AN ALGEBRAIC PROPERTY OF JOININGS

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Abstract. We show that an ergodic automorphism is semisimple if and only if the set of ergodic self-joinings is a subsemigroup of the semigroup of self-joinings.

1. Introduction

Assume that $T$ is an ergodic automorphism of a probability standard Borel space $(X, \mathcal{B}, \mu)$. By $J(T)$ we denote the set of all self-joinings of $T$ that are all $T \times T$-invariant measures defined on $(X \times X, \mathcal{B} \otimes \mathcal{B})$, both of whose natural projections are equal to $\mu$. On the set $J(T)$ there is a natural structure of a semitopological compact affine semigroup (see the next section for this and some further basic notions and results). By $J^e(T)$ we denote the set of ergodic members of $J(T)$.

In [3], A. del Junco, M.K. Mentzen and the second author introduced a notion of semisimplicity. We say that $T$ is semisimple if for any $\lambda \in J^e(T)$ the automorphism $(T \times T, \lambda)$ is relatively weakly mixing over $T$ ($T$ is given by the projection on the first coordinate). The notion of semisimplicity generalized the notion of minimal self-joinings [7] and of simplicity [11, 9]. Moreover, some Gaussian automorphisms turned out to be semisimple (see [5]). It follows from basic properties of relative products that $J^e(T)$ is stable under composition whenever $T$ is semisimple. The aim of this note is to prove that the converse also holds.

Theorem 1. Let $T$ be an ergodic automorphism of $(X, \mathcal{B}, \mu)$. Then $T$ is semisimple if and only if the set of ergodic self-joinings is a subsemigroup of $J(T)$.

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2. Notation and basic results

Suppose that $\pi : (Z, \mathcal{D}, \rho) \to (Y, \mathcal{C}, \eta)$ is a homomorphism of two standard probability spaces. Given $f \in L^1(Z, \rho)$, by $E(f|Y)$ or $E^\eta(f|Y)$ we denote the conditional expectation of $f$ with respect to $Y$, i.e. the function in $L^1(Y, \eta)$ given by

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$E(f|\pi^{-1}(C)) \circ \pi^{-1}$. If

$$\rho = \int_Y \rho_y \, d\eta(y)$$

denotes the disintegration of $\rho$ over $\eta$, then $E(\cdot|Y)(y) = \rho_y(\cdot)$ for a.a. $y \in Y$ (see [2], Th. 5.8). If $\pi' : (Z', D', \rho') \rightarrow (Y, C, \eta)$ is another homomorphism and $\rho' = \int_Y \rho'_y \, d\eta(y)$, then the measure

$$\rho \otimes_Y \rho' = \int_Y \rho_y \otimes \rho'_y \, d\eta(y)$$

defined on $D \otimes D'$ is called the relative product of $\rho$ and $\rho'$ over $(Y, \eta)$ (see [2], Chapter 5, §5). The resulting space will be denoted $(Z \times_Y Z', \rho \otimes_Y \rho')$. In what follows we will need the following.

**Lemma 1.** Consider a sequence of homomorphisms $(Z, D, \rho) \rightarrow (Y, C, \eta) \rightarrow (X, B, \mu)$. Whenever $f, g \in L^2(Z, \rho)$, then $E(f \otimes g|Y \times X) = E(f|Y) \otimes E(g|Y)$, $\eta \otimes_X \eta$ a.s.

**Proof.** Let $\rho = \int_Y \rho_y \, d\eta(y)$, $\rho = \int_X \hat{\rho}_x \, d\mu(x)$ and $\eta = \int_X \eta_x \, d\mu(x)$ stand for the relevant disintegrations. We have

$$\rho(A) = \int_Z \chi_A \, d\rho = \int_Y E(\chi_A | Y)(y) \, d\eta(y)$$

$$= \int_X \left( \int_Y E(\chi_A | Y)(y) \, d\eta_y(x) \right) \, d\mu(x) = \int_X \left( \int_Y \rho_y(A) \, d\eta_x(y) \right) \, d\mu(x);$$

thus

$$\hat{\rho}_x = \int_Y \rho_y \, d\eta_x(y).$$

Hence

$$\hat{\rho}_x \otimes \hat{\rho}_x = \int_{Y \times Y} \rho_\eta \otimes \rho_\eta' \, d\eta_x \otimes \eta_x(y, y').$$

It follows that

$$\rho \otimes_X \rho = \int_X \hat{\rho}_x \otimes \hat{\rho}_x \, d\mu(x)$$

$$= \int_X \left( \int_{Y \times Y} \rho_y \otimes \rho_\eta \, d\eta_x \otimes \eta_x(y, y') \right) \, d\mu(x) = \int_{Y \times Y} \rho_\eta \otimes \rho_\eta' \, d\eta \otimes_X \eta(y, y').$$

Hence, if $f, g \in L^2(Z, \rho)$, then

$$E(f \otimes g|Y \times X) = \int_{Y \times X} f \otimes g \, d\rho_\eta \otimes \rho_\eta'$$

$$= \int_Y f \, d\rho_y \int_Y g \, d\rho_y' = E(f|Y) \cdot E(g|Y)$$

and therefore $E(f \otimes g|Y \times X) = E(f|Y) \otimes E(g|Y)$ a.s. with respect to $\eta \otimes_X \eta$. $\square$

Assume now that $T$ is an ergodic automorphism on a standard probability Borel space $(X, B, \mu)$. To each element $\lambda \in J(T)$ we associate a Markov operator $\Phi_\lambda : L^2(X_1, \mu_1) \rightarrow L^2(X_2, \mu_2)$ (where $(X_i, \mu_i) = (X, \mu)$, for $i = 1, 2$) given by

$$\int_{X_2} \Phi_\lambda(f)g \, d\mu_2 = \int_{X_1} \int_{X_2} f \, d\lambda \, \mu_2.$$

By Markov property we mean that $\Phi_\lambda$ is positive and $\Phi_\lambda 1 = \Phi_\lambda^* 1 = 1$. We also have $\Phi_\lambda \circ T = T \circ \Phi_\lambda$. Moreover, for each $f \in L^2(X, \mu)$

$$\Phi_\lambda f(x_2) = E^\lambda(f|X_2)(x_2).$$
Furthermore, each Markov operator on $L^2(X, \mu)$ that commutes with $T$ is necessarily of the form $\Phi_\lambda$ (see e.g. [5] or [8]). The latter observation introduces a semigroup law on $J(T)$ by the formula $\Phi_{\lambda_2 \circ \lambda_1} = \Phi_{\lambda_2} \circ \Phi_{\lambda_1}$. Together with the weak topology and the natural simplex structure on $J(T)$ we obtain that $J(T)$ is a compact semitopological affine semigroup.

Suppose now $\lambda_1, \lambda_2 \in J(T)$. We will treat $\lambda_1$ as defined on $X_1 \times X_2$, while $\lambda_2$ is defined on $X_2 \times X_3$. By $\lambda_3^*$ we mean the joining corresponding to $\Phi_{\lambda_3}^*$, that is, the self-joining given by

$$\lambda_3^*(A_2 \times A_3) = \lambda_2(A_3 \times A_2).$$

Disintegrate $\lambda_1$ and $\lambda_3^*$ over the common factor $X_2$:

$$\lambda_1 = \int_{X_2} \lambda_{1,x_2} d\mu_2(x_2), \quad \lambda_3^* = \int_{X_2} \lambda_{3^*,x_2}^* d\mu_2(x_2).$$

Consider the relative product of $\lambda_1$ and $\lambda_3^*$ over the common factor $X_2$ that is the measure defined on $X_1 \times X_2 \times X_3$ given by

$$\lambda_1 \otimes_{X_2} \lambda_3^* = \int_{X_2} \lambda_{1,x_2} \otimes \lambda_{3^*,x_2}^* d\mu_2(x_2).$$

Take $f, g \in L^2(X, \mu)$. Using [1] we then have

$$\int_{X_1 \times X_3} f(x_1)g(x_3) d\lambda_1 \otimes_{X_2} \lambda_3^*(x_1, x_2, x_3)$$

$$= \int_{X_2} \left( \int_{X_1 \times X_3} f(x_1)g(x_3) d\lambda_{1,x_2} \otimes \lambda_{3^*,x_2}^* (x_1, x_3) \right) d\mu_2(x_2)$$

$$= \int_{X_2} (\Phi_{\lambda_1} f)(x_2)(\Phi_{\lambda_3}^* g)(x_2) d\mu_2(x_2) = \int_{X_3} (\Phi_{\lambda_2} \circ \Phi_{\lambda_3})(f) g d\mu_3.$$

We have shown the following:

(2)

$$\lambda_2 \circ \lambda_1 = \lambda_1 \otimes_{X_2} \lambda_3^* |_{X_1 \times X_3}.$$

In particular, if $\lambda \otimes_{X_2} \lambda^*$ is ergodic, then $\lambda \circ \lambda$ is ergodic and the key observation for the proof of Theorem 1 is that the converse is also true (see Proposition 1 below).

Let $T$ acting on $(X, \mathcal{B}, \mu)$ be a factor of an ergodic automorphism $S$ acting on $(Y, \mathcal{C}, \eta)$. Following [2] (see condition C5 on p. 132), we say that $S$ is a compact extension of $T$ if for each $0 \neq f \in L^2(Y, \eta)$ the limit of ergodic averages of $f \otimes T$ for $S \times S$ acting on $(Y \times Y, \mathcal{C} \otimes \mathcal{C}, \eta \otimes \eta)$ is also non-zero.

Remark 1. Usually a compact extension is defined in terms of relative eigenvectors (see [1] [10]). R. Zimmer proved in [10] that $S$ is a compact extension of $T$ if and only if $S$ is an isometric extension of $T$. Another proof of Zimmer’s result follows easily from the joining characterization of isometric extensions given in [10].

Assume that $R$ acting on $(Z, \mathcal{D}, \rho)$ is an ergodic extension of $T$ acting on $(X, \mathcal{B}, \mu)$. Then (see [2], Chapter 6):

(A) there exists a biggest factor, called the relative Kronecker factor, $S$ acting on $(Y, \mathcal{C}, \eta)$ between $R$ and $T$ such that $S$ is a compact extension of $T$;

(B) the relative Kronecker factor $S$ is trivial (i.e. $S = T$) iff the relative product $R \times R$ on $(Z \times Z, \rho \otimes \rho)$ is ergodic (the latter condition means that $R$ is a relatively weakly mixing extension of $T$).
Finally, recall that an ergodic automorphism $T$ on $(X, B, \mu)$ is called semisimple \(^3\) if for each $\lambda \in J^c(T)$, the relative product $\lambda \otimes_{X_2} \lambda^*$ is ergodic, that is (using (B)), $T \times T$ on $(X_1 \times X_2, \lambda)$ is a relatively weakly mixing extension of $X_2$.

### 3. Proof of Theorem 1

We will need a lemma which is a simple consequence of the $L^1$-convergence in the pointwise ergodic theorem.

**Lemma 2.** Let $S$ be an automorphism on $(Y, C, \eta)$. Denote by $I$ the $\sigma$-algebra of $S$-invariant sets. Assume that $E \subset C$ is a factor of $S$. Then:

(i) If the action of $S$ on $E$ is ergodic, then $E(f|I) = \int_Y f \, d\eta$ for each $f \in L^1(E)$.

(ii) If $f \in L^1(Y, \eta)$ and the sequence $(\frac{1}{n} \sum_{i=0}^{n-1} f \circ S^i)_{n \geq 1}$ converges to a constant $c(= \int_Y f \, d\eta)$, then $E(E(f|I)|I) = c$.

**Proof.** (i) Since $f \in L^1(Y, \eta)$, $\frac{1}{n} \sum_{i=0}^{n-1} f \circ S^i$ converges to $E(f|I)$ in $L^1(C)$ by the ergodic theorem. However $f$ is measurable with respect to $C$ which is $S$-invariant. The result follows by the ergodicity of $S$ on $E$.

(ii) Put $g = E(f|I)$. Then by the ergodic theorem, $\frac{1}{n} \sum_{i=0}^{n-1} g \circ S^i$ converges to $E(g|I)$ in $L^1(C)$ and hence in $L^1(E)$. Therefore for all $h \in L^\infty(E)$,

$$\frac{1}{n} \sum_{i=0}^{n-1} \int f \circ S^i \cdot \eta \, d\eta \rightarrow c \int \eta \, d\eta.$$

We have

$$\frac{1}{n} \sum_{i=0}^{n-1} \int f \circ S^i \cdot \eta \, d\eta = \frac{1}{n} \sum_{i=0}^{n-1} \int E(f \circ S^i|E) \cdot \eta \, d\eta = \frac{1}{n} \sum_{i=0}^{n-1} \int E(f|E) \circ S^i \cdot \eta \, d\eta;$$

thus $\frac{1}{n} \sum_{i=0}^{n-1} g \circ S^i$ converges weakly to $c$ in $L^1(E)$. Hence $E(g|I) = c$. \(\square\)

The following lemma is a direct consequence of our definition of compact extension, the $L^1$-convergence of ergodic averages and the fact that $f \otimes f \in L^1(Y \times_X Y)$ whenever $f \in L^2(Y, \eta)$.

**Lemma 3.** Let $S$ be an ergodic automorphism on $(Y, C, \eta)$. Suppose that $S$ is a compact extension of $T$ acting on $(X, B, \mu)$ and $f \in L^2(Y)$. Then $f = 0$ if and only if $E(f \otimes f|I) = 0$ in the relative product $Y \times_X Y$.

The following result is of independent interest.

**Proposition 1.** Assume that $T$ is an ergodic automorphism of $(X, B, \mu)$ and let $\lambda \in J^c(T)$. If $\lambda \circ \lambda$ is ergodic, then $\lambda \otimes_X \lambda^*$ is ergodic.

**Proof.** Given a real function $f \in L^2(X, \mu)$ put $f \otimes f(x_1, x_2, x_3) = f(x_1) f(x_3)$. We have $f \otimes f \in L^1(X_1 \times X_2 \times X_3, \lambda_1 \otimes_X \lambda_2)$, where $\lambda_1 = \lambda_2 = \lambda$. If $I$ denotes the $\sigma$-algebra of $T \times T \times T$-invariant sets in the relative product, then our ergodicity assumption on $\lambda \circ \lambda$ and \(^2\) give rise to

$$E(f \otimes f|I) = \int f \otimes f \, d\lambda \otimes_X \lambda^*.$$  

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Let \((Y,C,\eta)\) denote the relative Kronecker factor of \(T \times T\) on \((X_1 \times X_2, \lambda_1)\) over \(X_2\) (see (A)). Then:

\[
(X \times X, \lambda)
\]

\[
Y
\]

\[
X
\]

Fix a real function \(f \in L^2(X,\mu)\). We will show that

\[
E(f(x_1)|Y) = E(f(x_1)|X_2).
\]  

(4) Let \(g = E(f(x_1)|Y) - E(f(x_1)|X_2)\). By Lemma 3 it is enough to prove that \(E(g \otimes g|\mathcal{I}) = 0\) with respect to \(\lambda_1 \otimes_{X_2} \lambda_2\). We have

\[
E(g \otimes g|\mathcal{I}) = E((E(f(x_1)|Y_1) - E(f(x_1)|X_2)) \otimes (E(f(x_3)|Y_2) - E(f(x_3)|X_2))|\mathcal{I})
\]

\[
= E(E(f(x_1)|Y_1) \otimes E(f(x_3)|Y_2) - E(f(x_1)|Y_1) \cdot E(f(x_3)|X_2) - E(f(x_1)|X_2) \cdot E(f(x_3)|Y_2) + E(f(x_1)|X_2) \cdot E(f(x_3)|X_2))|\mathcal{I})
\]

\[
= E(E(f(x_1)|Y_1) \otimes E(f(x_3)|Y_2) - \int_{Y_1} E(f(x_1)|Y_1) \cdot E(f(x_3)|X_2) \, d\eta_1
\]

\[
- \int_{Y_2} E(f(x_1)|X_2) \cdot E(f(x_3)|Y_2) \, d\eta_2 + \int_{X_2} E(f(x_1)|X_2) \cdot E(f(x_3)|X_2) \, d\mu_2.
\]

by Lemma 2(i) and the fact that \(X_2\) is a factor of \(Y\) and \(Y\) is ergodic. By taking in the latter three summands the conditional expectation with respect to \(X_2\), we obtain

\[
E(g \otimes g|\mathcal{I}) = E(E(f(x_1)|Y_1) \otimes E(f(x_3)|Y_2) - \int_{X_2} E(f(x_1)|X_2) \cdot E(f(x_3)|X_2) \, d\mu_2.
\]

Using consecutively Lemma 1 and 3, together with Lemma 2(ii), and finally the definition of the relative product, we obtain that

\[
E(f(x_1)|Y_1) \otimes E(f(x_3)|Y_2) = \int_{X \times X \times X} f(x_1)f(x_3) \, d\lambda \otimes_X \lambda^*(x_1, x_2, x_3)
\]

\[
= \int_{X_2} E(f(x_1)|X_2) \cdot E(f(x_3)|X_2) \, d\mu_2.
\]

We hence have proved \(E(g \otimes g|\mathcal{I}) = 0\) and (4) directly follows.

If \(h = h(x_2)\) is in \(L^2(X_2)\), then \(h\) is \(Y\)-measurable and by (4) we have

\[
E(f(x_1) \cdot h(x_2)|Y) = h(x_2) \cdot E(f(x_1)|Y)
\]

\[
= h(x_2) \cdot E(f(x_1)|X_2) = E(f(x_1) \cdot h(x_2)|X_2).
\]

Since the family of the function of the form \(f \otimes h\) as above forms a linearly dense subset in \(L^2(X \times X, \lambda)\), \(E(F|Y) = E(F|X_2)\) for all \(F \in L^2(X \times X, \lambda)\). Hence \(Y = X_2\) and the relative Kronecker factor of \((X_1 \times X_2, \lambda_1)\) over \(X_2\) is trivial. In view of (B), it follows that \(\lambda \otimes_{X_2} \lambda^*\) is ergodic.

\[\square\]
Proof of Theorem 1. At first, assume that $T$ is semisimple. Consider $\lambda_1, \lambda_2 \in J_e(T)$. Then, by using Proposition 6.3 from [2], $\lambda_1 \otimes X_2 \perp \lambda_2^*$ is ergodic by semisimplicity of $T$. Therefore $\lambda_2 \circ \lambda_1$ remains ergodic.

If $J_e^2(T)$ is a subsemigroup, then directly from Proposition 1 it follows that $T$ is semisimple.

Remark 2. The proof of Proposition 1 gives a slightly more general result: Assume that $\lambda$ is an ergodic joining of $S$ (acting on $(Y, C, \eta)$) and $T$ (acting on $(X, B, \mu)$). Then the relative product $\lambda \otimes_X \lambda$ is ergodic if and only if the measure $\lambda^* \circ \lambda$ on $Y \times Y$ (given by the Markov operator $\Phi^*_\lambda \circ \Phi_\lambda$ on $L^2(Y, \eta)$) is ergodic. Therefore we obtain an answer to the question by Ryzhikov from [8], p. 95.

References


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