

RANDOM MENSHOV SPECTRA

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ABSTRACT. We show that the spectra Λ of frequencies λ obtained by random perturbations of the integers allows one to represent any measurable function f on \mathbb{R} by an almost everywhere converging sum of harmonics:

$$f = \sum_{\Lambda} c_{\lambda} e^{i\lambda t}.$$

1. INTRODUCTION

This paper concerns the representation of functions by series of exponentials which converge almost everywhere (a.e.). According to Menshov's theorem (1941, see [1]) every 2π -periodic measurable function f admits a representation as

$$(1) \quad f(t) = \sum_{k \in \mathbb{Z}} c(k) e^{ikt} \quad \text{a.e.}$$

Among the many generalizations and analogs of this fundamental result, there exists a version for the non-periodic case: Davtjan [2] proved that the corresponding representation on \mathbb{R} can be obtained if instead of the sum over integers one considers a "trigonometric integral" which involves all real frequencies.

In our paper [4] it was shown that most of the frequencies are redundant. Namely, by appropriate small perturbations of the integers we constructed a spectrum of frequencies $\Lambda = \{\lambda(k), k \in \mathbb{Z}\}$ such that any $f \in L^0(\mathbb{R})$ (that is, a measurable function on \mathbb{R}) can be decomposed as

$$(2) \quad f(t) = \sum_{k \in \mathbb{Z}} c(k) e^{i\lambda(k)t} \quad \text{a.e.}$$

The aim of this note is to show that this is not an exceptional feature of the constructed spectrum. In fact, by choosing the perturbations *randomly* one gets such a property with probability 1.

Analogously to the periodic case (see [5]) we introduce the following

Definition. A sequence $\Lambda = \{\dots < \lambda(k) < \dots < \lambda(-1) < \lambda(0) < \lambda(1) < \dots\}$ is called a Menshov spectrum for \mathbb{R} if for any $f \in L^0(\mathbb{R})$ there are coefficients $\{c(k)\}$ such that the decomposition (2) holds (convergence in (2) is understood in the sense of the limit of symmetric partial sums, i.e. $\lim_{x \rightarrow \infty} \sum_{|\lambda| < x}$).

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Theorem. *Let $r(n)$ be independent variables uniformly distributed on the segment $[-\frac{1}{2}, \frac{1}{2}]$. Then the sequence*

$$(3) \quad \lambda(n) = n + r(n), \quad n \in \mathbb{Z},$$

is almost surely a Menshov spectrum for \mathbb{R} .

The proof is based on the technique used in our recent paper [5]. We refer the reader to this paper for some historical comments and additional references.

2. PRELIMINARIES

By a trigonometric polynomial P we mean a finite linear combination of exponentials with real (not necessarily integer) frequencies $\dots < \lambda(-1) < \lambda(0) < \dots$. We call the set of λ 's involved, the spectrum of P and denote it by $\text{spec } P$. The corresponding coefficients are denoted as $\widehat{P}(\lambda)$, so

$$P = \sum_{\text{spec } P} \widehat{P}(\lambda) e^{i\lambda x}, \quad x \in \mathbb{R}.$$

We denote

$$\text{deg } P = \max_{\text{spec } P} |\lambda|.$$

As usual

$$\|\widehat{P}\|_1 := \sum |\widehat{P}(\lambda)|, \quad \|\widehat{P}\|_\infty := \max |\widehat{P}(\lambda)|.$$

Let P^* be the (non-symmetric) majorant of partial sums:

$$P^*(x) := \max_{a < b} \left| \sum_{\text{spec } P \cap [a, b]} \widehat{P}(\lambda) e^{i\lambda x} \right|.$$

For a given P and $l \in \mathbb{Z}^+$ we denote by $P_{[l]}$ the ‘‘contracted’’ polynomial:

$$P_{[l]}(x) = P(lx).$$

The following ‘‘special products’’ are used:

$$H = Q_{[l]} P.$$

If $\text{spec } Q \subset \mathbb{Z}$ and $l > 2 \text{deg } P$, then this product has a simple structure which provides the following estimate (compare with (10) in [5]):

$$(4) \quad H^*(x) \leq |P(x)| \cdot \|Q^*\|_{L^\infty(-\pi, \pi)} + 2P^*(x) \cdot \|\widehat{Q}\|_\infty.$$

We will use the following

Lemma 1 (see [5], Lemma 2.1). *Given any $\epsilon > 0$, $\delta > 0$, there exists a trigonometric polynomial $P = P_{\epsilon, \delta}$ with integer spectrum such that*

- (i) $\widehat{P}(0) = 0$, $\|\widehat{P}\|_\infty < \delta$;
- (ii) $\mathbf{m}\{x \in [-\pi, \pi] : |P(x) - 1| > \delta\} < \epsilon$;
- (iii) $\|P^*\|_\infty < C\epsilon^{-1}$.

3. PROOF OF THE THEOREM

3.1. The result is an easy consequence of the following (nonstochastic)

Proposition. *Let $\Lambda = \{\lambda(n)\}$, $\lambda(n) = n + r(n)$, $n \in \mathbb{Z}$, and suppose that for every $k \in \mathbb{Z}^+$ there exists a number $l = l(k)$ in \mathbb{Z}^+ such that*

$$(5) \quad |r(sl + q) - 2^{-|q|+1}| < \frac{1}{k^2}, \quad 0 < |s| < k, |q| < k.$$

Then Λ is a Menshov spectrum for \mathbb{R} .

To deduce the theorem from the proposition it is enough to mention that if we fix k and run l over a sufficiently fast increasing sequence $\{l_j\}$, then the events B_j that the inequalities above are fulfilled for $l = l_j$ are mutually independent and each has a positive probability $p(k)$ which does not depend on j . So for a random spectrum H the condition of the proposition is true almost surely.

3.2. Now we pass to the proof of the proposition. Denote $I(k) := \{sl(k) + q : 0 < |s| < k, |q| < k\}$ and $M(k) = kl(k) + k$. Clearly (passing to a subsequence if necessary) we may suppose that

$$I(k+1) \cap [-M(k), M(k)] = \emptyset.$$

Let $f \in L^0(\mathbb{R})$ be given. We shall define by induction an increasing sequence $\{k_j\}$ and “blocks” of coefficients $\{c(n)\}$, $n \in I(k_j)$; all other coefficients of the expansion (2) will be zero. We denote:

$$A_j := \sum_{I(k_j)} c(n) \exp(i\lambda(n)x),$$

$$S_N := \sum_{j \leq N} A_j \quad N \in \mathbb{Z}^+, S_0 := 0.$$

Fix N and suppose that the polynomials A_j are already defined for $j < N$. Let us describe the N 'th step of the induction. Set

$$R_N := f - S_{N-1}.$$

We need the following result proved in [3]:

if $0 < |r(q)| = o(1)$, then the system of exponentials $\exp i(q + r(q))$, $q \in \mathbb{Z}$, is complete in $L^0(\mathbb{R})$, that is, the set of linear combinations is dense with respect to convergence a.e.

Using this we find a polynomial

$$F_N(x) = \sum a_q \exp i(q + 2^{-|q|+1})x$$

so that

$$(6) \quad \mathbf{m} \left\{ x \in [-N\pi, N\pi] : |F_N(x) - R_N(x)| > \frac{1}{N^4} \right\} < \frac{1}{N^2}.$$

Next we use Lemma 1 with

$$(7) \quad \delta = \delta_N = \frac{1}{N^4 \|\widehat{F_N}\|_1}, \quad \epsilon = \frac{1}{N^3}$$

and find the corresponding polynomial Q_N . Fix a number k_N large enough:

$$(8) \quad k_N > k_{N-1}, \quad 3 \deg F_N, \quad \frac{\|\widehat{Q_N}\|_1}{\delta_N}$$

and set

$$(9) \quad H_N := F_N \cdot (Q_N)_{[l(k_N)]} .$$

One can easily see that

$$\text{spec } H \subset \left\{ sl(k_N) + q + 2^{-|q|+1} : 0 < |s| < k, |q| < k \right\} ,$$

so we can write

$$(10) \quad H_N = \sum_{\substack{0 < |s| < k_N \\ |q| < k_N}} b(N, s, q) \exp i(sl(k_N) + q + 2^{-|q|+1})x .$$

Finally we set:

$$(11) \quad A_N := \sum_{\substack{0 < |s| < k_N \\ |q| < k_N}} b(N, s, q) \exp i\lambda(sl(k_N) + q)x \equiv \sum_{I(k_N)} c(n) \exp i\lambda(n)x .$$

3.3. Now we show that

$$(12) \quad S_N \rightarrow f \quad \text{a.e.}$$

For this first we get from (ii) of Lemma 1:

$$(13) \quad \mathbf{m} \left(\left\{ x \in [-N\pi, N\pi] : |H_N(x) - F_N(x)| > \frac{1}{N^4} \right\} \right) \leq N \cdot \mathbf{m}(\{x \in [-\pi, \pi] : |Q_N - 1| \geq \delta_N\}) = O(N^{-2}) .$$

Further, (9), (8) and (7) imply:

$$(14) \quad \|\widehat{H}_N\|_1 = \|\widehat{F}_N\|_1 \cdot \|\widehat{Q}_N\|_1 < \|\widehat{F}_N\|_1 k_N \delta_N = \frac{k_N}{N^4} ,$$

so we can estimate, using (10), (11), (5), (14), (3) and (8):

$$(15) \quad \begin{aligned} & \|A_N - H_N\|_{L^\infty(-\pi N, \pi N)} \\ & \leq \|\widehat{H}_N\|_1 \cdot \max_{\substack{0 < |s| < k_N \\ |q| < k_N}} \|\exp i(r(sl(k_N) + q) - 2^{-|q|+1})x - 1\|_\infty \\ & < \frac{k_N}{N^4} \cdot \frac{1}{k_N^2} \cdot \pi N = O(N^{-4}) . \end{aligned}$$

Finally, we have from (6), (13) and (15):

$$\mathbf{m} \left\{ x \in [-N\pi, N\pi] : |A_N(x) - R_N(x)| > \frac{C}{N^4} \right\} = O(N^{-2}) ,$$

so

$$(16) \quad R_{N+1} = R_N - A_N = O(N^{-4}) \quad \text{a.e.,}$$

and (12) follows.

3.4. At last:

$$(17) \quad A_N^* \rightarrow 0 \quad \text{a.e.}$$

Indeed, estimating as in (15) we see:

$$A_N^* < H_N^* + O(N^{-4}) \quad \text{a.e.}$$

Since $l(k_N) > k_N > 3 \deg F_N$ we can use (4) and get:

$$H_N^*(x) < |F_N(x)| \cdot \|Q_N^*\|_{L^\infty(-\pi, \pi)} + 2\|\widehat{Q}\|_\infty \cdot \|\widehat{F}\|_1.$$

The first term on the right hand side is $O(N^{-1})$ a.e. due to (6), (16) and (iii). The last term is $O(N^{-4})$ because of (i) and (7). Clearly (12) and (17) imply the decomposition (2) and this completes the proof. \square

4. REMARKS

4.1. One can see that the result holds for $r(n)$ uniformly distributed on any fixed neighbourhood of zero. Moreover, it holds true for $r(n)$ uniformly distributed on $[-d_n, d_n]$ if d_n decrease slowly enough. This allows one to cover in full generality the result from [4] where a Menshov spectrum $\{n + o(1)\}$ was constructed. But d_n really must decay slowly. In contrast to the completeness property which occurs for **any** (nonzero) perturbation $r(n) = o(1)$ (the result from [3] used above), the following simple observation is true:

If $\lambda(n) = n + O(n^{-\alpha})$, $\alpha > 0$, then it is not a Menshov spectrum.

Indeed, if $\Delta f := f(x) - f(x - 2\pi)$, then

$$\Delta^k \left(\sum c_n e^{i\lambda_n x} \right) = \sum c_n O(n^{-\alpha k}) e^{i\lambda_n x}.$$

Hence for k sufficiently large (depending only on α) the representation (2) implies that $\Delta^k f$ equals a smooth function a.e., so $\lambda(n)$ cannot be a Menshov spectrum for \mathbb{R} .

4.2. In [5] we studied Menshov spectra in the periodic case. The main results of that paper can be extended to Menshov spectra in \mathbb{R} . For example, Menshov spectra for \mathbb{R} may be quite sparse, up to “almost Hadamarian lacunarity”. More precisely:

For any $\epsilon(n)$ decreasing to zero one can construct a (symmetric) Menshov spectrum Λ for \mathbb{R} such that $\lambda(n+1)/\lambda(n) > 1 + \epsilon(n)$, $n \in \mathbb{Z}^+$.

This is an analog of Theorem 1 from [5] and the proof is basically the same.

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