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NORMAL SUBSPACES OF PRODUCTS OF FINITELY MANY ORDINALS

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Abstract. Let X be a subspace of the product of finitely many ordinals. If X is normal, then X is strongly zero-dimensional, collectionwise normal, and shrinking. The proof uses $(\kappa_1, \ldots, \kappa_n)$-stationary sets.

1. Preliminary

We use $(\kappa_1, \ldots, \kappa_n)$-stationary sets to prove the following theorem which extends results of Kemoto, Nogura, Smith, and Yajima in [4] and Stanley in [8].

Theorem 1.1. *Let X be a subspace of the product of finitely many ordinals. The following are equivalent:*

1. *X is normal.*
2. *X is normal and strongly zero-dimensional.*
3. *X is collectionwise normal.*
4. *X is shrinking.*

This theorem differs from the theorem of [4] in two ways. First, that theorem applies to subspaces of the product of *two* ordinals. Second, that theorem does not include "strongly zero-dimensional". Instead, the fourth condition asserts that nine types of pairs of closed sets are separated. For example, if $(\mu, \nu) \notin X \subseteq \lambda^2$, then $\{(\alpha, \nu) \in X : \alpha < \lambda\}$ and $\{(\mu, \beta) \in X : \beta < \lambda\}$ are separated. Stanley's theorem asserts that if X is a normal subspace of the product of finitely many ordinals, then X is collectionwise Hausdorff. The condition that X is strongly zero-dimensional cannot be added to Theorem 1.1 because the paper [3] describes a subspace of $\omega + 1 \times \mathfrak{c}$ which is not strongly zero-dimensional.

First we define the notions in Theorem 1.1, and then introduce notation for tuples and products.

Definition 1.2. A *shrinking* of a cover $\mathcal{A}$ of a space X is a cover $\mathcal{B} = \{B_A : A \in \mathcal{A}\}$ such that $clB_A \subseteq A$ for all $A \in \mathcal{A}$. Every finite cozero cover of a space X has a cozero shrinking. If X is normal, then every finite open cover of X has an open shrinking (see [1], p. 386). We say that a space X is *shrinking* if every open cover

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of X has an open shrinking. We say that a space X is *normal and strongly zero-dimensional* if whenever H_0 and H_1 are disjoint closed subsets of X, then there are disjoint clopen sets W_0 and W_1 such that $H_0 \subseteq W_0$, $H_1 \subseteq W_1$, and $W_0 \cup W_1 = X$. The following characterization will be useful: a space X is normal and strongly zero-dimensional iff every finite open cover of X has a clopen shrinking. We say that a space X is *collectionwise normal* if whenever $\mathcal{H}$ is a discrete collection of closed subsets of X, then there is a discrete collection of open sets $\mathcal{U} = \{U(H) : H \in \mathcal{H}\}$ such that $H \subseteq U(H)$ for all $H \in \mathcal{H}$.

Definition 1.3. Let a be an n-tuple of ordinals. Let Πa abbreviate $\prod_{i \leq n} a_i$ and let Πa^* abbreviate $\prod_{i \leq n}(\{-1\} \cup a_i)$. If $y \in \Pi a$ and $z \in \Pi a^*$ we define

$$\begin{aligned} z < y \quad &\text{iff} \quad z_i < y_i \text{ for all } i \leq n, \\ z \leq y \quad &\text{iff} \quad z_i \leq y_i \text{ for all } i \leq n, \\ (z, y] \quad &= \quad \{t \in \Pi a : z < t \leq y\}, \\ b \prec a \quad &\text{iff} \quad b \leq a \text{ and } b_i < a_i \text{ for some } i. \end{aligned}$$

"Intervals" $(z, y]$ form a basis for the product topology on Πa. We will prove Theorem 1.1 by induction on the well founded order $\prec$.

We set notation for concatenating n-tuples. If $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_n)$, then $a^\frown b$ is the $(m+n)$-tuple $c = (c_1, \ldots, c_{m+n})$, where $c_i = a_i$ for $i \leq m$ and $c_{m+j} = b_j$ for $j \leq n$. If $X \subseteq \Pi(a^\frown b)$ and $t \in \Pi a$, define $X_t = \{y \in \Pi b : t^\frown y \in X\}$.

In Section 2, we introduce the notion k-stationary, where k is a strictly increasing n-tuple of regular uncountable cardinals. In section 3, we describe a class of nonnormal spaces. The fact that these spaces are not normal leads to a dicotomy for normal subspaces X of Πa: either they are "reducible" to a free sum of spaces which are homeomorphic to subspaces of Πb with $b \prec a$ (Lemma 3.7), or we can apply a general Pressing Down Lemma (Lemma 2.2, Lemma 3.10). Preparations complete, we prove Theorem 1.1. The implication [(3) or (4)] → (1) is obvious; we prove (1) → (2) in Section 4; and we prove (2) → [(3) and (4)] in Section 5.

2. STATIONARY

The original proof of Theorem 1.1 used a more general theory of $(\kappa_1, \ldots, \kappa_n)$-stationary sets, developed in Section 3 of [2]. Later, we realized that Lemma 3.5 allows us to specialize to the case where the κ_i's are strictly increasing, where the theory is simpler. We prove only what is needed for Theorem 1.1, using a different definition from that of [2]. The theory of $(\kappa_1, \ldots, \kappa_n)$-stationary sets, for non-decreasing n-tuples of regular, uncountable cardinals, is presented in [3], where we show that $A_1 \times A_2 \times \ldots \times A_n$ is strongly zero-dimensional when each A_i is a subspace of an ordinal.

Definition 2.1. For $X \subseteq \Pi b \times \kappa$, we define the *stationary projection*

$$\operatorname{st}\pi[X] = \{t \in \Pi b : X_t \text{ is stationary in } \kappa\}.$$

Let $k = (\kappa_1, \ldots, \kappa_n)$ be a strictly increasing n-tuple of regular uncountable cardinals. We define Y to be *k-stationary* by induction on n. For $Y \subseteq \Pi(\kappa_1) = \kappa_1$, we say that Y is k-stationary iff Y is stationary in κ_1. For $Y \subseteq \Pi k \times \kappa$, we say that Y is $k^\frown\kappa$-stationary iff $\operatorname{st}\pi[Y]$ is k-stationary. Sometimes it is convenient to say that $Y = \{0\}$ is 0-stationary, where 0 is the empty sequence.

Lemma 2.2. *Let $Y \subseteq \Pi k$ be k-stationary, where $k = (\kappa_1, \dots, \kappa_n)$ is a strictly increasing n-tuple of regular uncountable cardinals. Then:*

1. *(Kemoto) If $f : Y \to \Pi k^*$ satisfies $f(y) < y$ for all $y \in Y$, then there are $q \in \Pi k$ and Y', a k-stationary subset of Y such that $f(y) < q \leq y$ for all $y \in Y'$.*
2. *If $\theta < \kappa_1$ and $g : Y \to \theta$, then there are $\beta \in \theta$ and Y', a k-stationary subset of Y such that $g(y) = \beta$ for all $y \in Y'$.*
3. *Y is a directed, cofinal subset of Πk.*
4. *$S = \{y \in Y : \Pi y \cap Y$ is not cofinal in $\Pi y\}$ is not k-stationary.*

Proof. By induction on n. If $n = 1$, this is the usual Pressing Down Lemma (PDL) (see [7], II, 6.15). Let Y be $k^\frown\kappa$-stationary. For each $t \in \operatorname{st}\pi[Y]$, use PDL to find $s_t < t$, $\xi_t < \kappa$, and Z_t, a κ-stationary subset of Y_t such that $f(t^\frown\xi) = s_t{}^\frown\xi_t$ for all $\xi \in Z_t$. By induction hypothesis, there are $r \in \Pi k$ and T', a k-stationary subset of T such that $s_t < r \leq t$ for all $t \in T'$. Let $\xi = \sup\{\xi_t : t \in T'\}$. Set $q = r^\frown\xi$.

The similar proof of clause 2 is left to the reader. Clause 3 is obvious.

Towards a contradiction, assume that S is k-stationary. For each $y \in S$, let $f(y) \in \Pi y$ satisfy $[f(y), y) \cap Y = \emptyset$. Apply clause 1 to get q and a stationary subset Y' of S. Apply clause 3 twice to find y and y' in Y' such that $q < y' < y$. Then $y' \in [f(y), y) \cap Y$. Contradiction! □

3. Normal

Lemma 3.1. *Let $X \subseteq \Pi b \times \kappa$, where $|\Pi b| < \kappa$ and κ is a regular, uncountable cardinal. If X is normal, then $T = \operatorname{st}\pi[X]$ is normal.*

Proof. For each $s \in \Pi b \setminus T$, choose C_s club in κ such that $\{s\} \times C_s \cap X = \emptyset$. Set $C = \bigcap\{C_s : s \in \Pi b \setminus T\}$. Let H_0 and H_1 be disjoint closed subsets of T. Set $\tilde{H}_0 = (H_0 \times C) \cap X$ and $\tilde{H}_1 = (H_1 \times C) \cap X$. Let $\tilde{U}_0$ and $\tilde{U}_1$ be disjoint, open in X with $\tilde{H}_0 \subseteq \tilde{U}_0$ and $\tilde{H}_1 \subseteq \tilde{U}_1$. We will show that $U_0 = \operatorname{st}\pi[\tilde{U}_0]$ and $U_1 = \operatorname{st}\pi[\tilde{U}_1]$ are disjoint open subsets of T.

Let $t \in U_e$. For each γ such that $t^\frown\gamma \in \tilde{U}_e$, find $s_{t,\gamma} < t$ and $\beta_{t,\gamma} < \gamma$ satisfying $X \cap ((s_{t,\gamma}, t] \times (\beta_{t,\gamma}, \gamma]) \subseteq \tilde{U}_e$. By PDL, there are Y_t, stationary in κ, $s_t < t$ and $\beta_{t(e)} \in \kappa$ so that $s_{t,\gamma} = s_t$ and $\beta_{t,\gamma} = \beta_{t(e)}$ for all $\gamma \in Y_t$. Then $(s_t, t] \cap T \subseteq U_e$, so U_e is open. If $t \in U_0 \cap U_1$, then $\gamma \in \tilde{U}_0 \cap \tilde{U}_1$ for all $\gamma > \max\{\beta_t(0), \beta_t(1)\}$. Contradiction! □

Definition 3.2. For sets S and T, let the diagonal map $\operatorname{Dg}_T : S \to S^T$ be defined by $\operatorname{Dg}_T(s)(t) = s$ for all $t \in T$. For $X \subseteq S^T$, set $\Delta_T(X) = \{s \in S : \operatorname{Dg}_T(s) \in X\}$. We will omit subscripts when it is clear from the context. Let Lim be the set of countable limit ordinals. For A a set of ordinals, let $L(A)$ be the set of ordinals ξ (not necessarily in A) such that every neighborhood of ξ meets A in an infinite set.

Example 3.3. Let $X = \{(\mu, \nu) \in \omega_1 \times \omega_1 : \mu < \nu\}$. Let $H_0 = \{(\mu, \mu + 1) \in X : \mu \in \mathrm{Lim}\}$ and $H_1 = \{(\mu, \nu) \in X : \nu \in \mathrm{Lim}\}$.

This space X is a specific instance of a class of nonnormal spaces (implicit in Lemma 4 of [4], and Lemma 5.1.3 of [5]). Let κ be an uncountable regular cardinal. Let $S \subseteq C \subseteq \kappa$, where S is stationary and C is club. Suppose that $X \subseteq \{(\mu, \nu) \in \kappa^2 : \mu \leq \nu\} \setminus C^2$ and that X_μ is stationary for every $\mu \in S$. Finally, suppose that $h(\mu) \in X_\mu \cap \bigcap_{\mu' < \mu} L(X_{\mu'})$ satisfies $(\mu, h(\mu)] \cap C = \emptyset$ for each

$\mu \in S$. We claim that X is not normal because $H_0 = \{(\mu, h(\mu)) : \mu \in S\}$ and $H_1 = \{(\mu, \nu) \in X : \nu \in C\}$ cannot be separated in X.

Towards proving the claim, let $H_0 \subseteq U_0$ open. For each $\mu \in S$, find $f(\mu) < \mu$ and $\mu \leq g(\mu) < h(\mu)$ so that $(f(\mu), \mu] \times (g(\mu), h(\mu)] \subseteq U_0$. Apply PDL to f and S to get S' and β so that $f(\mu) = \beta$ for all $\mu \in S'$. Fix $\mu_0 \in S \setminus (\beta + 1)$. Let $\nu_0 \in L(S') \cap X_{\mu_0}$. Then $(\mu_0, \nu_0) \in H_1$. We claim that $(\mu_0, \nu_0) \in \operatorname{cl} U_0$.

For every neighborhood N of (μ_0, ν_0), there is $\gamma \in [\mu_0, \nu_0)$ such that $\{\mu_0\} \times (\gamma, \nu_0] \subseteq N$. Choose $\mu_1 \in S' \cap (\gamma, \nu_0)$. Because $h(\mu_1) \in L(X_{\mu_0})$, there is $\nu_1 \in (g(\mu_1), h(\mu_1)) \cap X_{\mu_0}$. Then $(\mu_0, \nu_1) \in N \cap U_0$.

Lemma 3.4. *Let X be a normal subspace of κ^2, where κ is an uncountable regular cardinal. If $\Delta(X)$ is not stationary, then there is C, club in κ, such that $X \cap C^2 = \emptyset$.*

Proof. Let C' be club so that $\mathrm{Dg}[C'] \cap X = \emptyset$. Set $X' = \{(\mu, \nu) \in X : \mu \leq \nu\}$. Towards a contradiction, assume that $S' = \{\mu \in \kappa : X_\mu \text{ is stationary}\}$ is stationary. (In the terminology of [2], assume that X' is κ^2-stationary.) For each $\mu \in S'$, choose $h(\mu) \in X'_\mu \cap \bigcap_{\mu' < \mu} L(X_{\mu'})$. Set $C = \{\gamma \in C' : (\forall \mu < \gamma)(h(\mu) < \gamma)\}$. Then $S' \cap C$, C, X', and h satisfy the conditions of Example 3.3. Hence X is not normal. Contradiction!

Therefore there are clubs C^* and $(C_\mu)_{\mu \in C^*}$ such that $C_\mu \cap X_\mu = \emptyset$ for all $\mu \in C^*$. Set $E_1 = \{\gamma \in C^* : (\forall \mu < \gamma)(\gamma \in C_\mu)\}$. Then E_1 is club and $E_1^2 \cap (X')^2 = \emptyset$.

By a similar argument, we get a club E_2 so that $E_2^2 \cap \{(\mu, \nu) \in X : \mu \geq \nu\} = \emptyset$. Then $C' \cap E_1 \cap E_2$ is the desired club. ☐

Lemma 3.5. *Let X be a normal subspace of κ^n, where κ is an uncountable regular cardinal and $n \in \omega$. If $\Delta(X)$ is not stationary, then there is C, club in κ, such that $X \cap C^n = \emptyset$.*

Proof. By induction on n. The cases $n = 0$ and $n = 1$ are trivial; the case $n = 2$ is Lemma 3.4. Let X be a normal subspace of κ^n where $n = m + 1$. For each $e \in m^n$, let $X_e = \{x \in X : \text{ if } e(i) = e(j), \text{ then } x_{i+1} = x_{j+1}\}$. Each X_e is a closed, hence normal, subspace of X and is homeomorphic to a subspace of κ^m, so by induction hypothesis, there is a club C_e such that $C_e^m \cap X_e = \emptyset$. Set $G = \bigcap\{C_e : e \in m^n\}$.

Fix $\mu \in G$ and $i \leq n$. Set $X(\mu, i) = \{x \in X : x_i = \mu\}$, a closed, normal subspace of X. Let h be the natural homeomorphism of $X(\mu, i)$ into κ^m ($-$ h "deletes μ in the i^{th} place"). It is straightforward to verify that $\mathrm{Dg}_m[G] \cap h[X(\mu, i)] = \emptyset$. (Consider the $e \in 2^n$ satisfying $e(i) = 0$ and $e(j) = 1$ if $j \neq i$.) By induction hypothesis there is a club $C(\mu, i)$ such that if $z_j \in C(\mu, i)$ for all $j \neq i$, then $z \notin X(\mu, i)$. Unfix μ and i. Then $C = \{\gamma \in G : (\forall \mu \in \gamma \cap G)(\forall i \leq n)(\gamma \in C(\mu, i))\}$ is the desired club. ☐

Now we introduce the notion "reducible" and prove a lemma justifying the name.

Definition 3.6. If $\alpha = \beta + 1$, we say that $\{\beta\}$ is club in α and that $\operatorname{cof} \alpha = 1$. Let $X \subseteq \Pi a$, where a is an n-tuple of ordinals. We say X *is reducible in* a, and we write $red_a(X)$, when either $\operatorname{cof} a_i = \omega$ for some $i \leq n$, or there are C_i, $i \leq n$, each C_i club in a_i such that $C_1 \times \ldots \times C_n \cap X = \emptyset$.

Lemma 3.7. *If X is a normal and strongly zero-dimensional subspace of Πa, and $red_a(X)$, then X is homeomorphic to a free sum $\bigoplus\{Y_\lambda : \lambda \in \Lambda\}$ where $Y_\lambda \subseteq \Pi b_\lambda$ and $b_\lambda \prec a$ for each λ.*

Proof. It is clear if $\operatorname{cof} a_i = \omega$ for some i.

Otherwise, let C_i, $i \leq n$, be clubs witnessing $red_a(X)$. For each i, set $Z_i = \{z \in \Pi a : z_i \notin C_i\}$. Because X is normal and strongly zero-dimensional, the finite open cover $\{Z_i \cap X : i \leq n\}$ has a disjoint clopen refinement $\{W_i : i \leq n\}$. For each $i \leq n$, $W_i \subseteq Z_i$ is homeomorphic to a subspace of the free sum of spaces Πb with $b \prec a$. $\square$

Let $X \subseteq \Pi a$, where a is an n-tuple of ordinals. If $\operatorname{cof} a_i = \omega$ for some $i \leq n$, then $red_a(X)$. If $a = (\beta_1 + 1, \ldots, \beta_n + 1)$, then $red_a(X)$ iff $(\beta_1, \ldots, \beta_n) \notin X$. There remains the case where $\operatorname{cof} a_i \neq \omega$ for all $i \leq n$ and $\operatorname{cof} a_i > \omega$ for some $i \leq n$. Now our goal is to show that in this case, it suffices to consider nondecreasing m-tuples of regular, uncountable cardinals.

Let us say that $(d, \rho, (\mu_j : j \leq m), \psi)$ is *good* for a if the following are satisfied:

1. d is a nondecreasing m-tuple of regular, uncountable cardinals,
2. $\rho : \{1, \ldots, m\} \to \{1, \ldots, n\}$ is one to one,
3. for all $j \leq m$, $\mu_j : d_j \to a_{\rho(j)}$ is increasing, continuous, and cofinal,
4. for all $i \leq n$, if $i \notin \operatorname{ran} \rho$, then $a_i = \beta_i + 1$,
5. $\psi : \Pi d \to \Pi a$ satisfies for all $y \in \Pi d$, $(\psi(y))_i = \mu_j(y_j)$ if $\rho(j) = i$; $(\psi(y))_i = \beta_i$ otherwise.

Lemma 3.8. *Assume that $(d, \rho, (\mu_j : j \leq m), \psi)$ is good for a. Let $d = (\kappa_1, \ldots, \kappa_m)$. If X is a normal subspace of Πa, then $X \cap \operatorname{ran} \psi$ is closed in X, hence normal, and then $\psi^{\leftarrow}[X]$ is normal. If $C_1 \times \ldots \times C_m$, each C_i club in κ_i witnesses $red_d(\psi^{\leftarrow}[X])$, then $\psi[C_1 \times \ldots \times C_m]$ witnesses $red_a(X)$.* $\square$

Lemma 3.9. *Let a be an n-tuple of ordinals. If X is a normal subspace of Πa, and not $red_a(X)$, then there are Y and φ such that Y is a $(\kappa_1, \ldots, \kappa_p)$-stationary set and φ is an order preserving homeomorphism from Y into X such that $\varphi[Y]$ is closed and cofinal in X.*

Proof. If $a = (\beta_1 + 1, \ldots, \beta_n + 1)$, let $Y = \{0\}$. Otherwise, by Lemma 3.8, we may assume that a is a nondecreasing n-tuple of regular cardinals. Let $(\kappa_1, \ldots, \kappa_p)$ be the distinct values of a_i in strictly increasing order. We proceed by induction on p. When $p = 1$, then $\Pi a = \kappa_1^{n_1}$, and Lemma 3.5 says that $Y = \Delta(X)$ is stationary in κ_1. Let $\varphi = \mathrm{Dg}$; then $\varphi[Y]$ is closed in X.

When $p = p' + 1$, then $a = b^\frown c$, where $c_i = \kappa_p > |\Pi b|$ for all $i \leq n_p$. Consider $T = \{t \in \Pi b : \Delta(X_t) \text{ is stationary in } \kappa_p\}$. If $red_b(T)$, then $red_a(X)$. Contradiction! So we may apply the induction hypothesis to T to obtain a $(\kappa_1, \ldots, \kappa_{p'})$-stationary set T', closed in Πb, and a homeomorphism φ'. Set $Y = \{z^\frown \xi \in T' \times \kappa_p : \mathrm{Dg}(\xi) \in X_z\} = (T' \times \mathrm{Dg}[\kappa_p]) \cap X$. Define φ by $\varphi(z^\frown \xi) = \varphi'(z)^\frown \mathrm{Dg}(\xi)$. $\square$

Lemma 3.10. *Let X, Y and φ satisfy the conclusion of Lemma 3.9. Let $f : \varphi[Y] \to \Pi a^*$ satisfy $f(\varphi(y)) < \varphi(y)$ for all $y \in Y$. Then there are $q \in \Pi a$ and R, directed and cofinal in X, such that $f(x) < q \leq x$ for all $x \in R$.*

Proof. It is easy to verify the conclusions when $Y = \{0\}$. Otherwise, let $Y_0 = \{y \in Y : \Pi y \cap Y \text{ is cofinal in } \Pi y\}$. By Lemma 2.2, Y_0 is k-stationary. For each $y \in Y_0$, find $g(y) \in Y$ such that $f(\varphi(y)) \leq \varphi(g(y)) < \varphi(y)$. Apply Lemma 2.2(1) to Y_0 and g to get q' and Y'. Set $q = \varphi(q')$ and $R = \varphi[Y']$.

Of course, if X satisfies the hypothesis of Lemma 3.9 and f is defined on all of X, we may apply this Lemma without explicitly naming Y and φ. $\square$

4. Strongly zero-dimensional

In this section we prove that (1) implies (2) of Theorem 1.1 following the method of [2]. Let $H(Z)$ abbreviate "For all $X \subseteq Z$, if X is normal, then X is strongly zero-dimensional". The next lemma lists some methods to prove $H(Z)$ for "big" spaces from $H(Z')$ for "small" spaces.

Lemma 4.1. *Each of the following are sufficient to imply $H(Z)$:*

1. *Z is homeomorphic to a subset of Z' and $H(Z')$.*
2. *$Z = \bigoplus_{i \in I} Z_i$, and $H(Z_i)$ for all $i \in I$.*
3. *$Z = Z_1 \cup Z_2$, $H(Z_1)$, $H(Z_2)$, and Z_1 is closed in Z.*
4. *$Z = \bigcup_{i \in I} Z_k$, I is finite, and Z_i is open in Z for all $i \in I$.*

Proof. Clauses 1 and 2 are obvious. Clause 4 follows from clause 3.

Towards clause 3, let H_0 and H_1 be disjoint closed subsets of a normal subspace X of Z. Then $X_1 = X \cap Z_1$ is normal, so by $H(Z_1)$ there are disjoint closed K_0 and K_1 satisfying

$$H_0 \cap X_1 \subseteq K_0, \quad H_1 \cap X_1 \subseteq K_1, \quad \text{and } K_0 \cup K_1 = X_1.$$

Since X is normal, there are open U_0 and U_1 such that $H_0 \cup K_0 \subseteq U_0$, $H_1 \cup K_1 \subseteq U_1$, and $\operatorname{cl} U_0 \cap \operatorname{cl} U_1 = \emptyset$. Set $X_2 = X \setminus (U_0 \cup U_1)$, a closed, hence normal, subspace of X. By $H(Z_2)$, there are disjoint W_0 and W_1, clopen in X_2, satisfying

$$(\operatorname{cl} U_0 \cup H_0) \cap X_2 \subseteq W_0, \quad (\operatorname{cl} U_1 \cup H_1) \cap X_2 \subseteq W_1, \quad \text{and } W_0 \cup W_1 = X_2.$$

Then $U_0 \cup W_0$ and $U_1 \cup W_1$ are the desired clopen subsets of X. □

Because every product of finitely many ordinals is a subspace of α^n for some n and α, to prove (1) implies (2) of Theorem 1.1, it suffices to prove $(\forall n \in \omega) H(\alpha^n)$ for all ordinals α. We proceed by induction. The base step is easy: if α is countable, then every subspace of α^n is strongly zero-dimensional. For the induction steps, we use Lemma 4.3 when α is an uncountable regular cardinal, and Lemma 4.2 otherwise.

Lemma 4.2. *Let α be either a successor ordinal or a singular limit ordinal. If $(\forall m \in \omega) H(\beta^m)$ for all $\beta < \alpha$, then $(\forall n \in \omega) H(\alpha^n)$.*

Proof. We prove $(\forall m \in \omega) H(\beta^m \times \alpha^n)$ by induction on n. The base step $n = 0$ follows from hypothesis. Let $n = p + 1$.

If α is a successor, $\alpha = \gamma + 1$, say, then set $C = \{\gamma\}$. If α is a limit, let $C = \{\gamma_\nu : \nu < \operatorname{cof} \alpha\}$ be increasing, closed, and cofinal in α, with $\gamma_0 = 0$. Set $Z_1 = \beta^m \times \alpha^p \times C$. Then Z_1 is closed in $\beta^m \times \alpha^n$, and $H(Z_1)$ holds because Z_1 is homeomorphic to a subspace of $\zeta^{m+1} \times \alpha^p$, where $\zeta = \max\{\beta, \operatorname{cof} \alpha\}$. Set $Z_2 = (\beta^m \times \alpha^n) \setminus Z_1$. If $\alpha = \gamma + 1$, then $H(Z_2)$ follows from $H(\gamma^{m+n})$. If α is a singular limit ordinal, then $Z_2 = \bigoplus_{\nu < \operatorname{cof} \alpha} \beta^m \times \alpha^p \times (\gamma_\nu, \gamma_{\nu+1})$. In this case, $H(Z_2)$ holds by induction hypothesis and Lemma 4.1(2). Having shown $H(Z_1)$ and $H(Z_2)$, we may conclude $H(\beta^m \times \alpha^n)$ because of Lemma 4.1(3). □

Lemma 4.3. *Assume that α is regular. If $(\forall m \in \omega) H(\beta^m)$ for all $\beta < \alpha$, then $(\forall n \in \omega) H(\alpha^n)$.*

Proof. We prove $(\forall m \in \omega) H(\beta^m \times \alpha^n)$ by induction on n. The base step $n = 0$ follows from hypothesis.

Let $n = 1$. Note that for any C club in κ, $H(\beta^m \times (\alpha \setminus C))$ holds by induction hypothesis and Lemma 4.1(2). Let H_0 and H_1 be disjoint closed subsets of X, a normal subspace of $\beta^m \times \alpha$. For each $b \in \beta^m$, find C_b, club in α, such that for $Y \in \{X, H_0, H_1\}$, if Y_b is not stationary, then $C_b \cap Y_p = \emptyset$. Set $C = \bigcap\{C_b : b \in \beta^m\}$ and $X_1 = X \cap (\beta^m \times C)$. Let π be the projection: $\pi(b^\frown\xi) = b$.

Set $T = \pi[X_1] = \operatorname{st}\pi[X]$. Lemma 3.1 implies that T is normal. By $H(\beta^m)$, there is W clopen in T satisfying $\pi[H_0 \cap X_1] \subseteq W \subseteq T \setminus \pi[H_1 \cap X_1]$. Set $K_0 = W \times C \cap X_1$ and $K_1 = (T \setminus W) \times C \cap X_1$. Because X is normal, there are open U_0 and U_1 such that $H_0 \cup K_0 \subseteq U_0$, $H_1 \cup K_1 \subseteq U_1$, and $\operatorname{cl} U_0 \cap \operatorname{cl} U_1 = \emptyset$. Now $X_2 = X \setminus (U_0 \cup U_1)$ is a closed, hence normal, subspace of X. By $H(\beta^m \times (\alpha \setminus C))$, there are disjoint W_0 and W_1, clopen in X_2, satisfying

$$(\operatorname{cl} U_0 \cup H_0) \cap X_2 \subseteq W_0, \quad (\operatorname{cl} U_1 \cup H_1) \cap X_2 \subseteq W_1, \quad \text{and } W_0 \cup W_1 = X_2.$$

Then $U_0 \cup W_0$ and $U_1 \cup W_1$ are the desired clopen subsets of X.

Let $n = p + 1$. We need

Sublemma 4.4. *Let $\beta < \alpha$, let $m < \omega$, and let C be club in α. Then we have $H(\beta^m \times (\alpha^n \setminus C^n))$.*

Proof. Let X be a normal subspace of $\beta^m \times (\alpha^n \setminus C^n)$. We will find a finite family of clopen, strongly zero-dimensional, subspaces which cover X. Set $n^* = \{m+1, \dots, m+n\}$. By induction hypothesis and Lemma 4.1, $H(Z_i)$ for all $i \in n^*$, where $Z_i = \{z \in \beta^m \times \alpha^n : i \notin C\}$.

For $x \in X$, set $\sigma(x) = |\{i \in n^* : x_i \notin C\}|$. For each nonempty $s \subseteq n^*$, define closed sets $H_s = \{x \in X : (\forall i \notin s)(x_i \in C)\}$ and $K_s = \{x \in X : x_{\min(s)} \in C\}$. Note that $H_s \cap K_s \cap \{x : |s| \leq \sigma(x)\} = \emptyset$.

We will define X_j, a clopen subspace of X satisfying $\sigma(x) \geq j$ for all $x \in X_j$, by induction for $1 \leq j \leq n$. Set $X_1 = X$. For each $s \in [n^*]^j$, use the normality of X_j to find U_s, open in X_j, satisfying

$$X_j \cap H_s \subseteq U_s \subseteq \operatorname{cl} U_s \subseteq X_j \setminus K_s.$$

Because $\operatorname{cl} U_s$ is a normal subspace of $Z_{\min s}$, there is W_s, clopen in $\operatorname{cl} U_s$ such that $H_s \subseteq W_s \subseteq U_s$. Then W_s is clopen in X. Set $X_{j+1} = X_j \setminus \bigcup\{W_s : |s| = j\}$. Set $W_{n^*} = X_n$. Then $X = \bigcup\{W_s : \emptyset \neq s \subseteq n^*\}$, as promised. □

Returning to the proof of the case $n = p + 1$, set $Z_1 = \beta^m \times \operatorname{Dg}[\alpha]$. We have $H(Z_1)$ because Z_1 is homeomorphic to $\beta^m \times \alpha$. Set $Z_2 = \beta^m \times \alpha^n \setminus Z_1$. Towards showing $H(Z_2)$, let Y be a normal subspace of Z_2. For each $b \in \beta^m$, Y_b is a normal subspace of α^n with $\Delta(Y_b) = \emptyset$. Apply Lemma 3.5 to get a club C_b. Set $C = \bigcap\{C_b : b \in \beta^m\}$. From Sublemma 4.4 we conclude that Y is strongly zero-dimensional. Having established $H(Z_2)$, we apply Lemma 4.1(3) to finish the proof. □

5. Collectionwise normal and shrinking

Let $J(Z)$ denote "if $X \subseteq Z$ is normal and strongly zero-dimensional, then X is collectionwise normal and shrinking". To prove (2) implies (3) and (4) in Theorem 1.1, it suffices to prove $J(\Pi a)$ for all n-tuples a of ordinals, which we do by induction on $\prec$ (see Definition 1.3). The base step is easy: if a_i is countable for all $i \leq n$, then every subspace of Πa is collectionwise normal and shrinking.

Lemma 5.1. *If $J(\Pi b)$ for all $b \prec a$, X is a normal and strongly zero-dimensional subspace of Πa, and $red_a(X)$, then X is collectionwise normal and shrinking.*

Proof. It follows directly from Lemma 3.7. ☐

Lemma 5.2. *If $J(\Pi b)$ for all $b \prec a$, X is a normal and strongly zero-dimensional subspace of Πa, and not $red_a(X)$, then X is collectionwise normal.*

Proof. Let $\mathcal{H}$ be a discrete closed family in X. For each $x \in X$, choose $f(x) < x$ so that $(f(x), x]$ meets at most one H in $\mathcal{H}$. Let q and R be as in Lemma 3.10. Because R is directed and cofinal in X, there is at most one H, call it H^*, which meets $[q, a)$. Because X is strongly zero-dimensional, there is a clopen W such that $\bigcup(\mathcal{H} \setminus \{H^*\}) \subseteq W \subseteq X \setminus (H^* \cup [q, a))$. For each i, set $Z_i = \{z \in \Pi a : z_i < q_i\}$. By induction hypothesis, $J(Z_i)$ for each i, which gives that W is collectionwise normal. Let $\{U(H) : H \in \mathcal{H} \setminus \{H^*\}\}$ separate $\mathcal{H} \setminus \{H^*\}$ in W. Set $U(H^*) = X \setminus W$. Then $\{U(H) : H \in \mathcal{H}\}$ is the desired separation of $\mathcal{H}$. ☐

Lemma 5.3. *If $J(\Pi b)$ for all $b \prec a$, X is a normal and strongly zero-dimensional subspace of Πa, and not $red_a(X)$, then X is shrinking.*

Proof. Let $\mathcal{U}$ be an open cover of X. For each $x \in X$, choose $U_x \in \mathcal{U}$ and $f(x) < x$ satisfying $(f(x), x] \subseteq U_x$. Let q and R be as in Lemma 3.10. Apply Lemma 3.9 to $[q, a) \cap X$ to obtain Y, a k-stationary set, and φ, an order preserving homeomorphism of Y onto a closed, cofinal subset of X. For each $U \in \mathcal{U}$ set

$$G(U) = \{y \in Y : (\exists x \in R)(\varphi(y) \leq x \text{ and } U = U_x)\}.$$

If $y' \leq y \in G(U)$, then $y' \in G(U)$; hence, $G(U)$ is open.

Sublemma 5.4. *There is $\mathcal{F} = \{F_U : U \in \mathcal{U}\}$, a family of closed subsets of Y such that $F_U \subseteq G(U)$ for all $U \in \mathcal{U}$ and $\bigcup \mathcal{F} = Y'$, a final segment of Y.*

Proof. If $Y = \{0\}$, then set $F_U = \{0\}$ if $\varphi(0) \in U$ and $F_U = \emptyset$ if $0 \notin U$. If $Y \subseteq \kappa$, set $T = \emptyset = \operatorname{st} \pi[Y]$. For each $A \subseteq \kappa$, let $A_\emptyset = A$. Otherwise, Y is $k^\frown\kappa$-stationary. Set $T = \operatorname{st} \pi[Y]$. For each $U \in \mathcal{U}$, define $U_T = \{t \in T : \sup G(U)_t = \kappa\}$, open in T because $|\Pi k| < \kappa$. We split into subcases.

Subcase 1. $\bigcup\{U_T : U \in \mathcal{U}\} = T$. By Lemma 3.1, T is normal. So by induction hypothesis, T is shrinking. Let $\{V_T(U) : U \in \mathcal{U}\}$ be a shrinking of $\{U_T : U \in \mathcal{U}\}$, and set $F_U = (\operatorname{cl} V_T(U) \times \kappa) \cap Y$. Then $\bigcup \mathcal{F} = Y = Y'$.

Subcase 2. There is $q^* \in T \setminus \bigcup\{U_T : U \in \mathcal{U}\}$. Set $Y' = \{y' \in Y : q^* \leq \pi(y')\}$. Note that $[y, k) \setminus \bigcup\{G(U) : U \in \mathcal{U}'\} \neq \emptyset$ for all $\mathcal{U}' \in [\mathcal{U}]^{<\kappa}$ and $y \in Y'$. Well order Y' as $\{y_\nu : \nu < \kappa\}$. Inductively choose $U_\nu \in \mathcal{U} \setminus \{U_\mu : \mu < \nu\}$ so that $y_\nu \in G(U_\nu)$. Set $F_{U_\nu} = [q^*, y_\nu] \cap Y$; set $F_U = \emptyset$ if U is not a U_ν. Then $\bigcup \mathcal{F} = Y'$. ☐

For each $U \in \mathcal{U}$, use the normality of X to find an open $V_1(U)$ satisfying

$$\varphi[F_U] \subseteq V_1(U) \subseteq \operatorname{cl} V_1(U) \subseteq U.$$

For each $y \in Y'$, choose $U'_y \in \mathcal{U}$ and $f'(y) \in \Pi a^*$ which satisfy $(f'(y), \varphi(y)] \cap X \subseteq V_1(U'_y)$. Apply Lemma 3.10 to f' and Y' to obtain q'. Set $X_2 = X \setminus \bigcup\{V_1(U) : U \in \mathcal{U}\}$. Because $[q', a) \cap X_2 = \emptyset$, we have $red_a(X_2)$. Hence X_2 is shrinking, and there is a closed family $\{F'_U : U \in \mathcal{U}\}$ covering X_2 satisfying $F'_U \subseteq U$ for all $U \in \mathcal{U}$. By normality of X, find open sets $V_2(U)$ such that

$$F'_U \subseteq V_2(U) \subseteq \operatorname{cl} V_2(U) \subseteq U$$

for all $U \in \mathcal{U}$. Then $\{V_1(U) \cup V_2(U) : U \in \mathcal{U}\}$ is the desired shrinking of $\mathcal{U}$. ☐

Together, the lemmas of this section yield "If $J(\Pi b)$ for all $b \prec a$, then $J(\Pi a)$", from which we conclude $J(\Pi a)$ for all n-tuples of ordinals. This completes the proof of Theorem 1.1.

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