q})$ and $\text{NULL}(\mathbb{Q}) \upharpoonright A$ are Rudin–Keisler isomorphic. We shall prove this in two steps.

If $A, B$ are two subsets of $\mathbb{Q} \cap [0, 1]$ with the same closure, there is a bijection $f: A \to B$ such that $\lambda(f(X)) = \lambda(X)$ for all $X \subseteq A$. Let $A = \{a_i : i \in \mathbb{N}\}$ and $B = \{b_i : i \in \mathbb{N}\}$ be one-to-one enumerations. Find a bijection $f$ so that $\lim_i d(a_i, f(a_i)) = 0$, making sure that every isolated point is fixed by $f$. Such an $f$ satisfies the requirements because $\overline{f(X)} = \overline{X}$ is countable for every $X$.

In the second step we prove that for every $K \subseteq [0, 1]$ of positive measure there is $g: K \to [0, 1]$ such that $\lambda(X) = 0$ if and only if $\lambda(g''X) = 0$ for every closed $X \subseteq K$. The function defined by

$$g(a) = \frac{\lambda([0, a] \cap K)}{\lambda(K)}$$

has the property that $\lambda(g''U) = \lambda(U)/\lambda(K)$ for every interval $U$. Therefore this equality holds for all Lebesgue-measurable sets, and $g$ as required.

To conclude the proof, let $A \subseteq \mathbb{Q}$ be positive. By the above, we can find maps $g: [0, 1]$ and $f: (g'')A \to \mathbb{Q}$ such that $f \circ g$ is a Rudin–Keisler isomorphism. □

The proof of the following result is very similar to some arguments of [1].

Lemma 4. The sequential topology on $\mathcal{P}(\mathbb{Q})/\text{NWD}(\mathbb{Q})$ is not Hausdorff.

Proof. In this proof, by open we mean relatively open in $\mathbb{Q}$ unless otherwise stated. Let us write $\mathcal{I}$ for $\text{NWD}(\mathbb{Q})$. We claim that each open in $\mathcal{P}(\mathbb{Q})/\mathcal{I}$ set containing $[\emptyset]_\mathcal{I}$ contains $[\mathbb{Q}]_\mathcal{I}$ in its closure. Let $\mathcal{D}$ be an open neighborhood of $[\emptyset]_\mathcal{I}$ in $\mathcal{P}(\mathbb{Q})/\mathcal{I}$.

It is straightforward to verify the following two facts about convergence in $\mathcal{P}(\mathbb{Q})/\mathcal{I}$. (The second of these facts is of a rather general nature while the first one is characteristic to $\text{NWD}(\mathbb{Q})$.)

1. If $(U_n)$ is an increasing sequence of open sets, then $[U_n]_\mathcal{I} \to [\bigcup_n U_n]_\mathcal{I}$.
2. Let $U$ be open (perhaps empty), $q \in \mathbb{Q}$, and let $V_n$ be an open ball around $q$ of radius $1/n$. Then $[U \cup V_n]_\mathcal{I} \to [U]_\mathcal{I}$.

List elements of $\mathbb{Q}$: $q_0, q_1, q_2, \ldots$. By induction, using (2), we construct a sequence of open sets $(U_n)$ with $[U_n]_\mathcal{I} \in \mathcal{D}$ and with $U_{n+1}$ being the union of $U_n$ and an open ball around $q_{n+1}$. Then by (1), $[U_n]_\mathcal{I} \to [\bigcup U_n]_\mathcal{I} = [\mathbb{Q}]_\mathcal{I}$. □

Lemma 5. The sequential topology on $\mathcal{P}(\mathbb{Q})/\text{NULL}(\mathbb{Q})$ is Hausdorff.

Proof. Let $\lambda(A)$ be the Lebesgue measure of $A$. Let us write $\mathcal{F}$ for $\text{NULL}(\mathbb{Q})$, and let $X = [x]_\mathcal{F}$, $Y = [y]_\mathcal{F}$, etc. We claim that whenever $\lim_i X_i = Y$ in $\mathcal{P}(\mathbb{Q})/\text{NULL}(\mathbb{Q})$, we have $\lim_i \lambda(x_i \Delta y) = 0$. Assume the contrary, and fix a sequence $X_i$ converging to $Y$ such that $\liminf_i \lambda(x_i \Delta y) = \delta > 0$. Let $B_n = \bigvee_{i \geq n} X_i$
and \( C_n = \bigwedge_{i \geq n} X_i \). Since \( B_n \geq X_n \geq C_n \) and \( B_n \geq Y \geq C_n \) for all \( n \), for every \( n \) we have either \( \lambda(b_n \setminus y) \geq \delta/2 \) or \( \lambda(y \setminus c_n) \geq \delta/2 \).

Let us assume that \( \lambda(y \setminus c_n) \geq \delta/2 \) for infinitely many \( n \).

By making small changes to these sets, we may assume \( c_1 \subseteq c_2 \subseteq c_3 \subseteq \cdots \subseteq y \).
Therefore, we have \( \lambda(y \setminus c_n) \geq \delta/2 \) for all \( n \). The set \( F = \bigcap_{n=1}^{\infty} y \setminus c_n \) has measure at least \( \delta/2 \), since \( \lambda(y \setminus c_n) \geq \delta/2 \) for all \( n \) and this is a decreasing sequence of closed subsets of \([0, 1]\).

For each \( n \) find \( s_n \in y \setminus (c_1 \cup \cdots \cup c_n) \) such that \( \inf_{a \in F} d(s_n, a) \leq 1/n \), assuring that the closure of \( x = \{ s_n : n \in \mathbb{N} \} \) includes \( F \). Then \( x \cap c_n \) is finite for all \( n \); moreover, \( x \subseteq y \), and \([x]_\mathcal{I} \neq [\emptyset]_\mathcal{I} \). Therefore, the sequence \( C_n \) does not converge to \( y \), contrary to our assumption.

Therefore, we have \( \lambda(b_n \setminus y) \geq \delta/2 \) for every \( n \). The proof that this case leads to the contradiction is identical to the above.

An easy induction on the sequential rank shows that every \( \tau \)-closed set is closed in the metric topology induced by \( \lambda \). Therefore, for \( y \subseteq Q \) and \( \varepsilon > 0 \) the set
\[
\{ [a]_\mathcal{I} : \lambda(a \cap y) < \varepsilon \}
\]
includes an open neighborhood of \([y]_\mathcal{I}\), in turn implying the space is Hausdorff. \( \square \)

Since the sequential topology is defined in algebraic terms, an isomorphism between Boolean algebras is automatically a homeomorphism. Therefore, the two quotients are not isomorphic, and this concludes the proof. \( \square \)

Note that Lemma 4 and Proposition 2 together imply

**Proposition 6.** The quotient \( \mathcal{P}(Q)/\text{NWD}(Q) \) is not isomorphic to \( \mathcal{P}(\mathbb{N})/\mathcal{I} \) for any analytic \( P \)-ideal \( \mathcal{I} \).

During the course of proving Lemma 5 we have proved that the sequential topology on \( \mathcal{P}(Q)/\text{NULL}(Q) \) is stronger than a metric topology. It is not difficult to see that the two topologies differ, but even more is true. If \( \mathcal{I} \) is an ideal on \([0, 1]\) that contains all singletons, define the ideal \( I(Q) \) on \( Q \cap [0, 1] \) by
\[
I(Q) = \{ A \subseteq Q \cap [0, 1] : \overline{A} \in \mathcal{I} \}.
\]
(The closure \( \overline{A} \) is taken in \( \mathbb{R} \)).

**Theorem 7.** If \( \mathcal{I} \) is a \( \sigma \)-ideal on \( Q \cap [0, 1] \) containing all singletons, then the sequential topology on \( \mathcal{P}(Q)/I(Q) \) is not metric. Therefore, the quotient \( \mathcal{P}(Q)/I(Q) \) is not isomorphic to \( \mathcal{P}(\mathbb{N})/\mathcal{I} \) for any analytic \( P \)-ideal \( \mathcal{I} \).

**Proof.** Define a sequence \( a_n \ (n \in \mathbb{N}) \) of subsets of \( Q \cap [0, 1] \) by
\[
a_{n+1} = [i/n, (i+1)/n],
\]
if \( 0 \leq i < n \). Then \( \lim_{i \to \infty} \lambda(a_i) = 0 \). However, the sequence \( A_i = [a_i]_{I(Q)} \) does not converge to \([\emptyset]_{I(Q)}\) algebraically. This is because \( C_n = \bigwedge_{i \geq n} [a_i]_{I(Q)} = [\emptyset]_{I(Q)} \) and \( B_n = \bigvee_{j > n} [a_j]_{I(Q)} = [Q]_{I(Q)} \) for all \( n \).

We claim that every subsequence of \( \{ a_n \} \) has a further subsequence that converges to \([\emptyset]_{I(Q)}\). Once proved, this will imply that the topology is not metric.

For \( i \in \mathbb{N} \) let \( x_i \) and \( y_i \) be the left and right endpoints of the interval \( a_i \). For a subsequence \( a_{n_i} \ (i \in \mathbb{N}) \) we can find a subsequence \( a_{m_i} \) such that \( \lambda(a_{m_i}) < 2^{-i} \).
and \( \lim_i x_m = x \) and \( \lim_i y_m = y \) for some \( x, y \). We necessarily have \( x = y \), and therefore for \( k \in \mathbb{N} \) we have

\[
 b_k = \bigcup_{i \geq k} a_m_i = \bigcup_{i \geq k} a_m_i \cup \{ x \}.
\]

Therefore \( \lambda(b_k) < 2^{-k+1} \). Since \( b_k \supseteq b_{k+1} \) for all \( k \) and \( I \) is a \( \sigma \)-ideal, the sequence \( B_k = \bigvee_{i \geq k} [a_m_i]_{I(Q)} \) converges to \( [0]_{I(Q)} \). This proves our claim and concludes the proof. \( \square \)

There are analytic ideals that are not P-ideals whose quotients are metrizable. For example, all \( F_\sigma \) ideals are of this form, because their quotients are discrete in the sequential topology (this follows from [3]).

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**References**


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