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HYPERCYCLIC OPERATORS ON NON-LOCALLY CONVEX SPACES

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ABSTRACT. We transfer a number of fundamental results about hypercyclic operators on locally convex spaces (due to Ansari, Bès, Bourdon, Costakis, Feldman, and Peris) to the non-locally convex situation. This answers a problem posed by A. Peris [*Multi-hypercyclic operators are hypercyclic*, Math. Z. 236 (2001), 779-786].

During the past years much research has been done about hypercyclic operators; the article [6] contains a rather complete survey of results until 1999. A (continuous linear) operator $T : X \to X$ on a topological vector space X is called hypercyclic if it admits a vector $x \in X$ having dense orbit $\mathrm{Orb}(x) = \{x, Tx, T^2x, \ldots\}$ (x is then called a hypercyclic vector). The following theorem collects some of the recent fundamental results:

Theorem. *Let X be a locally convex space and let $T : X \to X$ be an operator.*

(1) Ansari [1]: *If T is hypercyclic, then so is T^n for each $n \in \mathbb{N}$.*
(2) Bourdon [3], Bès [2]: *If T is hypercyclic there is a dense invariant subspace of (except for 0) hypercyclic vectors.*
(3) Costakis [5], Peris [8]: *If T is multi-hypercyclic (i.e. there are finitely many vectors such that the union of their orbits is dense), then T is hypercyclic.*
(4) Bourdon, Feldman [4]: *Each orbit is either everywhere dense or nowhere dense.*

A. Peris asked in [8] whether in (3) local convexity is really needed and we now show that it is indeed not:

ALL PARTS OF THE THEOREM HOLD FOR TOPOLOGICAL VECTOR SPACES.

The only place in the proof of the Theorem where local convexity plays a role is the following lemma which, for hypercyclic operators, is due to P. Bourdon [3] (the complex case) and J. Bès [2] (the real case). Our proof for the non-locally convex case is quite similar to their arguments.

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Lemma. *Let T be a continuous linear operator on a topological vector space admitting a vector with somewhere dense orbit. Then for each non-zero polynomial p the operator $p(T)$ has dense range.*

Of course, the coefficients of the polynomial are assumed to be real if X is a real topological vector space.

Proof. We first consider a complex topological vector space X. Since the complex polynomial factorizes and the composition of dense range operators has dense range we may assume $p(z) = z - \lambda$ for some $\lambda \in \mathbb{C}$.

We assume $L = \overline{(T - \lambda\,\mathrm{id})(X)} \neq X$ and consider the quotient map $q : X \to X/L$ which clearly vanishes on L and thus satisfies $q \circ (T - \lambda\,\mathrm{id}) = 0$. Inductively this yields $q \circ T^n = \lambda^n q$ for all $n \in \mathbb{N}$ and therefore

$$q(\mathrm{Orb}(x)) = \{\lambda^n q(x) : n \in \mathbb{N}\} =: M$$

where x is a vector whose orbit is somewhere dense. Since q is a quotient map, $q(\mathrm{Orb}(x))$ is somewhere dense, too. On the other hand, M is contained in a one-dimensional subspace of the separated (since L is closed) topological vector space X/L, hence M is nowhere dense if the dimension of X/L is at least two. Otherwise X/L is isomorphic to $\mathbb{C}$ and then (depending on $|\lambda|$) M either consists of a null sequence, is contained in some circle, or is closed, and in any case nowhere dense, a contradiction.

Now let X be a real topological vector space. If there is a polynomial p such that $p(T)$ does not have dense range we could use similar arguments as in [2] to produce a finite-dimensional factor of the dynamical system (X, T) with a somewhere dense orbit – indeed, by factorization it is enough to consider $p(t) = t^2 - at - b$ and then we would obtain that the linear map given by $\left(\begin{smallmatrix} a & 1 \\ b & 0 \end{smallmatrix}\right)$ on $\mathbb{R}^2$ has a somewhere dense orbit – and elementary arguments show that this is impossible.

However, there is a simpler proof which was generously provided by A. Peris. Let $\widetilde{X} = X + iX$ and $\widetilde{T}(x + iy) = T(x) + iT(y)$ be the complexifications. Since $p(T)$ has dense range if and only if $p(\widetilde{T}) = \widetilde{p(T)}$ has dense range, it is again enough to show that $\widetilde{T} - \lambda\,\mathrm{id}$ has dense range for each $\lambda \in \mathbb{C}$. Assuming the contrary, we define as before L as the closure of $(\widetilde{T} - \lambda\,\mathrm{id})(\widetilde{X})$ and denote the quotient map $\widetilde{X} \to \widetilde{X}/L$ by q. If $\mathrm{Orb}(x)$ is somewhere dense in X, then $A = \mathrm{Orb}(x) + i\mathrm{Orb}(x)$ is somewhere dense in $\widetilde{X}$. On the other hand, for $z = T^n(x) + iT^m(x) \in A$ we have $q(z) = (\lambda^n + i\lambda^m)q(x)$, hence the somewhere dense set $q(A)$ is contained in a one-dimensional subspace which implies $\widetilde{X}/L \cong \mathbb{C}$ (or, in other words, that $q \in \widetilde{X}'$ is an eigenvector of $\widetilde{T}^*$). Now, we can argue as in [8]: $Q(y) = |q(y)|$ defines a continuous and open map $X \to [0, \infty)$, hence $Q(\mathrm{Orb}(x))$ is somewhere dense contradicting

$$Q(T^n(x)) = |q(T^n(x))| = |q(\widetilde{T}^n(x))| = |\lambda^n q(x)| = |\lambda|^n |q(x)|.$$

□

The results about hypercyclicity stated in the theorem above have counterparts for supercyclic operators which, by definition, have an orbit whose scalar multiples are dense, i.e. there is $x \in X$ such that $\mathrm{Orb}(\langle x \rangle) = \{\alpha T^n(x) : n \in \mathbb{N}, \alpha \in \mathbb{K}\}$ is dense ($\langle x \rangle$ denotes the linear span of $\{x\}$). For locally convex spaces, Peris [8] proved that (in the obvious sense) multi-supercyclic operators are supercyclic and Bourdon and N. Feldman [4] even showed that $\mathrm{Orb}(\langle x \rangle)$ is either everywhere dense or nowhere dense for each vector individually. As for the hypercyclic case, local convexity was only used in the proof of the locally convex version of:

Lemma. *Let T be an operator on a topological vector space X admitting a vector x such that $\mathrm{Orb}(\langle x\rangle)$ is somewhere dense. Then there exists $\lambda \in \mathbb{C}$ such that $p(T)$ has dense range for each polynomial p with $p(\lambda) \neq 0$.*

Proof. Let us show the real case; the complex one is similar but simpler. If p is a polynomial with $p(T)$ having non-dense range, there is a root $\lambda_1 \in \mathbb{C}$ of p such that $\widetilde{T} - \lambda_1\,\mathrm{id}$ does not have dense range (where as before, $\widetilde{X}$ and $\widetilde{T}$ denote complexifications) and if the lemma were false we could find $\lambda_2 \notin \{\lambda_1, \overline{\lambda_1}\}$ such that $\widetilde{T} - \lambda_2\,\mathrm{id}$ has non-dense range, too. Again, we denote by L_j the closures of $(\widetilde{T} - \lambda_j\,\mathrm{id})(\widetilde{X})$ and the corresponding quotient maps by q_j.

Since $\mathrm{Orb}(\langle x\rangle) + i\mathrm{Orb}(\langle x\rangle)$ is somewhere dense in $\widetilde{X}$, we again obtain $\widetilde{X}/L_j \cong \mathbb{C}$, i.e. q_j is an eigenvector of $\widetilde{T}^*$ with respect to λ_j. If $q_j = \varphi_j + i\psi_j$ with *real* continuous linear functionals φ_j and ψ_j we obtain that either φ_1 or ψ_1 is linear independent of $\{\varphi_2, \psi_2\}$, since otherwise we could find $a, b \in \mathbb{C}$ such that $q_2 = aq_1 + b\overline{q_1}$ where $\overline{q_1} = \varphi_1 - i\psi_1$ is an eigenvector with respect to $\overline{\lambda_1}$, contradicting the fact that eigenvectors with respect to different eigenvalues are linearly independent. Hence there is $y \in X$ such that $q_1(y) \neq 0$ and $q_2(y) = 0$.

We fix a non-zero $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $u = \alpha T^n(x) \in A$ where A is the interior of $\overline{\mathrm{Orb}(\langle x\rangle)}$. Since $A - u$ is a 0-neighbourhood in X there is $\varepsilon > 0$ such that for $0 \le \delta \le \varepsilon$ we have $u + \delta y \in A \subseteq \overline{\mathrm{Orb}(\langle x\rangle)}$. For fixed δ with $q_1(u+\delta y) \neq 0$ we can thus choose sequences $(\beta_l)_{l\in\mathbb{N}}$ in $\mathbb{R}$ and $(k_l)_{l\in\mathbb{N}}$ in $\mathbb{N}$ such that $\beta_l T^{k_l}(x) \to u + \delta y$. From $\lambda_1 \neq 0$ (as $\widetilde{T}$ has dense range) and $q_1(x) \neq 0$ (as $q_1(\mathrm{Orb}(\langle x\rangle) + i\mathrm{Orb}(\langle x\rangle))$ is somewhere dense) we obtain for l large enough

$$\left|\frac{\lambda_2}{\lambda_1}\right|^{k_l} \left|\frac{q_2(x)}{q_1(x)}\right| = \left|\frac{q_2(\beta_l T^{k_l}(x))}{q_1(\beta_l T^{k_l}(x))}\right| \longrightarrow \left|\frac{q_2(u+\delta y)}{q_1(u+\delta y)}\right| = \left|\frac{q_2(u)}{q_1(u) + \delta q_1(y)}\right|.$$

Since $q_2(u) = \alpha\lambda_2^n q_2(x) \neq 0$ this implies that $|q_1(u) + \delta q_1(y)|$ is independent of δ which contradicts $q_1(y) \neq 0$. ☐

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