PERIODIC SOLUTIONS FOR PLANAR 2N-BODY PROBLEMS

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Abstract. In this paper we study some necessary conditions and sufficient conditions for the nested polygonal solutions of planar 2N-body problems.

1. Main results

This paper uses the same notations as those used in [6]. For \( n \geq 2 \) the equations of motion of the planar \( n \)-body problem ([1], [2], [3], [5], [6], [7]) can be written in the form

\[
\ddot{z}_k = -\sum_{j=1}^{n} m_j \frac{z_k - z_j}{|z_k - z_j|^3},
\]

where \( z_k \) is the complex coordinate of the \( k \)th mass \( m_k \) in an inertial coordinate system. Let \( \rho_k \) denote the \( N \) complex \( k \)th roots of unity, i.e.,

\[
\rho_k = \exp(2\pi ik/N).
\]

This equation will also serve to define \( \rho_k \) for any number \( k \). We assume that the mass \( m_k \) \((k = 1, \cdots, N)\) is located at the vertex \( \rho_k \) of a regular polygon inscribed on the unit circle, and \( \tilde{m}_k \) \((k = 1, \cdots, N)\) is located at

\[
\tilde{\rho}_k = a \rho_k \cdot e^{i\theta},
\]

where \( a > 0, 0 \leq \theta \leq 2\pi \), and \( a \neq 1 \) when \( \theta = 0 \) or \( 2\pi \).

Then the center of masses \( m_1, \cdots, m_N; \tilde{m}_1, \cdots, \tilde{m}_N \) is

\[
z_0 = \sum_j (m_j \rho_j + \tilde{m}_j \tilde{\rho}_j)/M
\]

where \( M = \sum_j (m_j + \tilde{m}_j) \). In (1.4) and throughout this paper, unless specially restricted, all indices and summations will range from 1 to \( N \). The functions describing their rotation about \( z_0 \) with angular velocity \( \omega \) are then given by

\[
z_k(t) = (\rho_k - z_0) \exp(i\omega t), \quad k = 1, \cdots, N,
\]

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and
\[ z_k(t) = (aρ_k e^{iθ} - z_0) \exp(iωt), \quad k = 1, \ldots, N. \]

Then the equations of motion of the planar 2N-body problem can be written in the following form:
\[ \ddot{z}_k = \sum_{j \neq k} m_j \frac{z_j - z_k}{|z_j - z_k|^3} + \sum_j \ddot{m}_j \frac{\ddot{z}_j - \ddot{z}_k}{|\ddot{z}_j - \ddot{z}_k|^3}. \]

R. Moeckel and C. Simó ([5]) proved the following result:

**Theorem (Moeckel-Simó).** If \( θ = 0 \) and \( m_1 = \cdots = m_N, \ m_1 = \cdots = \tilde{m}_N \), then for every mass ratio \( b = \frac{\tilde{m}_1}{m_1} \), there are exactly two planar central configurations consisting of two nested regular \( N \)-gons. For one of these, the ratio of the sizes of the two polygons is less than 1, and for the other it is greater than 1.

In this paper, we study the inverse problem of the theorem (Moeckel-Simó) and the following results are established.

**Theorem 1.** If, for \( N ≥ 2 \), the functions \( z_k(t) \) and \( \ddot{z}_k(t) \) given by (1.5) and (1.6) are solutions of the 2N-body problem (1.7) and (1.8), it follows that \( ω^2 \) satisfies
\[ \left[ \frac{1}{4} \sum_{j \neq N} \csc \left( \frac{πj}{N} \right) - \frac{ω^2}{M} N \right] \cdot \left[ \frac{1}{4} a^{-2} \sum_{j \neq N} \csc \left( \frac{πj}{N} \right) - \frac{ω^2}{M} a \cdot N \right] e^{iθ} \]
\[ = \left[ \sum_{j} \frac{1 - aρ_j e^{iθ}}{|1 - aρ_j e^{iθ}|^3} - Nω^2 \frac{N}{M} \right] \cdot \left[ \sum_{j} \frac{ae^{iθ} - ρ_j}{|ae^{iθ} - ρ_j|^3} - ω^2 \frac{Nae^{iθ}}{M} \right]. \]

**Theorem 2.** For \( N ≥ 2 \), \( m_k, \tilde{m}_k > 0 \), if the functions \( z_k(t) \) and \( \ddot{z}_k(t) \) given by (1.5) and (1.6) with \( ω^2 \) given by (1.9) are solutions of the 2N-body problem (1.7) and (1.8), then \( m_1 = m_2 = \cdots = m_N \) and \( \tilde{m}_1 = \tilde{m}_2 = \cdots = \tilde{m}_N \).

Conversely, if \( m_1 = m_2 = \cdots = m_N \) and \( \tilde{m}_1 = \tilde{m}_2 = \cdots = \tilde{m}_N \), let \( b = \frac{\tilde{m}_1}{m_1} \). If \( b, a \) and \( ω^2 \) satisfy
\[ \frac{Nω^2}{M} = \frac{1}{1 + b} \left[ \sum_{j=1}^{N-1} \frac{1 - ρ_j}{|1 - ρ_j|^3} + \sum_{j=0}^{N-1} \frac{b(1 - aρ_j e^{iθ})}{|1 - aρ_j e^{iθ}|^3} \right], \]
\[ \frac{Nω^2}{M} = \frac{e^{-iθ}}{a(1 + b)} \left[ \sum_j \frac{ae^{iθ} - ρ_j}{|ae^{iθ} - ρ_j|^3} + \sum_{j \neq N} \frac{b(a - aρ_j e^{iθ})}{|a - aρ_j|^3} \right], \]
then \( z_k(t) \) and \( \ddot{z}_k(t) \) are solutions of (1.7) and (1.8).

**Remark.** It is not difficult to prove that “\( ω^2 \)” and “\( a \)” are positive real numbers if \( θ = 0 \) or \( \frac{π}{2} \), and it seems that only \( θ = 0 \) or \( \frac{π}{2} \), “\( ω^2 \)” and “\( a \)” are positive real numbers, but the proof seems difficult.
Corollary 1 (MacMillan-Bartky [3]). Under the above assumptions,

(i) if \( N = 2, \theta = 0, a > 1 \) and \( z_k(t) \) and \( \tilde{z}_k(t) \) given by (1.5) and (1.6) with \( \omega^2 \) satisfying

\[
\left[ \frac{1}{4} - \frac{2\omega^2}{M} \right] \cdot \left[ \frac{1}{4} a^2 - \omega^2 M \cdot 2a \right]
\]

are solutions of the \( 2 \times 2 \)-body problems (1.7) and (1.8), then \( m_1 = m_2, \tilde{m}_1 = \tilde{m}_2 \).

Conversely, if \( \omega^2 = (m_1 + m_2 + \tilde{m}_1 + \tilde{m}_2) \gamma / N \) satisfying (1.12) and

\[
\frac{2\omega^2}{M} = 1 + b \left[ \frac{1}{4} - \frac{4ab}{(a^2 - 1)^2} \right]
\]

where

\[
b = \frac{a^7 - 2a^5 - 8a^4 + a^3 - 8a^2}{17a^4 - 2a^2 + 1},
\]

then \( z_k(t) \) and \( \tilde{z}_k(t) \) given by (1.5) and (1.6) are solutions of (1.7) and (1.8).

(ii) if \( N = 2, \theta = 0, 0 < a < 1 \) and \( z_k(t) \) and \( \tilde{z}_k(t) \) given by (1.5) and (1.6) with \( \omega^2 \) satisfying

\[
\left[ \frac{1}{4} - \frac{2\omega^2}{M} \right] \cdot \left[ \frac{1}{4} a^2 - \omega^2 M \cdot 2a \right]
\]

are solutions of (1.7) and (1.8), then \( m_1 = m_2, \tilde{m}_1 = \tilde{m}_2 \). Conversely, if \( \omega^2 = M \gamma / N \) satisfying (1.15) and

\[
\gamma = \frac{1}{1 + b} \left[ \frac{1}{4} + \frac{2b(a^2 + 1)}{(a^2 - 1)^2} \right]
\]

where

\[
b = \frac{a^7 - 2a^5 + 17a^3}{-8a^3 + a^4 - 8a^3 - 2a^2 + 1},
\]

then \( z_k(t) \) and \( \tilde{z}_k(t) \) given by (1.5) and (1.6) are solutions of (1.7) and (1.8).

Corollary 2 (MacMillan-Bartky [3]). (i) For \( N = 2 \) and \( \theta = \frac{\pi}{2} \), then (1.10) and (1.11) imply

\[
b = \left[ (1 + a^2)^{-3/2} - 2^{-3} \right] \cdot \left[ (1 + a^2)^{-3/2} - (2a)^{-3} \right]^{-1}.
\]

(ii) For \( N = 2 \) and \( \theta = \frac{\pi}{2} \), \( b = 1 \) if and only if \( a = 1 \). That is, \( z_k, \tilde{z}_k \) are solutions of (1.7) and (1.8). If \( m_1 = m_2, \tilde{m}_1 = \tilde{m}_2 \), then \( m_1 = m_2 = \tilde{m}_1 = \tilde{m}_2 \) if and only if \( m_1, m_2, \tilde{m}_1, \tilde{m}_2 \) are at the vertices of a square.

(iii) For \( N = 2 \) and \( \theta = \frac{\pi}{2} \), \( m_1 = m_2, \tilde{m}_1 = \tilde{m}_2 \) are at the vertices of a diamond.

Then \( m_1, m_2, \tilde{m}_1, \tilde{m}_2 \) form a central configuration if and only if \( \sqrt{3} < a < \sqrt{5} \), where \( a \) stands for the ratio of its diagonals and the ratio \( b \) of masses is determined uniquely by (1.18): especially considering the degenerate central configuration, when \( b = \infty, a = \sqrt{3} \), and when \( a = \sqrt{3}, b = 0 \).
Corollary 3 (Perko-Walter [6]). For \( N \geq 2, m_k, \tilde{m}_k > 0 \), if \( a = 1 \) and \( \theta = \frac{\pi}{2} \), then \( z_k(t) \) and \( \tilde{z}_k(t) \) given by (1.5) and (1.6) with \( \omega^2 = M\gamma/N \) and \( \omega^2 \) given by (1.9) are solutions of the 2N-body problem (1.7) and (1.8). Then \( m_1 = m_2 = \cdots = m_N = \tilde{m}_1 = \cdots = \tilde{m}_N \).

Corollary 4. If \( N = 3, \theta = 0 \). If \( z_k(t) \) and \( \tilde{z}_k(t) \) given by (1.5) and (1.6) with \( \omega^2 \) satisfying

\[
\omega^2 = \left[ \sqrt{3}M(a + a^{-2}) - \frac{3}{M} \sum_{j=1}^{3} \frac{a - \rho_j^{-1}}{|a - \rho_j^{-1}|^3} + a \sum_{j=1}^{3} \frac{1 - a\rho_j^{-1}}{|1 - a\rho_j^{-1}|^3} \right]^{-1}
\]

(1.19)

are the solution of the nested 2 \times 3-body problems (1.7) and (1.8), then \( m_1 = m_2 = m_3, \tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3 \).

Conversely, if \( m_1 = m_2 = m_3, \tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3 \), let \( b = \frac{\tilde{m}_1}{m_1} \). If \( \omega^2 \) satisfies (1.19) and \( a \) and \( b \) have the relation

\[
b = \left( \frac{\sqrt{3}}{3}a^{-3} - \frac{3}{a} \sum_{j=1}^{3} \frac{1 - a\rho_j^{-1}}{|a - \rho_j^{-1}|^3} \right)^{-1} \left( \sqrt{3} - a^{-1} \sum_{j=1}^{3} \frac{a - \rho_j^{-1}}{|a - \rho_j^{-1}|^3} \right),
\]

(1.20)

then \( z_k(t) \) and \( \tilde{z}_k(t) \) defined by (1.5) and (1.6) are solutions of (1.7) and (1.8).

Corollary 5. Assume \( N = 3, \theta = \frac{\pi}{2} \). If \( z_k(t) \) and \( \tilde{z}_k(t) \) given by (1.5) and (1.6) with \( \omega^2 \) satisfying

\[
\omega^2 = \left[ \frac{M}{3} \sqrt{3}(a + a^{-2}) - \sum_{j=1}^{3} \frac{a - e^{-\frac{\pi}{2}\sqrt{-1}}\rho_j^{-1}}{|a - e^{-\frac{\pi}{2}\sqrt{-1}}\rho_j^{-1}|^3} + a \sum_{j=1}^{3} \frac{1 - ae^{-\frac{\pi}{2}\sqrt{-1}}\rho_j^{-1}}{|1 - ae^{-\frac{\pi}{2}\sqrt{-1}}\rho_j^{-1}|^3} \right]^{-1}
\]

(1.21)

are the solution of the twisted 2 \times 3-body problems (1.7) and (1.8), then \( m_1 = m_2 = m_3, \tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3 \).

Conversely, if \( m_1 = m_2 = m_3, \tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3 \), let \( b = \frac{\tilde{m}_1}{m_1} \). If \( \omega^2 \) satisfies (1.21) and \( a \) and \( b \) have the following relation:

\[
b = \left( \frac{\sqrt{3}}{3}a^{-2} - \sum_{j=1}^{3} \frac{a(1 - ae^{-\frac{\pi}{2}\sqrt{-1}}\rho_j^{-1})}{|1 - ae^{-\frac{\pi}{2}\sqrt{-1}}\rho_j^{-1}|^3} \right)^{-1} \left( \sqrt{3}a - \sum_{j=1}^{3} \frac{a - e^{-\frac{\pi}{2}\sqrt{-1}}\rho_j^{-1}}{|a - e^{-\frac{\pi}{2}\sqrt{-1}}\rho_j^{-1}|^3} \right),
\]

(1.22)

then \( z_k(t) \) and \( \tilde{z}_k(t) \) defined by (1.5) and (1.6) are solutions of (1.7) and (1.8).

2. Eigenvalues and eigenvectors for circulant matrices

Definition 2.1 ([1]). If \( N \times N \) matrix \( A = (a_{i,j}) \) satisfies

\[
a_{i,j} = a_{i-j,N-1}, \quad 1 \leq i, j \leq N,
\]

(2.1)

where we assume \( a_{i,0} = a_{i,N} \) and \( a_{0,j} = a_{N,j} \). Then we call \( A \) a circulant matrix.
Lemma 2.1 ([4]). If $A$ and $B$ are $N \times N$ circulant matrices, then $A + B$, $A - B$, $A \cdot B$ are also circulant matrices and $AB = BA$.

Lemma 2.2 ([4]). Let $A = (a_{i,j})$ be an $N \times N$ circulant matrix. The eigenvalues $\lambda_k$ and the eigenvectors $\vec{v}_k$ of $A$ are
\[
\lambda_k(A) = \sum_j a_{1,j} \rho_{k-1}^j
\]
and
\[
\vec{v}_k = (\rho_{k-1}, \rho_{k-1}^2, \ldots, \rho_{k-1}^N)^T.
\]

Lemma 2.3 ([4]). Let $A, B$ be circulant matrices, where $\lambda_k(A)$, $\lambda_k(B)$ are eigenvalues of $A, B$. Then the eigenvalues of $A + B$, $A - B$, $A \cdot B$ are $\lambda_k(A) + \lambda_k(B)$, $\lambda_k(A) - \lambda_k(B)$, $\lambda_k(A) \cdot \lambda_k(B)$.

The following useful lemma can be simply proved using the properties of circulant matrices.

Lemma 2.4. If $A = (a_{i,j})$ is an $N \times N$ circulant matrix, and $A \cdot X = 0$, where $X = (x_1, \ldots, x_n)^T$, $\sum_i x_i \neq 0$, then
\[
a_{1,j} + \cdots + a_{N,j} = 0, \quad 1 \leq j \leq N,
\]
\[
a_{i,1} + \cdots + a_{i,N} = 0, \quad 1 \leq i \leq N.
\]

3. Proof of the main results

For two nested regular polygons, we define
\[
\rho_k = \exp(2\pi ik/N),
\]
\[
\tilde{\rho}_k = a \exp(2\pi ik/N) e^{i\theta},
\]
\[
z_0 = \sum_j (m_j \rho_j + \tilde{m}_j \tilde{\rho}_j)/M,
\]
where
\[
M = \sum_j (m_j + \tilde{m}_j),
\]
\[
z_k(t) = (\rho_k - z_0) \exp(i\omega t), \quad k = 1, \ldots, N,
\]
and
\[
\tilde{z}_k(t) = (a \rho_k e^{i\theta} - z_0) \exp(i\omega t), \quad k = 1, \ldots, N.
\]

Proof of Theorem 1. (3.1)–(3.6) imply that the $z_k(t)$ and $\tilde{z}_k(t)$ are the solutions of (1.7) and (1.8) if and only if
\[
(\rho_k - z_0)\omega^2 \exp(i\omega t) = \left(\sum_{j \neq k} m_j \frac{\rho_k - \rho_j}{|\rho_k - \rho_j|^2} + \sum_j \tilde{m}_j \frac{\rho_k - \tilde{\rho}_j}{|\rho_k - \tilde{\rho}_j|^2}\right) \exp(i\omega t)
\]
\[(\hat{\rho}_k - z_0)\omega^2 \exp(i\omega t) = \left( \sum_j m_j \frac{\hat{\rho}_k - \rho_j}{|\rho_k - \rho_j|^3} + \sum_{j \neq k} \hat{m}_j \frac{\hat{\rho}_k - \hat{\rho}_j}{|\hat{\rho}_k - \hat{\rho}_j|^3} \right) \exp(i\omega t) \]

or if and only if

\[
\sum_{j \neq k} m_j \left( \frac{1}{|\rho_k - \rho_j|^3} - \frac{\omega^2}{M} \right) (\hat{\rho}_k - \rho_j) + \sum_j \hat{m}_j \left( \frac{1}{|\rho_k - \rho_j|^3} - \frac{\omega^2}{M} \right) (\hat{\rho}_k - \hat{\rho}_j) = 0
\]

and

\[
\sum_j m_j \left( \frac{1}{|\rho_k - \rho_j|^3} - \frac{\omega^2}{M} \right) (\hat{\rho}_k - \rho_j) + \sum_{j \neq k} \hat{m}_j \left( \frac{1}{|\rho_k - \rho_j|^3} - \frac{\omega^2}{M} \right) (\hat{\rho}_k - \hat{\rho}_j) = 0.
\]

Multiplying both sides by \(\rho_{N-k}\) and noting that \(|\rho_k - \rho_j| = |\rho_k||1-\rho_{j-k}| = |1-\rho_{j-k}|\) and using \(\hat{\rho}_k = a\rho_k e^{i\theta}\),

\[
\sum_{j \neq k} m_j \left( \frac{1}{|1-\rho_{j-k}|^3} - \frac{\omega^2}{M} \right) (1 - \rho_{j-k}) + \sum_j \hat{m}_j \left( \frac{1}{|1-a\rho_{j-k} e^{i\theta}|^3} - \frac{\omega^2}{M} \right) (1 - a\rho_{j-k} e^{i\theta}) = 0
\]

and

\[
\sum_j m_j \left( \frac{1}{|ae^{i\theta} - \rho_j|^3} - \frac{\omega^2}{M} \right) (ae^{i\theta} - \rho_j - k) + \sum_{j \neq k} \hat{m}_j \left( \frac{1}{|a - a\rho_{j-k} e^{i\theta}|^3} - \frac{\omega^2}{M} \right) (a - a\rho_{j-k} e^{i\theta}) = 0.
\]

Notice that every step from (3.7) to (3.12) can be reversed, respectively. Now we define the \(N \times N\) circulant matrices \(C = [c_{k,j}], A = [a_{k,j}], B = [b_{k,j}], D = [d_{k,j}]\) as follows:

\[
c_{k,j} = \begin{cases} 0, & \text{for } k = j, \\ \frac{1}{|1-\rho_{j-k}|^3} - \frac{\omega^2}{M} (1 - \rho_{j-k}), & \text{for } k \neq j, \end{cases}
\]

\[
a_{k,j} = \frac{1}{|1-a\rho_{j-k} e^{i\theta}|^3} - \frac{\omega^2}{M} (1 - a\rho_{j-k} e^{i\theta}),
\]

\[
b_{k,j} = \frac{1}{|ae^{i\theta} - \rho_j|^3} - \frac{\omega^2}{M} (ae^{i\theta} - \rho_j - k),
\]

\[
d_{k,j} = \begin{cases} 0, & \text{for } k = j, \\ \frac{1}{|a - a\rho_{j-k}|^3} - \frac{\omega^2}{M} (a - a\rho_{j-k}) e^{i\theta}, & \text{for } k \neq j. \end{cases}
\]
Then (3.11) and (3.12) hold if and only if the matrix equation

\[
\begin{pmatrix}
C & A \\
B & D
\end{pmatrix}
\begin{pmatrix}
m_1 \\
m_N \\
\tilde{m}_1 \\
\vdots \\
\tilde{m}_N
\end{pmatrix} = 0
\]

has a positive solution.

Let

\[
m = (m_1, \cdots, m_N)^T, \quad \tilde{m} = (\tilde{m}_1, \cdots, \tilde{m}_N)^T.
\]

Then (3.17) is equivalent to

\[
C \cdot m + A \cdot \tilde{m} = 0,
\]

\[
B \cdot m + D \cdot \tilde{m} = 0.
\]

By (3.19) and (3.20) we have

\[
(CD - BA) \cdot m = 0,
\]

\[
(AB - CD) \cdot \tilde{m} = 0.
\]

We notice that (3.21) and (3.22) have a solution is equivalent to that \(AB - CD\) has a positive real eigenvector corresponding to eigenvalue 0. By Lemma 2.3 we have

\[
\lambda_k(AB - CD) = \lambda_k(AB) - \lambda_k(CD) = \lambda_k(A)\lambda_k(B) - \lambda_k(C)\lambda_k(D).
\]

Hence

\[
\lambda_k(AB - CD) = 0
\]

for some \(1 \leq k \leq N\) if and only if

\[
\lambda_k(A)\lambda_k(B) = \lambda_k(C)\lambda_k(D).
\]

Also, we notice by Lemma 2.2 and Lemma 2.4 \(\lambda_1(AB - CD) = 0\). Hence by Lemma 2.2 we have

\[
\sum_{j \neq 1} \left( \frac{1}{|1 - \rho_{j-1}|^3} - \frac{\omega^2}{M} (1 - \rho_{j-1}) \right) \cdot \left( \sum_{j \neq 1} \left( \frac{1}{|a - a\rho_{j-1}|^3} - \frac{\omega^2}{M} (a - a\rho_{j-1})e^{i\theta} \right) + \sum_{j \neq 1} \left( \frac{1}{|a - a\rho_{j-1}|^3} - \frac{\omega^2}{M} (a - a\rho_{j-1})e^{-i\theta} \right) \right)
\]

\[
= \sum_{j} \left( \frac{1}{|1 - a\rho_{j-1}e^{i\theta}|^3} - \frac{\omega^2}{M} (1 - a\rho_{j-1}e^{i\theta}) \right) \cdot \sum_{j} \left( \frac{1}{|ae^{i\theta} - \rho_{j-1}|^3} - \frac{\omega^2}{M} (ae^{i\theta} - \rho_{j-1}) \right).
\]
We notice that

\[
\begin{align*}
\sum_{j \neq 1} & \left( \frac{1 - \rho_{j-1}}{1 - \rho_{j-1}^3} - \frac{\omega^2}{M} \sum_j (1 - \rho_{j-1}) \right) \\
\cdot & \left[ a^{-2} \sum_{j \neq 1} \frac{1 - \rho_{j-1}}{|1 - \rho_{j-1}|^3} - \frac{\omega^2}{M} a \cdot \sum_j (1 - \rho_{j-1}) \right] \cdot e^{i\theta} \\
= & \left[ \frac{1}{4} \sum_{j \neq N} \csc \left( \frac{\pi j}{N} \right) - \frac{\omega^2}{M} \cdot N \right] \cdot \left[ \frac{1}{4} a^{-2} \sum_{j \neq N} \csc \left( \frac{\pi j}{N} \right) - \frac{\omega^2}{M} \cdot a \cdot N \right] \cdot e^{i\theta},
\end{align*}
\]

\[
\text{(3.27)}
\]

\[
\begin{align*}
\sum_j & \left[ \frac{1 - a e^{i\theta} \rho_{j-1}}{|1 - a e^{i\theta} \rho_{j-1}|^3} - \frac{\omega^2}{M} \cdot (1 - a e^{i\theta} \rho_{j-1}) \right] \\
\cdot & \sum_j \left[ \frac{a e^{i\theta} - \rho_{j-1}}{a e^{i\theta} - \rho_{j-1}^3} - \frac{\omega^2}{M} (a e^{i\theta} - \rho_{j-1}) \right] \\
= & \left[ \sum_j \frac{1 - a e^{i\theta} \rho_{j-1}}{|1 - a e^{i\theta} \rho_{j-1}|^3} - \frac{\omega^2}{M} \cdot N \right] \cdot \left[ \sum_j \frac{a e^{i\theta} - \rho_{j-1}}{a e^{i\theta} - \rho_{j-1}^3} - \frac{\omega^2}{M} \cdot N e^{i\theta} \right].
\end{align*}
\]

\[
\text{(3.28)}
\]

**Proof of Theorem 2.** $\lambda_1(AB - CD) = 0$ is simple so $\vec{v}_1 = (1, 1, \cdots, 1)^T$ is the only positive real eigenvector for $\lambda_1 = 0$.

Assume $m_1 = m_2 = \cdots = m_N > 0$ and $\tilde{m}_1 = \tilde{m}_2 = \cdots = \tilde{m}_N > 0$, $\omega^2$ is determined by (1.9) and $a, b$ is determined by (1.10) and (1.11). Then $(m_1, \cdots, m_1)^T$ is a solution of (3.11) and (3.12) or (1.7) and (1.8), since

\[
\sum_j m_j \left( \frac{1}{|1 - \rho_{j-k}|^3} - \frac{\omega^2}{M} \right) (1 - \rho_{j-k}) \\
+ \sum_j m_j \left( \frac{1}{|1 - a \rho_{j-k} e^{i\theta}|^3} - \frac{\omega^2}{M} \right) (1 - a \rho_{j-k} e^{i\theta})
\]

\[
= m_1 \left[ \sum_{j \neq k} \left( \frac{1}{|1 - \rho_{j-k}|^3} - \frac{\omega^2}{M} \right) (1 - \rho_{j-k}) \\
+ \sum_j b \left( \frac{1}{|1 - a \rho_{j-k} e^{i\theta}|^3} - \frac{\omega^2}{M} \right) (1 - a \rho_{j-k} e^{i\theta}) \right]
\]

\[
\text{(3.29)}
\]

\[
\text{(3.30)}
\]
Proof of Corollary 2. (i) For \( N = 2 \) and \( \theta = \frac{\pi}{2} \), \( b \) and \( a \) have the following relationship:

\[
(3.32) \quad b = \frac{\left( \sum_{j=0}^{N-1} \frac{ae^{i\theta} - \rho_j}{|ae^{i\theta} - \rho_j|^3} - \sum_{j=1}^{N-1} \frac{ae^{i\theta}(1 - \rho_j)}{|1 - \rho_j|^3} \right)}{\left( \sum_{j=0}^{N-1} \frac{ae^{i\theta}(1 - a \rho_j e^{i\theta})}{|1 - a \rho_j e^{i\theta}|^3} - \sum_{j=1}^{N-1} \frac{ae^{i\theta}(1 - a \rho_j)}{a^3|1 - a \rho_j|^3} \right)}.
\]

Then

\[
(3.33) \quad \sum_{j=0}^{N-1} \frac{ae^{i\theta} - \rho_j}{|ae^{i\theta} - \rho_j|^3} = \sum_{j=1}^{N-1} \frac{ae^{i\theta}(1 - \rho_j)}{|1 - \rho_j|^3} = \frac{ai - 1}{|ai - 1|^3} + \frac{ai + 1}{|ai + 1|^3} - \frac{ai(1 + 1)}{|2|^3}
\]

\[
= \frac{ai - 1}{(a^2 + 1)^{3/2}} + \frac{ai + 1}{(a^2 + 1)^{3/2}} - \frac{2ai}{8}
\]

\[
= \frac{2ai}{(a^2 + 1)^{3/2}} - \frac{2ai}{8} = 2ai \left( \frac{1}{(a^2 + 1)^{3/2}} - \frac{1}{8} \right)
\]

and

\[
(3.34) \quad \sum_{j=0}^{N-1} \frac{ae^{i\theta}(1 - a \rho_j e^{i\theta})}{|1 - a \rho_j e^{i\theta}|^3} = \sum_{j=1}^{N-1} \frac{ae^{i\theta}(1 - a \rho_j)}{a^3|1 - a \rho_j|^3}
\]

\[
= \frac{ai(1 - ai)}{|1 - ai|^3} + \frac{ai(1 - a(-1)i)}{|1 - a(-1)i|^3} - \frac{ai(1 + 1)}{|2a|^3}
\]

\[
= 2ai \left( \frac{1}{(a^2 + 1)^{3/2}} - \frac{1}{8a^2} \right)
\]

so

\[
(3.35) \quad b = \left[ (1 + a^2)^{-3/2} - 2^{-3} \right] \cdot \left[ (1 + a^2)^{-3/2} - (2a)^{-3} \right]^{-1}.
\]

(ii) That \( b = 1 \) if and only if \( a = 1 \) can be obtained directly from (1.18).

(iii) Since (1.11) holds and \( b \) must be greater than zero, we have

\[
(3.36) \quad \begin{cases} (1 + a^2)^{-3/2} - 2^{-3} > 0, \\ (1 + a^2)^{-3/2} - (2a)^{-3} > 0, \end{cases}
\]

or

\[
(3.37) \quad \begin{cases} (1 + a^2)^{-3/2} - 2^{-3} < 0, \\ (1 + a^2)^{-3/2} - (2a)^{-3} < 0. \end{cases}
\]

Then the result of (iii) is easily obtained.

Proof of Corollary 3. Under the assumptions of Corollary 3, we have \( b = 1 \).
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