

RESIDUES FOR AKIZUKI'S ONE-DIMENSIONAL LOCAL DOMAIN

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ABSTRACT. For a one-dimensional local domain C_M constructed by Akizuki, we find residue maps which give rise to a local duality. The completion of C_M is described using these residue maps.

Injective hulls of a given module are all isomorphic. For this reason, people often speak of *the* injective hull to indicate its “*uniqueness*”. However, isomorphisms between these injective hulls are not canonical. In fact, they are a part of the structure of the given module. For instance, a local duality for a power series ring [2, (5.9)] is interpreted as an isomorphism between two injective hulls—one given by local cohomology and another by continuous homomorphisms. This isomorphism is induced by a residue map, which was not observed from the viewpoint of “*uniqueness*” of injective hulls. In this article, our philosophy is taken up again by a Noetherian local ring C_M constructed by Akizuki [1]. Although C_M behaves beyond geometric expectation, we can still define certain maps, which give rise to a local duality as an identification of local cohomology classes and continuous homomorphisms. These maps, also called residue maps, determine all endomorphisms of an injective hull of the residue field of C_M . So we are able to describe the completion of C_M .

We recall Akizuki’s construction. Let A be a discrete valuation ring with the maximal ideal $\mathfrak{m} = tA$, let \hat{A} be its completion, and let K (resp. \hat{K}) be the quotient field of A (resp. \hat{A}). Assume that there is an element

$$z = a_0 + a_1 t^{n_1} + a_2 t^{n_2} + \cdots \in \hat{A} \quad (a_i \in A \setminus \mathfrak{m})$$

transcendental over A with the condition

$$n_r \geq 2n_{r-1} + 2 \quad (r \geq 1)$$

on exponents, where $n_0 = 0$. Let

$$z_r = a_r + a_{r+1} t^{n_{r+1} - n_r} + \cdots \quad (r \geq 0)$$

and

$$C = A[t(z_0 - a_0), \{(z_i - a_i)^2\}_{i=0}^\infty].$$

C_M is defined to be the localization of C at the maximal ideal M generated by t and $t(z_0 - a_0)$. Akizuki [1] showed that C_M is a one-dimensional Noetherian local domain, whose normalization is not a finite C_M -module.

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The quotient field of C_M is $K(z)$, which equals $(C_M)_t$ as t is a system of parameter of C_M . We can use the exact sequence

$$(1) \quad 0 \rightarrow C_M \xrightarrow{\text{localization}} (C_M)_t \rightarrow H^1_{MC_M}(C_M) \rightarrow 0$$

to describe elements of the first local cohomology module $H^1_{MC_M}(C_M)$ of C_M supported at the maximal ideal of C_M : For $f \in C_M$ and $n > 0$, the generalized fraction

$$\begin{bmatrix} f \\ t^n \end{bmatrix}_{C_M}$$

is defined as the image of f/t^n in $H^1_{MC_M}(C_M)$ under the map in (1). The description of top local cohomology modules by generalized fractions is essentially used in the concrete realizations of Grothendieck duality [2, 4, 5]. It can be applied to combinatorial analysis [3, 6]. See also [9] for an alternate treatment to generalized fractions.

For $f \in \hat{A}$ and $n > 0$, we denote by

$$(2) \quad \begin{bmatrix} f \\ t^n \end{bmatrix}_A$$

the image of f/t^n in \hat{K}/\hat{A} under the map in the exact sequence

$$0 \rightarrow \hat{A} \xrightarrow{\text{inclusion}} \hat{K} \rightarrow \hat{K}/\hat{A} \rightarrow 0.$$

We remark that there is a canonical isomorphism between \hat{K}/\hat{A} and $H^1_{m\hat{A}}(\hat{A})$, with which the representation (2) of elements in \hat{K}/\hat{A} and the representation of elements in $H^1_{m\hat{A}}(\hat{A})$ by generalized fractions agree.

Vanishing of elements in \hat{K}/\hat{A} and $H^1_{MC_M}(C_M)$ can be described in terms of ideal membership:

$$\begin{bmatrix} f \\ t^n \end{bmatrix}_A = 0 \quad \left(\text{resp. } \begin{bmatrix} f \\ t^n \end{bmatrix}_{C_M} = 0 \right)$$

if and only if f is contained in the ideal of \hat{A} (resp. C_M) generated by t^n . For instance,

$$\begin{bmatrix} t(z_0 - a_0) \\ t \end{bmatrix}_A = 0$$

but

$$\begin{bmatrix} t(z_0 - a_0) \\ t \end{bmatrix}_{C_M} \neq 0.$$

The canonical map $K/A \rightarrow \hat{K}/\hat{A}$ is an isomorphism. We use the notation in (2) to represent elements in K/A under this isomorphism.

Lemma 1. *Every element of $H^1_{MC_M}(C_M)$ can be written as*

$$\begin{bmatrix} X + Yt(z_0 - a_0) \\ t^n \end{bmatrix}_{C_M}$$

for some $X, Y \in A$ and $n > 0$.

Proof. It suffices to show that, for any specified $n > 0$, any element $f \in C_M$ can be written as

$$(3) \quad f = X + Yt(z_0 - a_0) + t^n Z$$

with $X, Y \in A$ and $Z \in C_M$. Write f as

$$f = \frac{f_1}{1 - f_2}$$

for some $f_1 \in C$ and $f_2 \in M$. Then

$$f - f_1(1 + f_2 + f_2^2 + \cdots + f_2^n) = \frac{f_1 f_2^{n+1}}{1 - f_2} \in M^{n+1}C_M.$$

Since $M^{n+1}C_M \subset t^n C_M$,

$$f - f_1(1 + f_2 + f_2^2 + \cdots + f_2^n) = t^n Z_2$$

for some $Z_2 \in C_M$. By [8, Section 9.5, equation (5)], there exist $X, Y \in A$ and $Z_1 \in C$ such that

$$f_1(1 + f_2 + f_2^2 + \cdots + f_2^n) = X + Yt(z_0 - a_0) + t^n Z_1.$$

The representation

$$f = X + Yt(z_0 - a_0) + t^n(Z_1 + Z_2)$$

is of the required form. \square

Lemma 2. *If $X, Y \in A$ and*

$$\left[\begin{array}{c} X + Yt(z_0 - a_0) \\ t^n \end{array} \right]_{C_M} = 0,$$

then $X, Y \in t^n A$.

Proof.

$$X + Yt(z_0 - a_0) = t^n Z$$

for some $Z \in C_M$. Write $X = t^\ell X_1$ and $Y = t^m Y_1$ with invertible $X_1, Y_1 \in A$. If $\ell \leq m$, then $n \leq \ell$, otherwise $X_1 = -t^{m-\ell} Y_1 t(z_0 - a_0) + t^{n-\ell} Z \in t\hat{A}$. If $\ell > m$, then $m \geq n$, otherwise $t(z_0 - a_0) = -t^{\ell-m} X_1 Y_1^{-1} + t^{n-m} Z Y_1^{-1} \in tC_M$. In either case, $X, Y \in t^n A$. \square

With these lemmas, we are able to define the following map for any $\sigma, \rho \in \hat{A}$.

Definition 3.

$$\text{res}_{\sigma, \rho}: H_{MC_M}^1(C_M) \rightarrow K/A$$

is defined to be the A -linear map given by

$$\left[\begin{array}{c} X + Yt(z_0 - a_0) \\ t^n \end{array} \right]_{C_M} \mapsto \left[\begin{array}{c} X\sigma + Y\rho \\ t^n \end{array} \right]_A.$$

Adopting the terminology of [2], we call $\text{res}_{\sigma, \rho}$ a residue map. Let

$$\text{Hom}_A^c(C_M, K/A) = \{\varphi \in \text{Hom}_A(C_M, K/A) \mid \varphi(M^n C_M) = 0 \text{ for some } n\}$$

be the C_M -module of continuous homomorphism. As a special case of J. Lipman's result [2, Proposition 3.4], $\text{Hom}_A^c(C_M, K/A)$ is an injective hull of the residue field of C_M . Note that $t^{n+1}C_M \subset M^{n+1}C_M \subset t^n C_M$. Hence an A -linear map $\varphi: C_M \rightarrow K/A$ is continuous if and only if $\varphi(t^n C_M) = 0$ for some n . Using the representation

(3) of elements of C_M , we see that a continuous homomorphism is determined by an integer n with which $t^n C_M$ is in the kernel and by its values at 1 and $t(z_0 - a_0)$.

Definition 4.

$$\Phi_{\sigma,\rho}: H^1_{MC_M}(C_M) \rightarrow \text{Hom}_A^c(C_M, K/A)$$

is defined to be the C_M -linear map given by

$$\Phi_{\sigma,\rho}(\omega)(f) = \text{res}_{\sigma,\rho}(f\omega),$$

where $\omega \in H^1_{MC_M}(C_M)$ and $f \in C_M$.

Local Duality. If ρ is invertible, then $\Phi_{\sigma,\rho}$ is an isomorphism.

Proof. The inverse map of $\Phi_{\sigma,\rho}$ can be written explicitly. Let

$$s_r := a_1 t^{n_1} + a_2 t^{n_2} + \dots + a_r t^{n_r} \in A.$$

Then we have

$$t(z_0 - a_0) = t^{n_r+1}(z_r - a_r) + t s_r$$

and

$$(4) \quad t^2(z_0 - a_0)^2 + t^2 s_r^2 - 2t s_r t(z_0 - a_0) = t^{2n_r+2}(z_r - a_r)^2 \in t^r C_M.$$

Given $\varphi \in \text{Hom}_A^c(C_M, K/A)$ with $\varphi(t^r C_M) = 0$ and

$$\begin{cases} \varphi(1) = \begin{bmatrix} \alpha \\ t^r \end{bmatrix}_A, \\ \varphi(t(z_0 - a_0)) = \begin{bmatrix} \beta \\ t^r \end{bmatrix}_A, \end{cases}$$

the system of equations

$$\begin{cases} \begin{bmatrix} X\sigma + Y\rho \\ t^r \end{bmatrix}_A = \begin{bmatrix} \alpha \\ t^r \end{bmatrix}_A, \\ \begin{bmatrix} X\rho - Yt^2 s_r^2 \sigma + 2Yt s_r \rho \\ t^r \end{bmatrix}_A = \begin{bmatrix} \beta \\ t^r \end{bmatrix}_A \end{cases}$$

can be solved by choosing $X, Y \in A$ such that

$$\begin{cases} X - \frac{\alpha t s_r (\sigma t s_r - 2\rho) + \beta \rho}{(\rho - t s_r \sigma)^2} \in t^r \hat{A}, \\ Y - \frac{\alpha \rho - \beta \sigma}{(\rho - t s_r \sigma)^2} \in t^r \hat{A}. \end{cases}$$

We define

$$\Phi^{-1}(\varphi) := \begin{bmatrix} X + Yt(z_0 - a_0) \\ t^r \end{bmatrix}_{C_M},$$

which is independent of the choices of r, α, β, X or Y . Then

$$\Phi_{\sigma,\rho}(\Phi^{-1}(\varphi))(1) = \begin{bmatrix} \alpha \\ t^r \end{bmatrix}_A$$

and

$$\begin{aligned} & \Phi_{\sigma,\rho}(\Phi^{-1}(\varphi))(t(z_0 - a_0)) \\ = & \operatorname{res}_{\sigma,\rho} \left[\begin{array}{c} Xt(z_0 - a_0) - Yt^2s_r^2 + 2Yts_rt(z_0 - a_0) \\ t^r \end{array} \right]_{C_M} \quad (\text{by (4)}) \\ = & \left[\begin{array}{c} \beta \\ t^r \end{array} \right]_A. \end{aligned}$$

Hence $\Phi_{\sigma,\rho}(\Phi^{-1}(\varphi)) = \varphi$. For any $\omega \in H^1_{MC_M}(C_M)$, it is also straightforward to check that $\Phi^{-1}(\Phi_{\sigma,\rho}(\omega)) = \omega$. So Φ^{-1} is indeed the inverse of $\Phi_{\sigma,\rho}$. \square

Corollary 5. C_M is Gorenstein.

Proof. $H^1_{MC_M}(C_M)$ is injective. Hence (1) is a finite injective resolution of C_M . \square

We remark that a Noetherian local ring R whose maximal ideal is generated by $1 + \text{depth } R$ elements is always Gorenstein [7, p. 163, Exercise 1].

Proposition 6. Any C_M -linear map

$$\Phi: H^1_{MC_M}(C_M) \rightarrow \operatorname{Hom}^c_A(C_M, K/A)$$

equals $\Phi_{\sigma,\rho}$ for some $\sigma, \rho \in \hat{A}$.

Proof. For each n , there exist $\sigma_n, \rho_n \in A$ such that

$$\begin{cases} \left[\begin{array}{c} \sigma_n \\ t^n \end{array} \right]_A = \Phi \left(\left[\begin{array}{c} 1 \\ t^n \end{array} \right]_{C_M} \right) (1), \\ \left[\begin{array}{c} \rho_n \\ t^n \end{array} \right]_A = \Phi \left(\left[\begin{array}{c} 1 \\ t^n \end{array} \right]_{C_M} \right) (t(z_0 - a_0)). \end{cases}$$

Since $\sigma_n - \sigma_{n+1}$ and $\rho_n - \rho_{n+1}$ are contained in $t^n A$, the limits

$$\begin{cases} \sigma = \lim_{n \rightarrow \infty} \sigma_n, \\ \rho = \lim_{n \rightarrow \infty} \rho_n \end{cases}$$

exist in \hat{A} . For any $X, Y \in A$,

$$\Phi \left(\left[\begin{array}{c} X + Yt(z_0 - a_0) \\ t^n \end{array} \right]_{C_M} \right) (1) = \left[\begin{array}{c} X\sigma_n + Y\rho_n \\ t^n \end{array} \right]_A = \left[\begin{array}{c} X\sigma + Y\rho \\ t^n \end{array} \right]_A.$$

Hence

$$(\Phi(\omega))(f) = \Phi(f\omega)(1) = \Phi_{\sigma,\rho}(f\omega)(1) = (\Phi_{\sigma,\rho}(\omega))(f)$$

for any $\omega \in H^1_{MC_M}(C_M)$ and $f \in C_M$. That is, $\Phi = \Phi_{\sigma,\rho}$. \square

Now we fix a σ_0 and an invertible ρ_0 . All endomorphisms of the C_M -module $H^1_{MC_M}(C_M)$ are of the form $\Phi_{\sigma_0,\rho_0}^{-1} \circ \Phi_{\sigma,\rho}$. Since different pairs of σ and ρ determine different C_M -linear maps $\Phi_{\sigma,\rho}$, the completion $\widehat{C_M}$ of C_M can be described as the set $\{\Phi_{\sigma,\rho}\}_{\sigma,\rho \in \hat{A}}$ with addition

$$\Phi_{\sigma_1,\rho_1} + \Phi_{\sigma_2,\rho_2} = \Phi_{\sigma_1+\sigma_2,\rho_1+\rho_2},$$

unit Φ_{σ_0,ρ_0} , and multiplication

$$\Phi_{\sigma_1,\rho_1} * \Phi_{\sigma_2,\rho_2} = \Phi_{\sigma_1,\rho_1} \circ \Phi_{\sigma_0,\rho_0}^{-1} \circ \Phi_{\sigma_2,\rho_2}$$

given by composition of endomorphisms. If $\sigma_0 = 0$ and $\rho_0 = 1$, then

$$\Phi_{\sigma_1, \rho_1} * \Phi_{\sigma_2, \rho_2} = \Phi_{\sigma_1 \rho_2 + \sigma_2 \rho_1 - 2\sigma_1 \sigma_2 t(z_0 - a_0), \rho_1 \rho_2 - \sigma_1 \sigma_2 t^2(z_0 - a_0)^2}.$$

Identifying $\Phi_{\sigma, \rho}$ with $\rho + \sigma X$ in $\hat{A}[X]/(X + t(z - a_0))^2$, and comparing their additions and multiplications, we get the following description of \widehat{C}_M .

Corollary 7. $\widehat{C}_M \simeq \hat{A}[X]/(X + t(z - a_0))^2$.

For $\rho \in C_M$, the endomorphism $\Phi_{0,1}^{-1} \circ \Phi_{0,\rho}$ of $H_{MC_M}^1(C_M)$ is multiplication by ρ . Therefore, with respect to the isomorphism in Corollary 7, the embedding $C_M \rightarrow \hat{A}[X]/(X + t(z - a_0))^2$ of completion is the composition

$$C_M \xrightarrow{\text{inclusion}} \hat{A} \xrightarrow{\text{canonical}} \hat{A}[X]/(X + t(z - a_0))^2.$$

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