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# THE RATIONAL LS-CATEGORY OF $k$-TRIVIAL FIBRATIONS

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Abstract. We provide new upper and lower bounds for the rational LS-category of a rational fibration $\xi : F \to E \to K(\mathbf{Q}, 2n)$ of simply connected spaces that depend on a measure of the triviality of $\xi$ which is strictly finer than the vanishing of the higher holonomy actions. In particular, we prove that if $\xi$ is *$k$-trivial* for some $k \geq 0$ and $H^*(F)$ enjoys Poincaré duality, then

$$\operatorname{cat}_0 E \geq \operatorname{cat}_0 F + k.$$

## §1. Introduction

The Lusternik-Schnirelmann category of a space $S$, denoted $\operatorname{cat} S$, is the least number of open sets, less one, which cover $S$ and are contractible in $S$. It is a subtle measure of the complexity of $S$ which is difficult to compute except where it agrees with other well-known homotopy invariants, such as $\dim S$ or the cup length in the cohomology ring. If $S$ is simply connected and has the homotopy type of a CW complex of finite type, the *rational category of* $S$, $\operatorname{cat}_0 S := \operatorname{cat} S_{\mathbf{Q}}$, introduced by Berstein, is a lower bound for $\operatorname{cat} S$ that is more amenable to computation because Felix and Halperin [FH] provided a complete algebraic description of $\operatorname{cat}_0 S$ in terms of a Sullivan minimal model of $S$.

We now know that $\operatorname{cat}_0(S_1 \times S_2) = \operatorname{cat}_0 S_1 + \operatorname{cat}_0 S_2$[1] [FHL], so a natural question is to find conditions on a non-trivial fibration $\xi : F \to E \to B$ which permit estimates of $\operatorname{cat}_0 E$ in terms of data associated to the "twisting". Some recent upper bounds in this spirit are provided in [CFJP], [JS] and [C2]. Lower bounds include the Mapping Theorem [FH], which in particular guarantees that $\operatorname{cat}_0 E \geq \operatorname{cat}_0 F$ when $F \hookrightarrow E$ induces an injection in rational homotopy. Extending this, various estimates of the form $\operatorname{cat}_0 E \geq \operatorname{cat}_0 F + k$ for $k > 0$ have been obtained using additional hypotheses on either the holonomy action and $B$ [J], on the homotopy Lie algebra of $E$ [GJ], or on the higher holonomy actions of $\xi$ itself [C1], [C2].

In this paper we attempt to unify and extend some of these ideas by introducing a new measure of the triviality of $\xi$ which is strictly finer than the vanishing of the higher holonomy operations.

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[1] A fact *not* true for cat. See [I].

To describe this new measure of triviality, let $G_kE \to E$ denote the $k^{th}$ Ganea fibration over $E$, and let $F \bowtie_E G_kE$ be the join of $F$ and $G_kE$ over $E$, with induced map $F \xrightarrow{j_k} F \bowtie_E G_kE$. (Recall that the join $X \bowtie_Y Z$ of $X \to Y \leftarrow Z$ may be obtained by a pull-back followed by a push-out.) Then, we make the following

**Definition.** The fibration $\xi$ is *trivial of order* $k$ (or $k$*-trivial*) if $j_k$ has a homotopy retraction, i.e., there exists a map $r$ in the diagram

$$\begin{tikzcd} F \arrow[r, "j_k"] \arrow[rr, bend right, "\mathrm{id}_F"'] & F \bowtie_E G_kE \arrow[r, dotted, "r"] & F \end{tikzcd}$$

making it commute up to homotopy.

Since the map $j_k$ factorizes through $j_{k-1}$, $k$-trivial fibrations are also $(k-1)$-trivial. Moreover, a trivial fibration in the usual sense is trivial of arbitrarily high order.

In rational homotopy, 0-triviality of the fibration is exactly the (single) hypothesis of the Mapping Theorem. When $k = 1$, it is precisely the first hypothesis of [GJ, Thm. 1] in that particular case.

The vanishing of the $k^{th}$(higher) holonomy action [C2] implies, but is not equivalent to, the $k$-triviality of the fibration. To see this, we recall that the $k^{th}$ holonomy action of $F \hookrightarrow E \to B$ vanishes iff

$$i \,:\, F \to G_{k+1}(E, F)$$

has a homotopy retraction (where $G_k(E, F)$ is the homotopy pullback of $E \to B$ and $G_kB \to B$), and that there is a natural map $F \bowtie_E G_kE \to G_{k+1}(E, F)$ which is compatible with $i$ and $j_k$.

Recall the definition of relative LS-category in the sense of Fadell and Husseini [FaHu]. If $(M, A)$ is an NDR pair, the relative category of $(M, A)$, denoted $\mathrm{cat}(M, A)$, is the least integer $n$ such that there exists an open covering $(M_j)_{0 \le j \le n}$ of $M$ such that $M_0 \hookrightarrow M$ factorizes up to homotopy (relative to $A$) through $A$ and that the open sets $M_j$, $j \ge 1$, are contractible in $M$. Moreover, $\mathrm{cat}(M, A) \le n + 1$ iff the map $i \bowtie g_n : A \bowtie_M G_nM \to M$ has a homotopy section $s$ satisfying $s \circ i \simeq j_k$, where $j_k$ is the inclusion $A \hookrightarrow A \bowtie_M G_kM$ [M]. Note that we have

$$cat\, M/A \le \mathrm{cat}(M, A) \le cat\, M + 1.$$

*Remark* 1. Note that when $\mathrm{cat}(E, F) \le k + 1$ and the fibration $\xi \,:\, F \hookrightarrow E \to B$ is $k$-trivial, then $\xi$ is in fact homotopically trivial, since the composition of the two homotopy retractions guaranteed by these conditions shows that $F \hookrightarrow E$ has a homotopy retraction. This has two interesting consequences:

• If $\xi$ is a $k$-trivial fibration which is not homotopically trivial, then $cat\,(E, F) > k + 1$ and, in general, for any $k$-trivial fibration, we have

$$cat(E, F) > min(cat\, B - 1, k + 1).$$

Indeed, assume $cat\,(E, F) \le k + 1$. Then the fibration $\xi$ is homotopically trivial and

$$cat\,(E, F) = cat\,(B \times F, F) \ge cat\, B.$$

• Since $cat\,(E, F) \le cat\, E + 1$, if $\xi$ is a $k$-trivial fibration with $cat\, E \le k$, then the fibration is homotopically trivial in the usual sense.

With $K(\mathbf{Q}, 2n)$ denoting as usual an Eilenberg-Mac Lane space and $\mathrm{cat}_0(E, F)$ denoting $\mathrm{cat}(E_{\mathbf{Q}}, F_{\mathbf{Q}})$, we can now state

**Theorem 1.** *Suppose $F \to E \to K(\mathbf{Q}, 2n)$ is a rational $k$-trivial fibration for some $k \geq 0$. Then:*

- $\mathrm{cat}_0 E \leq \mathrm{cat}_0(E, F) + \max(\mathrm{cat}_0 F, k+1) - k - 1$.
- *If in addition $H^*(F; \mathbf{Q})$ satisfies Poincaré duality, we have* $\mathrm{cat}_0 E \geq \mathrm{cat}_0 F + k$.

In the general case, we have $cat\, M \leq cat\,(M, A) + cat\, A$ and so the first inequality is an improvement on this bound. Similary, the second point of Remark 1 implies that $cat\, E > k$ and the second inequality is an improvement on this bound.

Stated this way, the second inequality relies on the equality $\mathrm{cat}_0 = e_0$ for Poincaré duality spaces [FHL]. Here, $e_0$ denotes Toomer's invariant [T], which is the largest $p$ such that in the spectral sequence of Milnor and Moore, $E_\infty^{p,\star} \neq 0$. Under our assumptions, $H^*(E; \mathbf{Q})$ is also a P.d.a [FHT2, Thms. 4.3 & 3.1], so what we actually prove is that $\mathrm{e}_0 E \geq \mathrm{e}_0 F + k$. As already mentioned, second point of Theorem 1 is a generalization of [GJ, Thm. 1], and, once we have algebraically characterized the condition of $k$-triviality, it is proved in the same way.

Theorem 1 is proven using the standard methods of rational homotopy, and we remark that it is a straightforward matter to check for $k$-triviality when one has a minimal model of $E$, especially when $E$ is an elliptic space, i.e., when $\pi_* E \otimes \mathbf{Q}$ and $H^*(E; \mathbf{Q})$ are finite dimensional.

This paper is organized as follows. In the next section, we give a characterization of our hypotheses at the level of minimal models, and we state a proposition that will imply the second point of Theorem 1. In section 3, we will prove the two points of Theorem 1. In the final section, we present 3 examples to illustrate our results.

## §2. Rational homotopy and $\mathrm{cat}_0$

All our spaces will be simply connected with the homotopy type of CW complexes with rational cohomology of finite type. We will work with $\mathbf{Q}$ as ground field and our principal tools are Sullivan models. A detailed description of these and the standard tools of rational homotopy can be found in [FHT1]. For our purposes, we recall the following.

Sullivan [S] defined a contravariant functor $\mathcal{A}_{PL}$ which associates to each space $S$ a commutative graded differential algebra (hereafter cgda) $\mathcal{A}_{PL}(S)$ which represents the rational homotopy type of $S$. He also constructed, for each simply connected cgda $(A, d)$ (i.e. satisfying $H^0(A, d) = H^1(A, d) = 0$), another cgda $(\Lambda X, d)$ and a map

$$(\Lambda X, d) \xrightarrow{\simeq} (A, d)$$

which induces an isomorphism in cohomology (hereafter called a *quasi-isomorphism*), where $\Lambda X$ denotes the free commutative-graded algebra on the graded vector space $X = \sum_{n \geq 2} X^n$, which has a well ordered, homogeneous basis $\{x_\alpha\}$ such that, if $X_{<\alpha}$ denotes $\mathrm{span}\{x_\beta \mid \beta < \alpha\}$, we have $dx_\alpha \in \Lambda^{\geq 2}(X_{<\alpha})$. The cgda $(\Lambda X, d)$ is called a *(minimal) Sullivan model* of $(A, d)$ or a *Sullivan model* of $S$ if $(A, d) = \mathcal{A}_{PL}(S)$.

He also defined a *geometric realization* functor $|.|$ which converts a (minimal) Sullivan model $(\Lambda X, d)$ (of finite type) into a rational space $|(\Lambda X, d)|$ so that $(\Lambda X, d)$

is a Sullivan model for $|(\Lambda X, d)|$. These functors define bijections

$$\left\{\begin{matrix}\text{rational homotopy}\\ \text{types of spaces}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{isomorphism classes of}\\ \text{minimal Sullivan models}\end{matrix}\right\},$$

$$\left\{\begin{matrix}\text{homotopy classes of}\\ \text{maps between rational spaces}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{homotopy classes of maps}\\ \text{between minimal Sullivan models}\end{matrix}\right\}.$$

When two cgda's have isomorphic minimal models, we say that they are *quasi-isomorphic cgda's*, even though there may not be a quasi-isomorphism between them.

Let $\Lambda(t, dt)$ be the cgda generated as an algebra by $t$ in degree 0, and $dt$ in degree 1, with $d(t) = dt$ and $d(dt) = 0$, and let $\varepsilon_0, \varepsilon_1 : \Lambda(t, dt) \to (\mathbf{Q}, 0)$ denote the unique maps satisfying $\varepsilon_i(t) = it$. If $(\Lambda X, d)$ is a minimal model and $(A, d_A)$ is any 0-connected cgda, two maps $\phi_0, \phi_1 : (\Lambda X, d) \to (A, d_A)$ are *homotopic* if there is a map

$$\Phi : (\Lambda X, d) \to (A, d_A) \otimes \Lambda(t, dt)$$

with $\phi_i = \varepsilon_i \Phi$, $i = 0, 1$.

If $\phi : (A, d) \to (B, d)$ is a morphism of 1-connected cgda's, a *Sullivan* or *relative model* of $\phi$ is a factoring $\phi = \psi\, i$ in

$$(A, d_A) \xrightarrow{i} (A \otimes \Lambda X, d) \underset{\simeq}{\xrightarrow{\psi}} (B, d_B)$$

where $i(a) = a \otimes 1$ for $a \in A$, $\psi$ is a quasi-isomorphism and

$$(\Lambda X, \bar{d}) := (A \otimes \Lambda X, d)/(A^+ \otimes \Lambda X, d)$$

is a minimal Sullivan model.

For every Serre fibration $\xi : F \xrightarrow{i} E \xrightarrow{p} B$ of simply connected spaces, there is a commutative diagram of augmented cgda's

$$\begin{tikzcd}
\mathcal{A}_{PL}(B) \arrow[r, "\mathcal{A}_{PL}(p)"] & \mathcal{A}_{PL}(E) \arrow[r, "\mathcal{A}_{PL}(i)"] & \mathcal{A}_{PL}(F) \\
(\Lambda X, d) \arrow[u, "\simeq", "\phi_B"'] \arrow[r] & (\Lambda X \otimes \Lambda Y, d) \arrow[u, "\simeq"] \arrow[r] & (\Lambda Y, \tilde{d}) \arrow[u, "\simeq"]
\end{tikzcd}$$

in which $(\Lambda X, d)$ and $(\Lambda Y, \bar{d})$ are Sullivan models for $B$ and $F$ respectively, and the bottom row is the Sullivan model of $\mathcal{A}_{PL}(p) \circ \phi_B$. The bottom row of this diagram is called a minimal K-S extension and a minimal model of the fibration. In general, the middle cgda need not be a minimal model of $E$, but will be precisely when the kernel of the homomorphism $\pi_k(F) \to \pi_k(E)$ is strictly torsion.

Now let $S$ be a space and $(\Lambda X, d)$ a minimal model of $S$. The projection $\Lambda X \to \Lambda X/\Lambda^{>m} X$ induces a differential $D$ in $\Lambda X/\Lambda^{>m} X$ which makes it a map of differential algebras. Let

$$(\Lambda X, d) \to (\Lambda X \otimes \Lambda V, d) \xrightarrow{\simeq} (\Lambda X/\Lambda^{>m} X, D)$$

be a relative model for this projection. The *rational category* of $(\Lambda X, d)$, denoted $\mathrm{cat}_0 S$, is the least $m$ such that there is a map $r : (\Lambda X \otimes \Lambda V, d) \to (\Lambda X, d)$ of cgda's which satisfies $r(x) = x$ for all $x \in X$. Félix and Halperin [FH] proved that $\mathrm{cat}_0 S = \mathrm{cat} S_{\mathbf{Q}}$.

Toomer's invariant, denoted $\mathrm{e}_0 S$, may also be defined as the least $m$ for which there is a map $r : (\Lambda X \otimes \Lambda V, d) \to (\Lambda X, d)$ of graded differential *vector spaces* which satisfies $r(x) = x$ for all $x \in X$. Clearly, $\mathrm{e}_0 S \leq \mathrm{cat}_0 S$ and it is straightforward that

$$\mathrm{e}_0 S = \sup\{k \mid \exists \alpha \in \Lambda^{\geq k} X \text{ with } 0 \neq [\alpha] \in H^*(S; \mathbf{Q})\}.$$

Moreover, if $S$ is a Poincaré duality space, the top class is the 'longest' class, that is, we may assume there exists $\alpha \in \Lambda^{\geq \mathrm{e}_0 S} X$ such that $[\alpha] \neq 0$ and $H^{>|\alpha|}(S; \mathbf{Q}) = 0$.

We are now in a position to interpret the $k$-triviality of the fibration on the level of minimal models. Let $E$ and $F$ be simply connected rational spaces and $\xi : F \to E \to K(Q, 2n)$ a rational fibration. Suppose $k \geq 0$, that $g_k : G_k E \to E$ is the $k^{th}$ Ganea fibration of $E$ and let $F \bowtie_E G_k E$ be the join of $g_k$ and the inclusion $i : F \to E$. That is, if $P$ denotes the homotopy pullback of $g_k$ and $i$, then $F \bowtie_E G_k E$ is the homotopy pushout of the projection $P \to F$ and $P \to G_k E$. Let $i \bowtie g_k : F \bowtie_E G_k E \to E$ be the induced map as in the diagram

$$\begin{tikzcd}
P \arrow[rr, dotted] \arrow[dd, dotted] \arrow[dr, phantom, "\text{1. pull}"] & & F \arrow[dd] \arrow[dl, dotted, "j"'] \\
& F \bowtie_E G_k E \arrow[dr, dotted] \arrow[ur, phantom, "\text{2. push}"] & \\
G_k E \arrow[ur, dotted] \arrow[rr, "g_k"] & & E
\end{tikzcd}$$

where we have denoted by $j$ the inclusion $F \to F \bowtie_E G_k E$.

**Proposition 2.** *The fibration $\xi$ is* $k$-trivial *iff there exists a minimal model*

$$\Lambda(a; 0) \to \Lambda(a, X; d) \to \Lambda(X; \bar{d})$$

*of $\xi$ with*

$$d : X \to \Lambda X \oplus \Lambda^+ a . \Lambda^{>k}(a, X).$$

We remark [C2] that the vanishing of the $k^{th}$-*higher holonomy* is equivalent, on this level, to the existence of a model of $\xi$ as above with

$$d : X \to \Lambda X \oplus \Lambda^{>k+1} a . \Lambda(a, X),$$

and an example exhibiting that this is stronger than the above is presented in the last section.

The proof of Proposition 2 will follow directly from Lemmas 3–6. However, before proceeding with them, we indicate why 1-triviality of the fibration is equivalent to the hypotheses of [GJ, Theorem 1].

If $(\Lambda X, d)$ is the minimal model of $S$, then as graded vector spaces, $L_S^n := \mathrm{Hom}(X^n, \mathbf{Q}) \cong \pi_n(S) \otimes \mathbf{Q}$. Then, $L_S^n \cong \pi_n(\Omega S) \otimes \mathbf{Q}$, and the homotopy Lie algebra of $S_{\mathbf{Q}}$ is encoded in the minimal model $(\Lambda X, d)$ as follows.

The differential $d$ can be written as a sum of derivations $d = d_2 + d_3 + \dots$ where $d_i : X \to \Lambda^i X$. The fact that $d^2 = 0$ implies the same for $d_2$. The dual of $d_2 : X \to \Lambda^2 X$ induces a bilinear and antisymmetric $[\,,] : L_S \otimes L_S \to L_S$, which represents the Samelson product in $\pi(\Omega S) \otimes \mathbf{Q}$. The Jacobi identity for $[\,,]$ is equivalent to $d_2^2 = 0$. One calls $(L_S, [\,,])$ the *rational homotopy Lie algebra of $S$.*

In particular, if $\{x_\alpha\}$ is a K-S basis for $X$, and $\{\hat{x}_\alpha\}$ is its dual basis, then

1. $[\hat{x}_\alpha, \hat{x}_\beta] = 0$ iff the coefficient of $x_\alpha x_\beta$ in $d_2 x_\gamma$ is zero for all $\gamma$, so that

2. $\hat{x}_\alpha$ is in the centre of $L_S$ iff

$$d_2 : X \to \Lambda^2\langle x_\beta \mid \beta \neq \alpha\rangle.$$

When no confusion will arise, we shall say that $x_\alpha$ belongs to the centre of $L_S$ if $\hat{x}_\alpha$ does.

Thus, a rational fibration $F \hookrightarrow E \to K(\mathbf{Q}, 2n)$ is $k$-trivial iff the fundamental homotopy class of the base, when viewed in $L_E$, is in the center of $L_E$.

We can now proceed with the proof of Proposition 2.

**Lemma 3.** *Let*

$$\begin{tikzcd}
 & (A,d) \arrow[d, "p"] \\
(\Lambda V, \bar{d}) \arrow[ur, "f"'] \arrow[r, "g"'] & (B,d)
\end{tikzcd}$$

*be a commutative diagram with $p$ surjective, and suppose that $g'$ is an another map homotopic to $g$. Then there exists a map $f' : (\Lambda V, \bar{d}) \to (A,d)$, homotopic to $f$ satisfying $p \circ f' = g'$.*

The proof of this lemma is a standard lifting argument.

**Lemma 4.** *Let $\Lambda(a;0) \to \Lambda(a,X;d) \xrightarrow{p} \Lambda(X;\bar{d})$ be any minimal model of $\xi$ and suppose that $W_k$ is a geometric realization of $(\Lambda(a,X)/\Lambda^{>k}(a,X), D)$. Then,*

$$(\Lambda(a,X)/a \cdot \Lambda^{>k}(a,X), D)$$

*is a model of $F \bowtie_E W_k$.*

*Proof.* Consider a relative model of $(\Lambda(a,X),d) \xrightarrow{q} (\Lambda(a,X)/\Lambda^{>k}(a,X),D)$

$$\begin{tikzcd}
(\Lambda(a,X),d) \arrow[r, "q"] \arrow[dr, "j"'] & (\Lambda(a,X)/\Lambda^{>k}(a,X),D) \\
 & (\Lambda(a,X) \otimes \Lambda Y, \delta) \arrow[u, two heads, "\varphi", "\simeq"']
\end{tikzcd} \tag{1}$$

and let $(A,d)$ be the join of $\Lambda(X;\bar{d})$ and $(\Lambda(a,X) \otimes \Lambda Y, \delta)$ as in the diagram

$$\begin{tikzcd}
(\Lambda(a,X),d) \arrow[rr, "j"] \arrow[dr, dotted] \arrow[dd, "p"'] & & (\Lambda(a,X) \otimes \Lambda Y, \delta) \arrow[dd, dotted, "\text{1. push}"'] \\
 & (A,d) \arrow[ur, dotted] \arrow[dl, dotted, "\text{2. pull}"] & \\
\Lambda(X;\bar{d}) \arrow[rr, dotted] & & (B,d)
\end{tikzcd}$$

Thus, $(\Lambda X \otimes \Lambda Y, \bar{\delta}) := (\Lambda X, \bar{d}) \otimes_{(\Lambda(a,X),d)} (\Lambda(a,X) \otimes \Lambda Y, \delta) = (B,d)$ is a model of the pull back of $p$ and $j$, and

$$(A,d) = (\Lambda X \oplus_{\Lambda X \otimes \Lambda Y} \Lambda(a,X) \otimes \Lambda Y, \bar{d} + \delta),$$

which is isomorphic to the differential subspace $(\Lambda X \oplus (a\,\Lambda(a,X) \otimes \Lambda Y), \delta)$ of $(\Lambda(a,X) \otimes \Lambda Y, \delta)$. On the other hand,

$$(\Lambda(a,X)/a\,\Lambda^{>k}(a,X), D) = (\Lambda X \oplus a\,(\Lambda(a,X)/\Lambda^{>k}(a,X)), D)$$

is a differential subspace of $(\Lambda(a,X)/\Lambda^{>k}(a,X),D)$. Now, the surjective quasi-isomorphism $\varphi$ in diagram (1) above induces a surjective map of differential algebras

$$\begin{aligned}\psi := (\mathrm{id}_{\Lambda X} \oplus a\,\varphi)\ :\ &(\Lambda X \oplus (\,a\,\Lambda(a,X)\otimes \Lambda Y),\delta)\\ &\to (\Lambda X \oplus a\,(\Lambda(a,X)/\Lambda^{>k}(a,X)),D)\end{aligned}$$

whose kernel isomorphic to $a.\ker\varphi$, so that $H^*(\ker\psi) = a\,H^*(\ker\varphi) = 0$. Thus, $\psi$ is a quasi-isomorphism. Since the functor $\mathcal{A}_{PL}$ and the geometric realization functor interchange homotopy pushouts and homotopy pullbacks, $(\Lambda(a,X)/a\,\Lambda^{>k}(a,X),D)$ is indeed a model of $F\bowtie_E W_k$. □

To proceed, we retain the notation introduced in Lemma 4, and as usual, identify $x$ with $q(x)$ for all $x \in X$. Then, we can see that the condition on the differential $d$ is equivalent to $DX \subset \Lambda X$ in $(\Lambda(a,X)/\Lambda^{>k}(a,X),D)$. We next establish

**Lemma 5.** *Let* $\Lambda(a;0) \to \Lambda(a,V;d) \xrightarrow{p} \Lambda(V;\bar{d})$ *be any model of* $\xi$. *There exists a minimal model* $(\Lambda(a,X),d)$ *of* $E$ *with*

$$d : X \to \Lambda X \oplus \Lambda^+ a.\Lambda^{>k}(a,X)$$

*iff there exists a map* $\sigma$ *in the diagram*

$$\begin{tikzcd}\Lambda(V;\bar{d}) \arrow[r, "\sigma"] \arrow[rr, bend right, "\mathrm{id}_{\Lambda V}"'] & (\Lambda(a,V)/a\Lambda^{>k}(a,V),D) \arrow[r, "\tilde{p}"] & \Lambda(V;\bar{d})\end{tikzcd}$$

*making it commute up to homotopy. (Here,* $\tilde{p}$ *is the map induced by* $p$*.)*

*Proof.* Suppose the condition on $d$ is satisfied. By the remark preceding Lemma 5, if we define $\sigma(x) = q(x)$, $\sigma$ will be a map of differential algebras making the diagram above (with $V = X$) commute exactly.

Conversely, suppose we have such a $\sigma$. Because $\tilde{p}$ is surjective and $\Lambda(V;\bar{d})$ is minimal, we may suppose without loss of generality (by Lemma 3) that $\tilde{p}\sigma = \mathrm{id}_{\Lambda V}$. We will use $\sigma$ to make a change of basis as follows. Our assumptions imply that there is a subspace $X \subset \Lambda(a,V)$ such that

$$q : X \to \sigma(V)$$

and

$$\tilde{p}q : X \to V$$

are isomorphisms. This implies that $\Lambda(a,X) = \Lambda(a,V)$, and so

$$(\Lambda(a,V)/\Lambda^{>k}(a,V),D) = (\Lambda(a,X)/\Lambda^{>k}(a,X),D).$$

But then,

$$DX = D\sigma(V) = \rho\bar{d}(V) \subset \sigma(\Lambda V) = \Lambda\sigma(V) = \Lambda X,$$

so we are done. □

We now relate $F\bowtie_E W_k$ and $F\bowtie_E G_kE$. As before, let $g_k : G_kE \to E$ denote the $k^{th}$ Ganea fibration over $E$.

**Lemma 6.** *There exists a retraction $r$ in the diagram*

$$\begin{tikzcd} F \arrow[r] \arrow[rr, bend right, "\mathrm{id}_F"'] & F\bowtie_E G_kE \arrow[r, "r"] & F \end{tikzcd}$$

*making it commute up to homotopy iff there exists a retraction $s$ in the diagram*

$$\begin{tikzcd} F \arrow[r] \arrow[rr, bend right, "\mathrm{id}_F"'] & F\bowtie_E W_k \arrow[r, "s"] & F \end{tikzcd}$$

*rendering it homotopy commutative.*

*Proof.* Denote by $w_k : W_k \to E$ a geometric realization of the morphism $\Lambda(a,X) \to \Lambda(a,X)/\Lambda^{>k}(a,X)$ and by $j$ (resp. by $\bar{j}$) the inclusion of $F$ in $F\bowtie_E G_kE$ (resp. $F\bowtie_E W_k$). Recall [ST] that there exist maps $\beta : G_kE \to W_k$ and $\alpha : W_k \to G_kE$ such that $g_k \circ \alpha \simeq w_k$ and $w_k \circ \beta \simeq g_k$. These induce maps $\bar{\alpha} : F\bowtie_E G_kE \to F\bowtie_E W_k$ and $\bar{\beta} : F\bowtie_E W_k \to G_kE$ such that $\bar{j}\circ\bar{\alpha} \simeq j$ and $j \circ \bar{\alpha} \simeq \bar{j}$. Thus, $j$ has a homotopy retraction if and only if $\bar{j}$ does. ☐

*Proof of Proposition* 2. This is immediate from Lemmas 4, 5 and 6. ☐

## §3. PROOF OF THE THEOREM

With the results and notation of the previous section, the second point of Theorem 1 now follows directly from

**Proposition 7.** *Suppose the degree of $a$ is even, $\Lambda(a;0) \to \Lambda(a,X;d) \to \Lambda(X;\bar{d})$ is a minimal K-S extension and that $H^*\Lambda(X;\bar{d})$ is a Poincaré duality algebra. Then, if $k \geq 0$ and $d : X \to \Lambda X \oplus \Lambda^+ a.\Lambda^{>k}(a,X)$, we have*

$$\mathrm{cat}_0\Lambda(a,X;d) \geq \mathrm{cat}_0\Lambda(X;\bar{d}) + k.$$

*Proof.* Though this closely follows the proof of [GJ, Thm. 1], we present it here for the sake of completeness.

Since the fibre is a Poincaré duality space and the base is Gorenstein, by [FHT2, Thm. 4.3], the total space is also Gorenstein. Moreover, since we may assume that $\mathrm{cat}_0\Lambda(a,X;d) < \infty$, by Theorem 3.1 of the same article, we conclude that $H^*\Lambda(a,X;\bar{d})$ is a Poincaré duality algebra. Then, by Propositions 5.1 and 5.3, again of [FHT2], the formal dimension of the fibre is strictly greater than that of the total space.

Now note that we can write $d = \bar{d} + \sum_i a^i\eta_i$, where each $\eta_i$ is a derivation of $\Lambda X$ of odd degree. The assumption $d : X \to \Lambda X \oplus \Lambda^+ a.\Lambda^{>k}(a,X)$ then implies that $\eta_i : \Lambda^p X \to \Lambda^{\geq \max\{p-1,\, p+k+1-i\}}X$. In particular, if $\beta \in \Lambda^{\geq p}X$ satisfies $\bar{d}\beta = 0$, then $d\beta = a\gamma$, where $\gamma \in \Lambda^{\geq p+k}(a,X)$.

Denote $e_0\Lambda(X;\bar{d}) = e$, and let $\beta \in \Lambda^{\geq e}X$ be a cycle of degree equal to the formal dimension, representing the longest non-zero class. Since $\bar{d}\beta = 0$, $d\beta = a\alpha$, where $d\alpha = 0$ and $\alpha \in \Lambda^{\geq e+k}(a,X)$. Suppose that $\alpha$ is exact in $\Lambda(a,X;d)$, say $\alpha = d\gamma$. Then, $\beta - a\gamma$ is a $d$-cycle of degree greater than the formal dimension of $\Lambda(a,X;d)$, so $\beta - a\gamma = dz$ for some $z = z_0 + az_1$, where $z_0 \in \Lambda X$. Comparison of coefficients of powers of $a$ on both sides of $\beta - a\gamma = dz$ yields $\bar{d}z_0 = \beta$, which is a contradiction. Thus, the class of $\alpha$ is nonzero in $H^*\Lambda(a,X;d)$, and so $e_0\Lambda(a,X;d) \geq e + k$. ☐

It is hoped that one can eventually remove the assumption of Poincaré duality in the above. For the moment however, we content ourselves by remarking that Theorem 1 is *not* valid for a fibration over an *odd* Eilenberg-Mac Lane space (i.e. an odd rational sphere)

$$F \to E \to \mathbf{S}^{2n+1}_{\mathbf{Q}}.$$

Consider the fibration $\Lambda(u;0) \to \Lambda(u,v,w,x,y;d) \to \Lambda(v,w,x,y;0)$ where all generators are of odd degree and the only non-zero differential is $dy = uvwx$. This satisfies all the hypotheses of Theorem 2 with $k = 2$ except for the parity, but

$$\mathrm{cat}_0\Lambda(u,v,w,x,y;d) = 5 < 6 = \mathrm{cat}_0\Lambda(v,w,x,y;0) + 2.$$

We now proceed with the proof of the first part of Theorem 1. Let $(M,A)$ be an NDR pair, denote by $i$ the inclusion $A \hookrightarrow M$, and as before let $g_k : G_kM \to M$ denote the $k^{th}$ Ganea fibration over $M$. Recall that for a map $f : S \to M$, $cat f \leq n$ iff $f$ factors through $g_n$ iff there is an open cover $(S_l)_{1\leq l \leq n+1}$ of $S$ such that for each $l$, the composition $S_l \hookrightarrow S \xrightarrow{f} M$ is homotopic to the constant map.

The first part of Theorem 1 now follows immediately from

**Proposition 8.** *Suppose $F \to E \to K(\mathbf{Q}, 2n)$ is a fibration with model $\Lambda(a;0) \to \Lambda(a,X;d) \to \Lambda(X;\bar{d})$. If $k \geq 0$ and $d : X \to \Lambda X \oplus \Lambda^+ a.\Lambda^{>k}(a,X)$, then*

$$\mathrm{cat}_0 E \leq \mathrm{cat}_0(E,F) + \max(\mathrm{cat}_0 F, k+1) - k - 1.$$

*Proof.* Suppose that $\mathrm{cat}(E,F) = n+1$. By [M], $E$ is dominated by $F \bowtie_E G_nE$ and so $E$ is dominated by $F \bowtie_E G_nE \simeq F \bowtie_E G_kE \bowtie_E G_{n-k-1}E$ [C2]. Using the map $\beta$ used in the proof of Lemma 6, we also see that $E$ is dominated by $F \bowtie_E W_k \bowtie_E G_{n-k-1}E$. Hence,

$$\begin{aligned}\mathrm{cat}\, E &\leq \mathrm{cat}(F \bowtie_E W_k \bowtie_E G_{n-k-1}E)\\ &\leq \mathrm{cat}(F \bowtie_E W_k) + \mathrm{cat} G_{n-k-1}E + 1\\ &\leq \mathrm{cat}(F \bowtie_E W_k) + n - k.\end{aligned}$$

The above inequalities also apply to the $\mathbf{Q}$-localizations of all the spaces, so we may replace cat by $\mathrm{cat}_0$. Now suppose that $\mathrm{cat}_0 F = m$, so that $\Lambda(X;\bar{d})$ is a homotopy retract of $\Lambda X/\Lambda^{>m}X$. The condition on the differential implies that $A := (\Lambda X \oplus a \cdot (\Lambda(a,X)/\Lambda^{>k}(a,X)), D)$ is a homotopy retract of

$$B := (\Lambda X/\Lambda^{>m}X \oplus a \cdot (\Lambda(a,X)/\Lambda^{>k}(a,X)), D).$$

Since the nilpotency of $B$ is at most $\max(m, k+1)$, its rational category also has the same upper bound [FH]. Moreover, by Lemma 4, $A$ is a model of $F \bowtie_E W_k$, so we obtain $\mathrm{cat}_0(F \bowtie_E W_k) \leq \max(\mathrm{cat}_0 F, k+1)$, which completes the proof. □

Note that this proof does not use the relativity of the homotopy in the definition of $\mathrm{cat}(E,F)$.

## §4. Examples

**Example 1.** Consider the rational fibration $(\mathbf{CP}^2)^3 \times (\mathbf{S}^7)^2 \to E \to \mathbf{CP}^\infty$ with model

$$(\Lambda a, 0) \to (\Lambda\,(a,b,c,e,u,v,w,x,y), d) \to (\Lambda\,(b,c,e,u,v,w,x,y), \bar{d}), \tag{3}$$

where $a, b, c$ and $e$ are cycles of degree 2, $du = a^4$, $dv = b^3$, $dw = c^3$, $dx = e^3$, and $dy = abce$. The fibration is 2-trivial (while even the 0-holonomy does not vanish), so the second point of Theorem 1 implies that $\mathrm{cat}_0 E \geq 10$. The best lower bound that the Mapping Theorem or [C1] can provide is 8, while [J] and [GJ] can both be massaged to yield 9. In fact, if we apply [FH, lemma 6.6] to the fibration $\mathbf{S}^7 \to E \to \mathbf{CP}^3 \times (\mathbf{CP}^2)^3$, we obtain $\mathrm{cat}_0 E \leq 10$, so indeed $\mathrm{cat}_0 E = 10$ and the lower bound of Theorem 1 is sharp here.

Theorem 1 applied to (3) tells us that $\mathrm{cat}_0(E, (\mathbf{CP}^2)^3 \times (\mathbf{S}^7)^2) \geq 5$.

**Example 2.** Consider the rational fibration $\mathbf{S}^7 \times \mathbf{CP}^2 \to E \to \mathbf{CP}^\infty$ with model $(\Lambda a, 0) \to (\Lambda(a, b, x, y), d) \to (\Lambda(b, x, y), \bar{d})$ where $a$ and $b$ are cycles of degree 2 and $dx = b^3$ and $dy = a^4 + b^2 a^2$. Again, the fibration is 2-trivial and so Theorem 1 gives $\mathrm{cat}_0 E \geq 5$. In fact, $\mathrm{cat}_0 E = 5$, because a simple length-degree argument shows that $= \mathrm{e}_0 E = 5$. Theorem 1 implies also that $\mathrm{cat}_0(E, \mathbf{S}^7 \times \mathbf{CP}^2) \geq 5$.

**Example 3 .** Consider the rational fibration $\mathbf{S}^2 \times \mathbf{S}^7 \times \mathbf{S}^7 \to E \to \mathbf{CP}^\infty$ with Sullivan model given by $(\Lambda\, b, 0) \to (\Lambda\,(a, b, x, y), d) \to (\Lambda\,(a, x, y), \bar{d})$ where $|a| = |b| = 2$, $|x| = 3$, $|y| = |z| = 7$, $da = db = 0$, $dx = a^2$, $dy = b^4$ and $dz = a^3 b$. Theorem 1 implies that $\mathrm{cat}_0 E \geq 5$ and that $\mathrm{cat}_0(E, S^2 \times S^2 \times S^5) \geq 2$. If we apply [C2] to the fibration $S^5 \to E \to CP(2) \times S^2$ with model $(\Lambda\, b, a, x, y, d) \to (\Lambda\,(a, b, x, y), d) \to (\Lambda\, z, 0)$, we get $\mathrm{cat}_0 E \leq 5$.

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