

AN ALGEBRAIC PROPERTY OF JOININGS

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ABSTRACT. We show that an ergodic automorphism is semisimple if and only if the set of ergodic self-joinings is a subsemigroup of the semigroup of self-joinings.

1. INTRODUCTION

Assume that T is an ergodic automorphism of a probability standard Borel space (X, \mathcal{B}, μ) . By $J(T)$ we denote the set of all self-joinings of T that are all $T \times T$ -invariant measures defined on $(X \times X, \mathcal{B} \otimes \mathcal{B})$, both of whose natural projections are equal to μ . On the set $J(T)$ there is a natural structure of a semitopological compact affine semigroup (see the next section for this and some further basic notions and results). By $J^e(T)$ we denote the set of ergodic members of $J(T)$.

In [3], A. del Junco, M.K. Mentzen and the second author introduced a notion of semisimplicity. We say that T is semisimple if for any $\lambda \in J^e(T)$ the automorphism $(T \times T, \lambda)$ is relatively weakly mixing over T (T is given by the projection on the first coordinate). The notion of semisimplicity generalized the notion of minimal self-joinings [7] and of simplicity [4, 9]. Moreover, some Gaussian automorphisms turned out to be semisimple (see [5]). It follows from basic properties of relative products that $J^e(T)$ is stable under composition whenever T is semisimple. The aim of this note is to prove that the converse also holds.

Theorem 1. *Let T be an ergodic automorphism of (X, \mathcal{B}, μ) . Then T is semisimple if and only if the set of ergodic self-joinings is a subsemigroup of $J(T)$.*

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2. NOTATION AND BASIC RESULTS

Suppose that $\pi : (Z, \mathcal{D}, \rho) \rightarrow (Y, \mathcal{C}, \eta)$ is a homomorphism of two standard probability spaces. Given $f \in L^1(Z, \rho)$, by $E(f|Y)$ or $E^\eta(f|Y)$ we denote the conditional expectation of f with respect to Y , i.e. the function in $L^1(Y, \eta)$ given by

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$E(f|\pi^{-1}(\mathcal{C})) \circ \pi^{-1}$. If

$$\rho = \int_Y \rho_y d\eta(y)$$

denotes the disintegration of ρ over η , then $E(\cdot|Y)(y) = \rho_y(\cdot)$ for a.a. $y \in Y$ (see [2], Th. 5.8). If $\pi' : (Z', \mathcal{D}', \rho') \rightarrow (Y, \mathcal{C}, \eta)$ is another homomorphism and $\rho' = \int_Y \rho'_y d\eta(y)$, then the measure

$$\rho \otimes_Y \rho' = \int_Y \rho_y \otimes \rho'_y d\eta(y)$$

defined on $\mathcal{D} \otimes \mathcal{D}'$ is called the *relative product* of ρ and ρ' over (Y, η) (see [2], Chapter 5, §5). The resulting space will be denoted $(Z \times_Y Z', \rho \otimes_Y \rho')$. In what follows we will need the following.

Lemma 1. *Consider a sequence of homomorphisms $(Z, \mathcal{D}, \rho) \rightarrow (Y, \mathcal{C}, \eta) \rightarrow (X, \mathcal{B}, \mu)$. Whenever $f, g \in L^2(Z, \rho)$, then $E(f \otimes g|Y \times_X Y) = E(f|Y) \otimes E(g|Y)$, $\eta \otimes_X \eta$ a.s.*

Proof. Let $\rho = \int_Y \rho_y d\eta(y)$, $\rho = \int_X \tilde{\rho}_x d\mu(x)$ and $\eta = \int_X \eta_x d\mu(x)$ stand for the relevant disintegrations. We have

$$\begin{aligned} \rho(A) &= \int_Z \chi_A d\rho = \int_Y E(\chi_A|Y)(y) d\eta(y) \\ &= \int_X \left(\int_Y E(\chi_A|Y)(y) d\eta_x(y) \right) d\mu(x) = \int_X \left(\int_Y \rho_y(A) d\eta_x(y) \right) d\mu(x); \end{aligned}$$

thus

$$\tilde{\rho}_x = \int_Y \rho_y d\eta_x(y).$$

Hence

$$\tilde{\rho}_x \otimes \tilde{\rho}_x = \int_{Y \times Y} \rho_y \otimes \rho_{y'} d\eta_x \otimes \eta_x(y, y').$$

It follows that

$$\begin{aligned} \rho \otimes_X \rho &= \int_X \tilde{\rho}_x \otimes \tilde{\rho}_x d\mu(x) \\ &= \int_X \left(\int_{Y \times Y} \rho_y \otimes \rho_{y'} d\eta_x \otimes \eta_x(y, y') \right) d\mu(x) = \int_{Y \times Y} \rho_y \otimes \rho_{y'} d\eta \otimes_X \eta(y, y'). \end{aligned}$$

Hence, if $f, g \in L^2(Z, \rho)$, then

$$\begin{aligned} E(f \otimes g|Y \times_X Y)(y, y') &= \int_{Y \times Y} f \otimes g d\rho_y \otimes \rho_{y'} \\ &= \int_Y f d\rho_y \int_Y g d\rho_{y'} = E(f|Y)(y) \cdot E(g|Y)(y') \end{aligned}$$

and therefore $E(f \otimes g|Y \times_X Y) = E(f|Y) \otimes E(g|Y)$ a.s. with respect to $\eta \otimes_X \eta$. \square

Assume now that T is an ergodic automorphism on a standard probability Borel space (X, \mathcal{B}, μ) . To each element $\lambda \in J(T)$ we associate a Markov operator $\Phi_\lambda : L^2(X_1, \mu_1) \rightarrow L^2(X_2, \mu_2)$ (where $(X_i, \mu_i) = (X, \mu)$, for $i = 1, 2$) given by

$$\int_{X_2} \Phi_\lambda(f)\bar{g} d\mu_2 = \int_{X_1 \times X_2} f\bar{g} d\lambda.$$

By Markov property we mean that Φ_λ is positive and $\Phi_\lambda 1 = \Phi_\lambda^* 1 = 1$. We also have $\Phi_\lambda \circ T = T \circ \Phi_\lambda$. Moreover, for each $f \in L^2(X, \mu)$

$$(1) \quad (\Phi_\lambda f)(x_2) = E^\lambda(f|X_2)(x_2).$$

Furthermore, each Markov operator on $L^2(X, \mu)$ that commutes with T is necessarily of the form Φ_λ (see e.g. [5] or [8]). The latter observation introduces a semigroup law on $J(T)$ by the formula $\Phi_{\lambda_2 \circ \lambda_1} = \Phi_{\lambda_2} \circ \Phi_{\lambda_1}$. Together with the weak topology and the natural simplex structure on $J(T)$ we obtain that $J(T)$ is a compact semitopological affine semigroup.

Suppose now $\lambda_1, \lambda_2 \in J(T)$. We will treat λ_1 as defined on $X_1 \times X_2$, while λ_2 is defined on $X_2 \times X_3$. By λ_2^* we mean the joining corresponding to $\Phi_{\lambda_2}^*$, that is, the self-joining given by

$$\lambda_2^*(A_2 \times A_3) = \lambda_2(A_3 \times A_2).$$

Disintegrate λ_1 and λ_2^* over the common factor X_2 :

$$\lambda_1 = \int_{X_2} \lambda_{1,x_2} d\mu_2(x_2), \quad \lambda_2^* = \int_{X_2} \lambda_{2,x_2}^* d\mu_2(x_2).$$

Consider the relative product of λ_1 and λ_2^* over the common factor X_2 that is the measure defined on $X_1 \times X_2 \times X_3$ given by

$$\lambda_1 \otimes_{X_2} \lambda_2^* = \int_{X_2} \lambda_{1,x_2} \otimes \lambda_{2,x_2}^* d\mu_2(x_2).$$

Take $f, g \in L^2(X, \mu)$. Using (1) we then have

$$\begin{aligned} & \int_{X_1 \times X_3} f(x_1)g(x_3) d\lambda_1 \otimes_{X_2} \lambda_2^*(x_1, x_2, x_3) \\ &= \int_{X_2} \left(\int_{X_1 \times X_3} f(x_1)g(x_3) d\lambda_{1,x_2} \otimes \lambda_{2,x_2}^*(x_1, x_3) \right) d\mu_2(x_2) \\ &= \int_{X_2} (\Phi_{\lambda_1} f)(x_2)(\Phi_{\lambda_2^*} g)(x_2) d\mu_2(x_2) = \int_{X_3} (\Phi_{\lambda_2} \circ \Phi_{\lambda_1})(f)g d\mu_3. \end{aligned}$$

We have shown the following:

$$(2) \quad \lambda_2 \circ \lambda_1 = \lambda_1 \otimes_{X_2} \lambda_2^*|_{X_1 \times X_3}.$$

In particular, if $\lambda \otimes_{X_2} \lambda^*$ is ergodic, then $\lambda \circ \lambda$ is ergodic and the key observation for the proof of Theorem 1 is that the converse is also true (see Proposition 1 below).

Let T acting on (X, \mathcal{B}, μ) be a factor of an ergodic automorphism S acting on (Y, \mathcal{C}, η) . Following [2] (see condition C5 on p. 132), we say that S is a *compact extension* of T if for each $0 \neq f \in L^2(Y, \eta)$ the limit of ergodic averages of $f \otimes \bar{f}$ for $S \times S$ acting on $(Y \times_X Y, \eta \otimes_X \eta)$ is also non-zero.

Remark 1. Usually a compact extension is defined in terms of relative eigenvectors (see [1, 10]). R. Zimmer proved in [10] that S is a compact extension of T if and only if S is an isometric extension of T . Another proof of Zimmer’s result follows easily from the joining characterization of isometric extensions given in [6].

Assume that R acting on (Z, \mathcal{D}, ρ) is an ergodic extension of T acting on (X, \mathcal{B}, μ) . Then (see [2], Chapter 6):

- (A) there exists a biggest factor, called the *relative Kronecker factor*, S acting on (Y, \mathcal{C}, η) between R and T such that S is a compact extension of T ;
- (B) the relative Kronecker factor S is trivial (i.e. $S = T$) iff the relative product $R \times R$ on $(Z \times_X Z, \rho \otimes_X \rho)$ is ergodic (the latter condition means that R is a *relatively weakly mixing extension* of T).

Finally, recall that an ergodic automorphism T on (X, \mathcal{B}, μ) is called *semisimple* ([3]) if for each $\lambda \in J^e(T)$, the relative product $\lambda \otimes_{X_2} \lambda^*$ is ergodic, that is (using (B)), $T \times T$ on $(X_1 \times X_2, \lambda)$ is a relatively weakly mixing extension of X_2 .

3. PROOF OF THEOREM 1

We will need a lemma which is a simple consequence of the L^1 -convergence in the pointwise ergodic theorem.

Lemma 2. *Let S be an automorphism on (Y, \mathcal{C}, η) . Denote by \mathcal{I} the σ -algebra of S -invariant sets. Assume that $\mathcal{E} \subset \mathcal{C}$ is a factor of S . Then:*

- (i) *If the action of S on \mathcal{E} is ergodic, then $E(f|\mathcal{I}) = \int_Y f \, d\eta$ for each $f \in L^1(\mathcal{E})$.*
- (ii) *If $f \in L^1(Y, \eta)$ and the sequence $(\frac{1}{n} \sum_{i=0}^{n-1} f \circ S^i)_{n \geq 1}$ converges to a constant $c (= \int_Y f \, d\eta)$, then $E(E(f|\mathcal{E})|\mathcal{I}) = c$.*

Proof. (i) Since $f \in L^1(Y, \eta)$, $\frac{1}{n} \sum_{i=0}^{n-1} f \circ S^i$ converges to $E(f|\mathcal{I})$ in $L^1(\mathcal{C})$ by the ergodic theorem. However f is measurable with respect to \mathcal{C} which is S -invariant. The result follows by the ergodicity of S on \mathcal{E} .

(ii) Put $g = E(f|\mathcal{E})$. Then by the ergodic theorem, $\frac{1}{n} \sum_{i=0}^{n-1} g \circ S^i$ converges to $E(g|\mathcal{I})$ in $L^1(\mathcal{C})$ and hence in $L^1(\mathcal{E})$. Therefore for all $h \in L^\infty(\mathcal{E})$,

$$\frac{1}{n} \sum_{i=0}^{n-1} \int f \circ S^i \cdot \bar{h} \, d\eta \rightarrow c \int \bar{h} \, d\eta.$$

We have

$$\frac{1}{n} \sum_{i=0}^{n-1} \int f \circ S^i \cdot \bar{h} \, d\eta = \frac{1}{n} \sum_{i=0}^{n-1} \int E(f \circ S^i|\mathcal{E}) \cdot \bar{h} \, d\eta = \frac{1}{n} \sum_{i=0}^{n-1} \int E(f|\mathcal{E}) \circ S^i \cdot \bar{h} \, d\eta;$$

thus $\frac{1}{n} \sum_{i=0}^{n-1} g \circ S^i$ converges weakly to c in $L^1(\mathcal{E})$. Hence $E(g|\mathcal{I}) = c$. □

The following lemma is a direct consequence of our definition of compact extension, the L^1 -convergence of ergodic averages and the fact that $f \otimes \bar{f} \in L^1(Y \times_X Y)$ whenever $f \in L^2(Y, \eta)$.

Lemma 3. *Let S be an ergodic automorphism on (Y, \mathcal{C}, η) . Suppose that S is a compact extension of T acting on (X, \mathcal{B}, μ) and $f \in L^2(Y)$. Then $f = 0$ if and only if $E(f \otimes \bar{f}|\mathcal{I}) = 0$ in the relative product $Y \times_X Y$.*

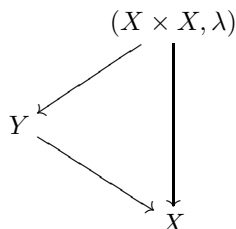
The following result is of independent interest.

Proposition 1. *Assume that T is an ergodic automorphism of (X, \mathcal{B}, μ) and let $\lambda \in J^e(T)$. If $\lambda \circ \lambda$ is ergodic, then $\lambda \otimes_X \lambda^*$ is ergodic.*

Proof. Given a real function $f \in L^2(X, \mu)$ put $f \otimes f(x_1, x_2, x_3) = f(x_1)f(x_3)$. We have $f \otimes f \in L^1(X_1 \times X_2 \times X_3, \lambda_1 \otimes_{X_2} \lambda_2^*)$, where $\lambda_1 = \lambda_2 = \lambda$. If \mathcal{I} denotes the σ -algebra of $T \times T \times T$ -invariant sets in the relative product, then our ergodicity assumption on $\lambda \circ \lambda$ and (2) give rise to

$$(3) \quad E(f \otimes f|\mathcal{I}) = \int f \otimes f \, d\lambda \otimes_{X_2} \lambda^*.$$

Let (Y, \mathcal{C}, η) denote the relative Kronecker factor of $T \times T$ on $(X_1 \times X_2, \lambda_1)$ over X_2 (see (A)). Then:



Fix a real function $f \in L^2(X, \mu)$. We will show that

$$(4) \quad E(f(x_1)|Y) = E(f(x_1)|X_2).$$

Let $g = E(f(x_1)|Y) - E(f(x_1)|X_2)$. By Lemma 3 it is enough to prove that $E(g \otimes g|\mathcal{I}) = 0$ with respect to $\lambda_1 \otimes_{X_2} \lambda_2^*$. We have

$$\begin{aligned}
 & E(g \otimes g|\mathcal{I}) \\
 &= E((E(f(x_1)|Y_1) - E(f(x_1)|X_2)) \otimes (E(f(x_3)|Y_2) - E(f(x_3)|X_2))|\mathcal{I}) \\
 &= E(E(f(x_1)|Y_1) \otimes E(f(x_3)|Y_2)|\mathcal{I}) - E(E(f(x_1)|Y_1) \cdot E(f(x_3)|X_2)|\mathcal{I}) \\
 &\quad - E(E(f(x_1)|X_2) \cdot E(f(x_3)|Y_2)|\mathcal{I}) + E(E(f(x_1)|X_2) \cdot E(f(x_3)|X_2)|\mathcal{I}) \\
 &= E(E(f(x_1)|Y_1) \otimes E(f(x_3)|Y_2)|\mathcal{I}) - \int_{Y_1} E(f(x_1)|Y_1) \cdot E(f(x_3)|X_2) d\eta_1 \\
 &\quad - \int_{Y_2} E(f(x_1)|X_2) \cdot E(f(x_3)|Y_2) d\eta_2 + \int_{X_2} E(f(x_1)|X_2) \cdot E(f(x_3)|X_2) d\mu_2
 \end{aligned}$$

by Lemma 2(i) and the fact that X_2 is a factor of Y and Y is ergodic. By taking in the latter three summands the conditional expectation with respect to X_2 , we obtain

$$\begin{aligned}
 & E(g \otimes g|\mathcal{I}) \\
 &= E(E(f(x_1)|Y_1) \otimes E(f(x_3)|Y_2)|\mathcal{I}) - \int_{X_2} E(f(x_1)|X_2) \cdot E(f(x_3)|X_2) d\mu_2.
 \end{aligned}$$

Using consecutively Lemma 1 and (3), together with Lemma 2(ii), and finally the definition of the relative product, we obtain that

$$\begin{aligned}
 & E(E(f(x_1)|Y_1) \otimes E(f(x_3)|Y_2)|\mathcal{I}) \\
 &= E(E(f \otimes f|Y_1 \times_{X_2} Y_2)|\mathcal{I}) = \int_{X \times X \times X} f(x_1)f(x_3) d\lambda \otimes_X \lambda^*(x_1, x_2, x_3) \\
 &= \int_{X_2} E(f(x_1)|X_2) \cdot E(f(x_3)|X_2) d\mu_2.
 \end{aligned}$$

We hence have proved $E(g \otimes g|\mathcal{I}) = 0$ and (4) directly follows.

If $h = h(x_2)$ is in $L^2(X_2)$, then h is Y -measurable and by (4) we have

$$\begin{aligned}
 & E(f(x_1) \cdot h(x_2)|Y) = h(x_2) \cdot E(f(x_1)|Y) \\
 &= h(x_2) \cdot E(f(x_1)|X_2) = E(f(x_1) \cdot h(x_2)|X_2).
 \end{aligned}$$

Since the family of the function of the form $f \otimes h$ as above forms a linearly dense subset in $L^2(X \times X, \lambda)$, $E(F|Y) = E(F|X_2)$ for all $F \in L^2(X \times X, \lambda)$. Hence $Y = X_2$ and the relative Kronecker factor of $(X_1 \times X_2, \lambda_1)$ over X_2 is trivial. In view of (B), it follows that $\lambda \otimes_{X_2} \lambda^*$ is ergodic. \square

Proof of Theorem 1. At first, assume that T is semisimple. Consider $\lambda_1, \lambda_2 \in J^e(T)$. Then, by using Proposition 6.3 from [2], $\lambda_1 \otimes_{X_2} \lambda_2^*$ is ergodic by semisimplicity of T . Therefore $\lambda_2 \circ \lambda_1$ remains ergodic.

If $J_2^e(T)$ is a subsemigroup, then directly from Proposition 1 it follows that T is semisimple.

Remark 2. The proof of Proposition 1 gives a slightly more general result: Assume that λ is an ergodic joining of S (acting on (Y, \mathcal{C}, η)) and T (acting on (X, \mathcal{B}, μ)). Then the relative product $\lambda \otimes_X \lambda$ is ergodic if and only if the measure $\lambda^* \circ \lambda$ on $Y \times Y$ (given by the Markov operator $\Phi_\lambda^* \circ \Phi_\lambda$ on $L^2(Y, \eta)$) is ergodic. Therefore we obtain an answer to the question by Ryzhikov from [8], p. 95.

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