

SECOND ORDER SINGULAR PERIODIC PROBLEMS IN THE PRESENCE OF DRY FRICTION

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ABSTRACT. We prove, via an approach by ordinary differential equations, the existence of oscillations for second order inclusions with restoring terms with singularities both of repulsive and attractive type and with a dry friction term.

1. INTRODUCTION

The existence and properties of periodic solutions for differential equations of type

$$(1) \quad u'' + g(u) + f(u') + \mu \operatorname{sign} u' = \varphi(t),$$

where f , g are continuous, φ is periodic and $\mu > 0$, have attracted the attention of several authors in the last decade. Equation (1) models a one-dimensional mechanical system subjected to dry friction and periodic forcing. Dry friction arises when two rigid bodies interact in such a way that contact phases are interrupted by slip phases. In one of its simplest forms, this non-smooth mechanical effect can be modelled by Coulomb's law, according to which the term $\mu \operatorname{sign} u'$ has been inserted. Additional viscous damping may be represented by $f(u')$, and $g(u)$ is a restoring force depending on position. We refer the reader to the survey by Kunze [9] for a wealth of information on this kind of problem.

Existence, nonexistence and properties of periodic solutions of such systems have been considered by Deimling [4], Deimling and Szilágyi [5] and Bothe [2]. In [4] problems of existence and uniqueness are discussed under the assumption that $f = 0$ and g is linear. In particular an existence result for resonance at the first nonzero eigenvalue is obtained. (See also section 2.)

The case when f and g are both linear and (1) may contain an extra bounded perturbation has been considered by Senkyrik [12], Senkyrik and Guenther [13] and more recently by Bothe [2]; in the latter article new interesting results are obtained for the case of u -dependent $\mu = \mu(u)$.

In all the papers mentioned above equation (1) is viewed as a differential inclusion where sign is interpreted in the usual way as a maximal monotone graph. The

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techniques used by these authors naturally involve multi-valued mappings and the theory of differential inclusions. In [7] results about (1) are given as corollaries of general existence theorems that use the concept of solution in the sense of Filipov's [6].

In this note our purpose is twofold. We address the problem of existence of periodic solutions to (1) in a singular setting, namely when $g(u)$ is defined for $u > 0$ only and becomes infinite near $u = 0$. In other words, we consider the one-dimensional motion of a particle in a singular force field (e.g. due to an attractive or repulsive electrical source) and affected by periodic forcing and dry friction. On the other hand, although we look at (1) as a differential inclusion, we exploit a hint of Deimling [4] and show that a simple approximation procedure can be carried out, with resource to well known tools to solve nonlinear problems in the field of ordinary differential equations, to obtain in the end a solution in the sense of inclusions.

The study of periodic solutions under one-dimensional singular fields was initiated by Lazer and Solimini [10] for conservative systems. Analogous results for the case where dissipation is present were obtained in [8] and more recently in [1].

We shall consider both attractive and repulsive forces and we take $f = 0$, so that damping is restricted to dry friction. In the attractive case we allow g to grow linearly below a certain slope related to the period T as has already been done, e.g. in [1], [15]. In the repulsive case we combine the use of upper and lower solutions with a device of Mawhin [11] concerning problems where the nonlinearities depend only on the derivative.

In order to show how our (simple) approximation procedure works we give in section 2 a slight improvement of a result of Deimling [4] for a nonsingular problem; in its proof we present the main idea to be used in the remaining proofs.

Finally it should be noted that, as in other papers already mentioned in connection with the dry friction problem, e.g. [2, 4], there are simple conditions that one can formulate in order to prevent (1) from possessing constant solutions, so that in such cases our results do not reduce to trivial statements.

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Notation. When we deal with functions u defined in a given interval $[0, T]$ we denote by $\|u\|_X$ the standard norm of u in the functional space X and by $\|u\|_p$ ($p \geq 1$) the $L^p(0, T)$ -norm of u .

2. APPROXIMATION PROCEDURE AND PERIODIC SOLUTION OF A RESONANT PROBLEM

In this section we consider the following periodic problem:

$$(2.1) \quad u'' + \mu(u)S(u') + u = \varphi(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

where

(A) S is a function defined, bounded and continuous in $\mathbb{R} \setminus 0$ with a jump discontinuity at 0, $S(0^-) < S(0^+)$, and

$$\alpha := \limsup_{z \rightarrow -\infty} S(z) < 0 < \beta := \liminf_{z \rightarrow +\infty} S(z),$$

(B) φ is continuous, 2π -periodic, and

(C) μ is a continuous function in \mathbb{R} such that there exist numbers μ_0 and μ_1 such that $0 < \mu_0 \leq \mu(u) \leq \mu_1$, $\forall u \in \mathbb{R}$.

By a *solution* of (2.1) we mean a 2π -periodic function $u \in W^{2,1}(0, 2\pi)$ such that there exists $w \in L^\infty(0, 2\pi)$ satisfying $S(0^-) \leq w(t) \leq S(0^+)$ a.e. in $B = \{t : u'(t) = 0\}$, $w(t) = S(u'(t))$ a.e. in $A := \{t : u'(t) \neq 0\}$ and

$$u'' + \mu(u)w(t) + u = \varphi(t), \quad \text{a.e.}, \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi).$$

The following theorem illustrates how we solve problems like (2.1) where a similar definition of solution is applicable. It may be related to results of Deimling [4] and Bothe [2].

Since (2.1) contains in the left-hand side a noninvertible operator, we shall use decompositions of functional spaces into its kernel and a complementary subspace: we set

$$L^2(0, 2\pi) = E_1 \oplus E_2,$$

where $E_1 := \text{span}\{\cos t, \sin t\}$ and \oplus denotes orthogonal direct sum. Accordingly, we split each $u \in L^2(0, 2\pi)$ as

$$u = u_1 + u_2, \quad u_i \in E_i, \quad i = 1, 2.$$

Theorem 1. *Suppose that (A), (B), (C) hold and $\|\varphi_1\|_\infty < 2\mu_0(\beta - \alpha)/\pi$. Then the problem (2.1) has at least one solution.*

Proof. 1st step: approximated problem. Let P_i denote the orthogonal projection onto E_i . For each small $\epsilon > 0$ define S_ϵ as the continuous function such that $S_\epsilon(z) = S(z)$ if $|z| \geq \epsilon$ and S_ϵ is linear in $[-\epsilon, \epsilon]$. Now consider the system

$$(2.2) \quad \begin{cases} u''_2 + u_2 + \lambda P_2 \mu(u) S_\epsilon(u'_1 + \lambda u'_2) = \varphi_2, \\ P_1 \mu(u_1 + \lambda u_2) S_\epsilon(u'_1 + \lambda u'_2) = \varphi_1, \end{cases}$$

where $\lambda \in [0, 1]$. Clearly, when $\lambda = 1$, this is equivalent to the approximated problem for an ordinary differential equation

$$(2.3) \quad u'' + \mu(u)S_\epsilon(u') + u = \varphi(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi).$$

To prove that (2.3) has a solution we show that (a) the solutions of (2.2) are a priori bounded independently of λ and (b) noting that system (2.2) may be written as $u - T(\lambda, u) = 0$, where T is compact in $[0, 1] \times C^1[0, 2\pi]$, the Leray-Schauder degree of $u \mapsto u - T(0, u)$ in a large ball is $\neq 0$.

Proof of (a): First we note that as a consequence of (A)–(C) fixing $\alpha < \alpha^* < 0 < \beta^* < \beta$ there is a constant C independent of ϵ so that for any function $v \in W^{1,1}(0, 2\pi)$ one has

$$(2.4) \quad \int_0^{2\pi} \mu(v)S_\epsilon(v')v' dt \geq \mu_0(\beta^* \|v'^+\|_1 - \alpha^* \|v'^-\|_1) - C.$$

Now from the first equation (2.2), our assumptions and the fact that the linear map $u_2 \mapsto u''_2 + u_2$ is invertible in E_2 , it follows that there is $C' > 0$ independent of λ and ϵ so that for any possible solution,

$$\|u_2\|_{C^2} \leq C'.$$

Let us take the second equation, multiply it by u'_1 and integrate in $[0, 2\pi]$. Note that if $z \in E_1$, then $\int_0^{2\pi} z^+ = \int_0^{2\pi} z^-$ and $z' \in E_1$ as well. Therefore, using (2.4) and the boundedness of μ and S , we obtain with another such constant D

$$\mu_0(\beta^* - \alpha^*) \frac{\|u'_1\|_1}{2} \leq D \|u'_2\|_1 + \frac{\pi}{4} \|\varphi_1\|_\infty \|u'_1\|_1 + C.$$

Since we can assume that $\|\varphi_1\|_\infty < 2\mu_0(\beta^* - \alpha^*)/\pi$, the boundedness of u follows.

Proof of (b): Note that, by well known facts of degree theory (see e.g. [3] for details), when $\lambda = 0$ the calculation of the degree for the compact perturbation of the identity involved in (2.2) reduces to that of the 2-dimensional operator $F(u_1) := P_1\mu(u_1)S_\epsilon(u'_1) - \varphi_1$ defined in E_1 . Using the natural inner product (denoted \cdot) in this space, multiplying by u'_1 and integrating as above it turns out that for all $u_1 \in E_1$ on the boundary of a sufficiently large ball B one has $F(u_1) \cdot d(u_1) > 0$, where d is the differentiation operator. Since d is an isomorphism we have $|\text{deg}(F, B)| = 1$.

2nd step: passing to the limit. As a consequence of the estimates in the 1st step, the family of problems (2.3) has solutions u_ϵ that are uniformly bounded with respect to ϵ . Hence, by elementary compactness arguments, along a sequence $\epsilon \rightarrow 0$ we may assume that for some functions u, w

$$u_\epsilon \rightarrow u \text{ in } W^{2,\infty}(0, 2\pi) \text{ and } S_\epsilon(u'_\epsilon) \rightarrow w \text{ in } L^\infty(0, 2\pi) - \text{weak}^*.$$

Consider the sets $A_\delta := \{t : |u'(t)| \geq \delta\}$ ($\delta > 0$) and $B := \{t : u'(t) = 0\}$. Since $u'_\epsilon \rightarrow u'$ uniformly, our definition of S_ϵ implies that for ϵ sufficiently small $S_\epsilon(u'_\epsilon) = S(u'_\epsilon)$ in A_δ and we conclude $w(t) = S(u'(t))$ a.e. in A_δ ; the arbitrariness of δ shows that the same is true a.e. in A . Given $\delta > 0$, there exist $\eta > 0, \epsilon_0 > 0$ such that

$$S(0^-) - \delta \leq S_\epsilon(z) \leq S(0^+) + \delta \text{ whenever } |z| \leq \eta \text{ and } \epsilon \leq \epsilon_0.$$

Again using the uniform convergence of u'_ϵ in the set $B_\eta := \{t : |u'(t)| \leq \eta/2\}$, we see that the restriction of $S_\epsilon(u'_\epsilon)$ to this set takes values in $[S(0^-) - \delta, S(0^+) + \delta]$ for $\epsilon \leq \epsilon_0$. By weak-star convergence, the same is true for w a.e. Considering a sequence $\delta \rightarrow 0$, we may assume that the corresponding $\eta \rightarrow 0$ as well. We can write B as a countable intersection of the B_η 's and it is straightforward to conclude that $w(t) \in [S(0^-), S(0^+)]$ a.e. in B . This completes the proof.

3. REPULSIVE FORCES

In this and in the following section we consider the problem

$$(3.1) \quad u'' + H(u') + g(u) = \varphi(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

($T > 0$). We decompose the continuous, T -periodic function $\varphi(t) = \bar{\varphi} + \tilde{\varphi}(t)$ where $\bar{\varphi}$ is the mean value of φ in $[0, T]$. We assume

(N) H is a continuous function in $\mathbb{R} \setminus 0$ so that $H(z)z \geq 0$ if $z \neq 0, H(0^-) < 0 < H(0^+)$ and normalized so that

$$(3.2) \quad \inf H = -1, \quad \sup H = 1.$$

In our first result $g(u)$ is bounded above:

Theorem 2. *Let $g :]0, +\infty[\rightarrow \mathbb{R}$ be a continuous function such that*

(i) *either there exists $R \in \mathbb{R}$ such that $g(R) > \varphi(t) + 1, \forall t \in \mathbb{R}$, or there exists an interval J with $g(u) > \bar{\varphi} + 1 \forall u \in J$ and the length of J is $> C_1 T^{3/2}/12$ where $C_1 = 1 + \sup(g - \bar{\varphi}) + \sup|\tilde{\varphi}|$;*

(ii) $\int_0^1 g(u) du = -\infty$.

Then the problem (3.1) admits at least one solution.

Proof. For each $\epsilon > 0$ define a continuous function $H_\epsilon : \mathbb{R} \rightarrow \mathbb{R}$ by

$$H_\epsilon(x) = \begin{cases} H(x) & \text{if } |x| \geq \epsilon, \\ H(\epsilon)x/\epsilon & \text{if } 0 \leq x \leq \epsilon, \\ -H(-\epsilon)x/\epsilon & \text{if } -\epsilon \leq x \leq 0, \end{cases}$$

and consider the approximated problem

$$(3.3) \quad u'' + H_\epsilon(u') + g(u) = \varphi(t).$$

We shall show that (3.3) has a T -periodic solution u_ϵ and we shall obtain bounds for this solution that are independent of ϵ . Once this is done we extract a converging subsequence as $\epsilon \rightarrow 0$ and we obtain a solution to (3.1) in the same way as has been done in the preceding section. To this end, let us consider the homotopy

$$(3.4) \quad u'' + H_\epsilon(u') + g(u, \lambda) = \bar{\varphi} + \lambda\tilde{\varphi}(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \quad 0 \leq \lambda \leq 1.$$

Here we define

$$g(u, \lambda) = (1 - \lambda)g_0(u) + \lambda g(u),$$

where $g_0 :]0, +\infty[\rightarrow \mathbb{R}$ is chosen as a strictly increasing C^1 function, with properties (i) (with the same R or J) and (ii), such that $\sup g_0 - \bar{\varphi} = \sup g - \bar{\varphi}$.

Fix $r > 0$ such that $g(r), g_0(r) < \inf \varphi - 1$. Let R be given by (i) and, in case the second condition there described holds, let R be the left endpoint of J . We consider the set S of solutions u of (3.4) for some ϵ and λ such that $\max u > r$ and $\min u < R$.

Claim. There exist $\alpha, \beta > 0$ independent of ϵ and λ such that for every $u \in S$ we have $u(t) > \alpha \forall t \in [0, T]$ and $\|u\|_{C^1[0, T]} < \beta$.

Proof of the Claim. For any given $u \in S$ let $m := \min u$. Multiplying (3.4) by $u - m$ and integrating in $[0, T]$ we obtain

$$(3.5) \quad \int_0^T u'(t)^2 dt \leq C_1 \int_0^T (u(t) - m) dt.$$

Now, for functions $w \in H_0^1(0, T)$ the inequality

$$\left| \int_0^T w(t) dt \right| \leq \frac{T^{3/2}}{12} \left(\int_0^T w'(t)^2 dt \right)^{1/2}$$

holds. Since periodic functions can be translated without affecting their integrals over a period, we have from (3.5)

$$\left(\int_0^T u'(t)^2 dt \right)^{1/2} \leq C_1 \frac{T^{3/2}}{12}$$

and it immediately follows that

$$(3.6) \quad u(t) \leq m + C_1 \frac{T^2}{12}, \quad \forall t \in [0, T].$$

Next, let $\tau \in [0, T]$ be such that $u(\tau) = m$; multiplying (3.4) by $u'(t)$ and integrating in $[\tau, t]$ it turns out that

$$\begin{aligned} \frac{u'(t)^2}{2} + \int_\tau^t H_\epsilon(u'(s))u'(s) ds + \int_\tau^t g(u(s), \lambda)u'(s) ds \\ = \bar{\varphi} \int_\tau^t u'(s) ds + \lambda \int_\tau^t \tilde{\varphi}(s)u'(s) ds. \end{aligned}$$

Since $H_\epsilon(z)z \geq 0 \forall z \in \mathbb{R}$ we infer from this that

$$\int_\tau^t g(u(s), \lambda)u'(s) ds = \lambda \int_{u(\tau)}^{u(t)} g(u) du + (1 - \lambda) \int_{u(\tau)}^{u(t)} g_0(u) du$$

is bounded above. Therefore, in view of (ii), we see that there exists $\alpha > 0$ (independent of ϵ and λ) such that

$$u(t) \geq \alpha, \quad \forall t \in [0, T].$$

From (3.4) we also deduce that

$$\int_0^T u''(t)^2 dt \leq \int_0^T |\tilde{\varphi}(t) - g(u(t), \lambda)| |u''(t)| dt,$$

and, since it is now known that $g(u(t), \lambda)$ is bounded independently of ϵ and λ , the same is true for $\int_0^T u''(t)^2 dt$. Together with previous estimates, this yields the desired bound on $\|u\|_{C^1[0, T]}$ and the Claim is proved.

Consider the bounded, open set $\Omega := \{u \in C^1[0, T] : \alpha < \min u < R, \max u > r, \|u\|_{C^1[0, T]} < \beta\}$. In view of the above Claim, to see that the homotopy (3.4) has no solutions $u \in \partial\Omega$ it suffices to show that no solution u of (3.4) can satisfy $\max u = r$ or $\min u = R$. The case $\max u = r$ is easily excluded by our choice of r . Take the case $\min u = R$: if the first condition in (i) holds, we see in the same manner that it is impossible; if the second condition in (i) holds, then, by (3.6) we would have $u(t) \in J \forall t \in [0, T]$, and, integrating (3.4) in $[0, T]$ we reach a contradiction.

For $\lambda = 0$ (3.4) has a unique (constant) solution in Ω with nonzero local index, and standard arguments of the Leray-Schauder degree theory (see [3]) allow us to conclude that (3.3) has a solution in Ω as well. This completes the proof.

Theorem 3. *Let $g :]0, +\infty[\rightarrow \mathbb{R}$ be a continuous function such that*

(i) *There exists $R \in \mathbb{R}$ such that either $g(R) > \varphi(t) + 1, \forall t \in \mathbb{R}$, or $g(u) > \bar{\varphi} + 1 \forall u \in]R, +\infty[$.*

(ii) *$\int_0^1 g(u) du = -\infty$.*

(iii) *There exist constants $0 < \gamma < \frac{\pi^2}{T^2}$ and $C > 0$ such that $g(u) \leq \gamma u + C \forall u > 0$.*

Then the problem (3.1) admits at least one solution.

Proof. The proof is similar to the preceding one and so we omit some details. Again a homotopy (3.4) is considered, where we now make a choice of g_0 to be a C^1 , strictly increasing function in $]0, +\infty[$ such that $g'_0(+\infty) = \gamma$.

The main difference now is in the way to obtain the estimates for u and u' . First, multiplying (3.4) by u and integrating we have

$$\int_0^T u'(t)^2 dt \leq \gamma \int_0^T u(t)^2 dt + \int_0^T (C + 1 + |\varphi|) |u| dt;$$

then since, as noted in [15], for all T -periodic functions $u \in H^1(0, T)$ with $m := \min u$,

$$\int_0^T (u(t) - m)^2 dt \leq \frac{T^2}{\pi^2} \int_0^T u'(t)^2 dt,$$

we arrive at the following inequality:

$$\|u\|_2^2 \leq 2R \|u\|_1 + \frac{T^2}{\pi^2} \|u'\|_2^2.$$

Since $\gamma < \frac{\pi^2}{T^2}$ we obtain a bound for $\|u\|_2$ and then for $\|u'\|_2$.

4. ATTRACTIVE FORCES

Theorem 4. *Assume in addition to (N) that*

$$\limsup_{u \rightarrow +\infty} g(u) < \limsup_{u \rightarrow 0^+} g(u) = +\infty.$$

Then for each periodic continuous $\varphi(t)$ such that

$$\bar{\varphi} > \limsup_{u \rightarrow +\infty} g(u) + 1,$$

the problem (3.1) has at least one solution.

Proof. We again consider the approximated problem (3.3). Let us construct lower and upper solutions for this problem. Clearly, we can select a small positive constant a such that $-1 + g(a) \geq \varphi(t) \forall t \in \mathbb{R}$ and this is a lower solution.

Now let us introduce the problem with a real parameter h

$$(4.1) \quad z'' + H_\epsilon(z') = h + \tilde{\varphi}(t), \quad z(0) = z(T), \quad z'(0) = z'(T).$$

By a theorem of Mawhin [11] we know that for fixed $\tilde{\varphi}$ there exists $h = h_\epsilon$ such that (4.1) is solvable. Moreover, it is easy to see that $|h_\epsilon| \leq 1$. Also, a solution of (4.1) is given by

$$(4.2) \quad z_\epsilon(t) = \int_0^t v_\epsilon(s) ds,$$

where v_ϵ is a fixed point of the nonlinear operator

$$(4.3) \quad v \mapsto \int_0^t V(s) ds - \frac{1}{T} \int_0^T V(s)(T - s) ds,$$

V being given by $V(s) = -H_\epsilon(v(s)) + \frac{1}{T} \int_0^T H_\epsilon(v(\tau)) d\tau + \tilde{\varphi}(s)$. Note that this operator has bounded range independent of ϵ .

Let $b > 0$ be a number such that $u \geq b$ implies $g(u) \leq \bar{\varphi} - 1$. Setting

$$-K_\epsilon = \min_t (z_\epsilon(t) - b)$$

we conclude, by (4.2) and (4.3), that the functions

$$y_\epsilon(t) = z_\epsilon(t) + K_\epsilon$$

are uniformly bounded with respect to ϵ . Moreover,

$$y''_\epsilon + H_\epsilon(y'_\epsilon) + g(y_\epsilon) = z''_\epsilon + H_\epsilon(z'_\epsilon) + g(y_\epsilon) \leq h_\epsilon + \tilde{\varphi}(t) + \bar{\varphi} - 1 \leq \varphi(t),$$

so that y_ϵ is the upper solution we look for. Therefore (4.1) has a solution u_ϵ such that

$$a \leq u_\epsilon(t) \leq y_\epsilon(t), \quad t \in \mathbb{R}.$$

Integrating (4.1) we infer that the family u_ϵ is uniformly bounded in C^1 -norm, so that we can extract a subsequence $u_\epsilon \rightarrow u$ as $\epsilon \rightarrow 0$ and the conclusion follows as in the preceding cases.

Remark. In the course of revising the manuscript, the authors have learned that in a paper just published, Torres [14] has used the same approximation procedure to study problems similar to those that concern us here.

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