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ALGEBRAIC STRUCTURES DETERMINED BY 3 BY 3 MATRIX GEOMETRY

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Dedicated to Masamichi and Kyoko Takesaki and the memory of Yuki

ABSTRACT. Using a "3 by 3 matrix trick" we show that multiplication (an algebraic structure) in a C^*-algebra $\mathcal{A}$ is determined by the geometry of the C^*-algebra of the 3 by 3 matrices with entries from $\mathcal{A}$, $M_3(\mathcal{A})$. This is an example of an algebra-geometry duality which, we claim, has applications.

INTRODUCTION

The motivation for this research was the question, posed about 25 years ago in [10], to find a "natural" generalization of the van Kampen–Pontriagin duality theorem to any locally compact group G, based on the order structure of the semigroup of continuous positive definite functions on G, $P(G)$. The nonabelian group duality we seek follows in part from a more general duality between the algebraic structure of a C^*-algebra and its intrinsic matrix geometry. Since the proof of this algebra-geometry duality for C^*-algebras is relatively simple and of independent interest, we present the result in this note.

KEY LEMMA

In [8] J. L. Smul'jan proved a version of our key lemma which was subsequently translated from Russian to English (and also into contemporary operator-theoretic language) by the late Domingo Herrero. A slight modification of our key lemma, more appropriate to numerical linear algebra because it does not involve taking the square root of operators, can be found in [1]. The version of our key lemma which we state below can be found in [2].

In the following lemma $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces, $\mathcal{B}(\mathcal{H}_i, \mathcal{H}_j)$ is the set of all bounded linear operators from $\mathcal{H}_i$ *to* $\mathcal{H}_j$, $i = 1, 2$, $j = 1, 2$, and $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$, $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, $B^* \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $D \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_2)$. Double vertical bars, $\|\cdot\|$, denote the usual uniform (sup) operator norm, e.g., $\|D\| = \sup\{\|D\zeta\|_{\mathcal{H}_2} \mid \|\zeta\|_{\mathcal{H}_2} \leq 1\}$.

Key lemma. *Let* $\widetilde{A} = \left[\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right]$. *If* $\widetilde{A}$ *is a positive operator in* $\mathcal{H}_1 \oplus \mathcal{H}_2$, *then there exists a bounded operator* $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ *satisfying the following conditions.*

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(1) $B = A^{1/2}X$,
(2) *Range* $(X) \subseteq \overline{Range\,(A)}$.

On the other hand, if B *is of the above form in* (1) *for some operator* X, *then* $\widetilde{A} \geq 0$ *if and only if* $D \geq X^*X$.

The duality between the product operation in C^*-algebra $\mathcal{A}$ and the geometry in $M_3(\mathcal{A})$

In the following $M_3(\mathcal{A})$ is the set of 3 by 3 matrices with entries from C^*-algebra $\mathcal{A}$. With the usual operations of matrix addition, multiplication and taking the adjoint, $M_3(\mathcal{A})$ is a C^*-algebra; cf. [9], p. 192. The definition of "the geometry of $M_3(\mathcal{A})$" will be evident in the subsequent discussion.

Theorem 1. *If* $a, b \in \mathcal{A}$, *a* C^**-algebra with unit* e, *then the product* ab *in* $\mathcal{A}$ *is determined by the geometry of* $M_3(\mathcal{A})$, *most notably the order structure of* $M_3(\mathcal{A})$.

The proof of Theorem 1 depends on the following lemma.

Lemma 1. *Let* $\mathcal{A}$ *be a* C^**-algebra with unit* e. *Let* a, b *and* c *be unitary operators in* $\mathcal{A}$, *e.g.,* $a^*a = e = aa^*$. *We have*

$$\begin{pmatrix} e & a & b \\ a^* & e & c \\ b^* & c^* & e \end{pmatrix} \geq 0 \quad in \quad M_3(\mathcal{A})$$

if and only if $c = a^{-1}b$ *if and only if* $b = ac$.

Proof of Lemma 1. In the notation of our key lemma above, $\widetilde{A}$ is the 3 by 3 matrix given in the statement of Lemma 1, and $A = \left(\begin{smallmatrix} e & a \\ a^* & e \end{smallmatrix}\right)$. Since a is unitary, $A^{1/2} = \frac{1}{\sqrt{2}}A$. We assume $\widetilde{A}$ is a positive operator in the C^*-algebra $M_3(\mathcal{A})$, hence it will be a positive operator in any realization of $M_3(\mathcal{A})$ on Hilbert space.

Thus by (1) of our key lemma, there will be bounded operators x_1, x_2 satisfying the following pair of equations:

$$ex_1 + ax_2 = \sqrt{2}b,$$
$$a^*x_1 + ex_2 = \sqrt{2}c.$$

Multiplying the first equation by $a^* = a^{-1}$ immediately yields $c = a^{-1}b$.

The converse is elementary, since $\left(\begin{smallmatrix} e & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)^* \left(\begin{smallmatrix} e & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \geq 0$; cf. [9], p. 193. □

Proof of Theorem 1. We consider the order structure on $M_3(\mathcal{A})$ to be part of "geometry". Furthermore, note that $\frac{1}{2}A = \frac{1}{2}\left(\begin{smallmatrix} e & a \\ a^* & e \end{smallmatrix}\right) \in M_2(\mathcal{A})$ is a projection in C^*-algebra $M_2(\mathcal{A})$, i.e., $(\frac{1}{2}A)^* = \frac{1}{2}A = (\frac{1}{2}A)^2$, if and only if a is a unitary in C^*-algebra $\mathcal{A}$. Projections in a C^*-algebra are characterized geometrically as the extreme points of the convex intersection of the unit ball and the positive cone of the C^*-algebra; cf. [7], p. 484. Similarly projections $\frac{1}{2}\left(\begin{smallmatrix} e & b \\ b^* & e \end{smallmatrix}\right)$ and $\frac{1}{2}\left(\begin{smallmatrix} e & c \\ c^* & e \end{smallmatrix}\right)$ determine (and are determined by) unitaries b and c, respectively. Since every element in a C^*-algebra with unit is a linear combination of unitaries (cf. [4]), we have determined the product in $\mathcal{A}$ by various aspects of the geometry of $\mathcal{A}$, $M_2(\mathcal{A})$ and $M_3(\mathcal{A})$, and all three of these C^*-algebras can be considered subalgebras of $M_3(\mathcal{A})$. □

Possible applications

In [12], which is in preparation, we apply Theorem 1 above to construct the duality for locally compact groups mentioned in the Introduction as our motivation for searching for Theorem 1. It is our as yet unrealized goal to make a significant contribution to (via a simplification of) the existing classification of finite groups using our duality theory.

In the "standard model" which appears in theoretical physics, 3 by 3 matrices play a role. This was brought to my attention by a lecture of Alain Connes. Whether the structures explored in this paper might contribute any insights in physics, however minor, remain to be discovered.

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