

HECKE ALGEBRAS OF SEMIDIRECT PRODUCTS

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ABSTRACT. We consider group-subgroup pairs in which the group is a semidirect product and the subgroup is contained in the normal part. We give conditions for the pair to be a Hecke pair and we show that the enveloping Hecke algebra and Hecke C^* -algebra are canonically isomorphic to semigroup crossed products, generalizing earlier results of Arledge, Laca and Raeburn and of Brenken.

INTRODUCTION

A group-subgroup pair (Γ, Γ_0) is called a *Hecke pair* if each double coset $\Gamma_0\gamma\Gamma_0$ is a finite union of left cosets $\gamma_i\Gamma_0$. By considering inverses, one sees that it is equivalent to require that each double coset is the union of finitely many right cosets. This situation is variously referred to by saying that Γ_0 is a *Hecke subgroup* or an *almost normal* subgroup of Γ . See [7] for more details.

To each Hecke pair one associates a *Hecke $*$ -algebra* $\mathcal{H}(\Gamma, \Gamma_0)$, by endowing the set of Γ_0 -bi-invariant complex valued functions on Γ that are supported in finitely many double cosets with a convolution product defined by the (finite) sum

$$f * g(\gamma) = \sum_{\gamma_1 \in \Gamma_0 \backslash \Gamma} f(\gamma\gamma_1^{-1})g(\gamma_1),$$

and an involution defined by $f^*(\gamma) = \overline{f(\gamma^{-1})}$. The characteristic function of Γ_0 is the identity element. Hecke algebras have been around for quite some time but their C^* -algebraic version has only recently begun to get some attention, initially in the work of Binder [2] and Bost and Connes [4], and more recently in that of Hall [6] and Tzanev [16]. One defines the *reduced Hecke C^* -algebra* $C_r^*(\Gamma, \Gamma_0)$ to be the closure of $\mathcal{H}(\Gamma, \Gamma_0)$ in the natural representation on $\ell^2(\Gamma_0 \backslash \Gamma)$ obtained using the above convolution formula with $g \in \ell^2(\Gamma_0 \backslash \Gamma)$. A universal version $C_u^*(\Gamma, \Gamma_0)$, through which all representations of $\mathcal{H}(\Gamma, \Gamma_0)$ on Hilbert space factor, is not so readily obtained because in some cases there are generators that do not have uniformly bounded norms in all $*$ -representations; see [6, Example 2.3.3]. To some extent this difficulty can be sidestepped by restricting the attention to representations that do not increase a certain Banach algebra norm on $\mathcal{H}(\Gamma, \Gamma_0)$; see [16].

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The present work is motivated by the realization of various Hecke C*-algebras as semigroup crossed products. Following [9, 1], but especially guided by [5], we are first led to consider the semigroup of elements of Γ whose left coset is right invariant, and hence a double coset. We observe that if this semigroup generates all of Γ by left quotients, then the problem with unbounded norms does not arise because $\mathcal{H}(\Gamma, \Gamma_0)$ has a generating set consisting of isometries, providing a simple proof of a result of Hall concerning the existence of the universal Hecke C*-algebra [6, Corollary 4.6].

Our main results are about pairs consisting of a semidirect product group $\Gamma = N \rtimes G$ and a subgroup $\Gamma_0 \subset N$. We give sufficient conditions on the action of G on N for (Γ, Γ_0) to be a Hecke pair, and we realize the Hecke algebra, and C*-algebra, as semigroup crossed products. These results unify and generalize previous realizations of Hecke C*-algebras as semigroup crossed products obtained in [9, 1, 5, 13].

Our approach relies freely on the basic results from Section 1 of [5] where the generality is appropriate for our purposes, but the techniques used to obtain our main results are independent of the rest of [5]. Indeed, we are able to give simpler proofs of more general statements. Notably, by considering a different set of generating monomials, inspired by those of [1], we have done away with the quasi-lattice structure on G . Perhaps more significantly, we have removed the assumption that Γ_0 be normal in N ; instead we assume that (N, Γ_0) is a Hecke pair, and replace the group algebra $\mathbb{C}(N/\Gamma_0)$ with the Hecke algebra $\mathcal{H}(N, \Gamma_0)$.

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1. HECKE ALGEBRAS OF SEMIDIRECT PRODUCTS

Definition 1.1 (cf. [4]). For each $\gamma \in \Gamma$, let $R(\gamma)$ (respectively, $L(\gamma)$) be the cardinality of the image of $\Gamma_0\gamma\Gamma_0$ in $\Gamma_0\backslash\Gamma$ (respectively, in Γ/Γ_0). It is clear that $L(\gamma) = R(\gamma^{-1})$.

Notation. Throughout we denote by $[\gamma]$ the characteristic function of the double coset of an element $\gamma \in \Gamma$.

Remark 1.2. The Hecke algebra $\mathcal{H}(\Gamma, \Gamma_0)$ is the linear span of $\{[\gamma] : \gamma \in \Gamma\}$. The product of two elements of this spanning set is given by

$$([\gamma][\delta])(x) = |\Gamma_0 \backslash (\Gamma_0 \gamma^{-1} \Gamma_0 x \cap \Gamma_0 \delta \Gamma_0)|,$$

from which one easily sees that the support of $[\gamma][\delta]$ is $\Gamma_0\gamma\Gamma_0\delta\Gamma_0$.

From Theorems 1.2 and 1.4 of [5] we know that the set

$$\Sigma := \{\sigma \in \Gamma : L(\sigma) = 1\}$$

is a semigroup on which R is multiplicative, and that the map

$$\sigma \in \Sigma \mapsto W_\sigma := R(\sigma)^{-1/2}[\sigma] \in \mathcal{H}(\Gamma, \Gamma_0)$$

is a representation of Σ by isometries in $\mathcal{H}(\Gamma, \Gamma_0)$. We will need the following supplement of this.

Lemma 1.3. *For every $\gamma \in \Gamma$ and $\sigma \in \Sigma$,*

- (i) *the product $[\gamma][\sigma]$ is a positive integer multiple of $[\gamma\sigma]$, and $L(\gamma\sigma) = L(\gamma)$;*

- (ii) the product $[\sigma]^*[\gamma]$ is a positive integer multiple of $[\sigma^{-1}\gamma]$, and $R(\sigma^{-1}\gamma) = R(\gamma)$.

Proof. Clearly $\sigma \in \Sigma$ if and only if $\Gamma_0\sigma\Gamma_0 = \sigma\Gamma_0$, from which it follows that $\Gamma_0\gamma\sigma\Gamma_0 = \Gamma_0\gamma\Gamma_0\sigma\Gamma_0 = (\bigsqcup_{i=1}^{L(\gamma)} \gamma_i\Gamma_0)\sigma\Gamma_0 = \bigsqcup_{i=1}^{L(\gamma)} \gamma_i\sigma\Gamma_0$. Since the support of $[\gamma][\sigma]$ is $\Gamma_0\gamma\Gamma_0\sigma\Gamma_0$, we obtain (i), from which (ii) follows on considering adjoints and replacing γ by γ^{-1} . \square

Proposition 1.4. *Suppose (Γ, Γ_0) is a Hecke pair such that $\Sigma^{-1}\Sigma = \Gamma$. Then $W_\beta^*W_\alpha = W_\tau^*W_\sigma$ whenever $\beta^{-1}\alpha = \tau^{-1}\sigma$ and the set $\{W_\tau^*W_\sigma : \tau, \sigma \in \Sigma\}$ is a linear basis of $\mathcal{H}(\Gamma, \Gamma_0)$. Moreover, the universal C^* -enveloping algebra $C_u^*(\Gamma, \Gamma_0)$ of $\mathcal{H}(\Gamma, \Gamma_0)$ is a quotient of the universal C^* -algebra of isometric representations of Σ .*

Remark 1.5. Since the semigroup Σ is cancellative, because it is contained in Γ , by assuming that $\Sigma^{-1}\Sigma = \Gamma$ (or equivalently that $\Sigma\gamma \cap \Sigma\gamma' \neq \emptyset$ for every pair $\gamma, \gamma' \in \Gamma$) we are saying that Σ is an Ore semigroup with Γ its group of left quotients. The proposition asserts in particular that $C_u^*(\Gamma, \Gamma_0)$ exists, proved in [6, Corollary 4.6]; see also [5, Proposition 2.8].

Proof. Let $\tau, \sigma \in \Sigma$; from the lemma we know that

$$[\tau]^*[\sigma] = K_{\sigma,\tau}[\tau^{-1}\sigma]$$

where $K_{\sigma,\tau}$ is a positive integer. Hence, up to positive constants, the products $W_\tau^*W_\sigma$ are the characteristic functions of all the double cosets, so their linear span is $\mathcal{H}(\Gamma, \Gamma_0)$.

From this it already follows that $\mathcal{H}(\Gamma, \Gamma_0)$ is generated by isometries and that the universal C^* -enveloping algebra for $\mathcal{H}(\Gamma, \Gamma_0)$ exists and is a quotient of the universal C^* -algebra of isometric representations of Σ . The key here is, of course, that isometries have norm 1 in any nontrivial Hilbert space representation, so the basic conditions for the existence of the universal C^* -algebra (see e.g. [3]) are satisfied.

Since the characteristic functions of distinct double cosets are disjoint, to conclude that the given set is a linear basis of $\mathcal{H}(\Gamma, \Gamma_0)$ all we have to prove is that $W_\beta^*W_\alpha = W_\tau^*W_\sigma$ whenever $\beta^{-1}\alpha = \tau^{-1}\sigma$, with α, β, σ and τ in Σ . For this it suffices to show that the expression

$$(1.1) \quad \frac{K_{\sigma,\tau}}{R(\tau)^{1/2}R(\sigma)^{1/2}}$$

depends only on the left quotient $\tau^{-1}\sigma$. Since Σ is directed, that is, $\Sigma\gamma \cap \Sigma\gamma' \neq \emptyset$, it is enough to show that for each $\gamma \in \Sigma$ the value of the above expression does not change when we replace σ and τ by $\gamma\sigma$ and $\gamma\tau$, respectively, for which it suffices to show that $K_{\gamma\sigma,\gamma\tau} = R(\gamma)K_{\sigma,\tau}$ because R is multiplicative on Σ . In order to prove this we first compute $K_{\sigma,\tau}$ by evaluating the convolution product at $\tau^{-1}\sigma$ using Remark 1.2:

$$\begin{aligned} K_{\sigma,\tau} &= ([\tau^{-1}]^*[\sigma])(\tau^{-1}\sigma) \\ &= |\Gamma_0 \setminus (\Gamma_0\tau\Gamma_0\tau^{-1}\sigma \cap \Gamma_0\sigma\Gamma_0)| \\ &= |\Gamma_0 \setminus (\tau\Gamma_0\tau^{-1}\sigma \cap \sigma\Gamma_0)| \\ &= |\Gamma_0 \setminus (\tau\Gamma_0\tau^{-1} \cap \sigma\Gamma_0\sigma^{-1})|. \end{aligned}$$

The claim then follows by applying the basic fact $|A \setminus C| = |A \setminus B| |B \setminus C|$, valid for every group inclusion $A \subset B \subset C$, to the inclusions

$$\Gamma_0 \subset \gamma \Gamma_0 \gamma^{-1} \subset \gamma \tau \Gamma_0 \tau^{-1} \gamma^{-1} \cap \gamma \sigma \Gamma_0 \sigma^{-1} \gamma^{-1},$$

and observing that $R(\gamma) = |\Gamma_0 \setminus (\gamma \Gamma_0 \gamma^{-1})|$ and that

$$|(\gamma \Gamma_0 \gamma^{-1}) \setminus (\gamma \tau \Gamma_0 \tau^{-1} \gamma^{-1} \cap \gamma \sigma \Gamma_0 \sigma^{-1} \gamma^{-1})| = |\Gamma_0 \setminus (\tau \Gamma_0 \tau^{-1} \cap \sigma \Gamma_0 \sigma^{-1})|.$$

□

We pause to recall that when α is an action of a semigroup S by endomorphisms of a unital C^* -algebra A , one defines a *covariant representation* to be a pair (π, V) in which π is a unital representation of A and V is a representation of S by isometries such that $\pi(\alpha_s(a)) = V_s \pi(a) V_s^*$ for every $a \in A$ and $s \in S$. The associated *semigroup crossed product* is the C^* -algebra generated by a universal covariant representation, which, by definition, is one through which every covariant representation factors; see [8, 15] for more details. The analogous concept in the category of $*$ -algebras is defined in the obvious way and we refer to it as the $*$ -algebraic semigroup crossed product.

Remark 1.6. Since W is a semigroup of isometries, there is a semigroup of endomorphisms (of $\mathcal{H}(\Gamma, \Gamma_0)$ and of $C_u^*(\Gamma, \Gamma_0)$) defined by $\alpha_s(b) := W_s b W_s^*$. Let B denote the $*$ -subalgebra of $\mathcal{H}(\Gamma, \Gamma_0)$ generated by the projections $W_s W_s^*$ with $s \in \Sigma$, in other words, B is the linear span of the products of such projections. Since $W_\sigma (\prod_j W_{s_j} W_{s_j}^*) W_\sigma^* = \prod_j W_{\sigma s_j} W_{\sigma s_j}^*$, the endomorphisms leave B invariant, indeed, B is the smallest invariant unital $*$ -subalgebra. It is easy to show that the associated semigroup crossed product $B \rtimes \Sigma$ maps onto the Hecke algebra in a canonical way. However, there is in principle no reason why this map should be injective in general, because the covariance relation (which takes place in the algebra B) might not capture all the relations satisfied by the generators of the Hecke algebra. By the same token, if N is a normal subgroup of Γ and Γ_0 is normal in N , there is a surjective homomorphism of $C(N/\Gamma_0) \rtimes \Sigma$ onto $\mathcal{H}(\Gamma, \Gamma_0)$. This shows that the conclusion of [5, Proposition 2.10] holds without any assumptions on least upper bounds.

When Γ is a semidirect product we will be able to say more. Recall that if the group G acts on the group N by automorphisms ψ_g , one defines the semidirect product group $\Gamma = N \rtimes G$ by endowing the set $N \times G$ with the twisted product $(n, g)(m, h) = (n\psi_g(m), gh)$. One can then view G and N as subgroups of Γ , with N normal, and write the automorphisms ψ_g as inner automorphisms obtained by conjugation by elements of G , namely, $\psi_g(m) = gm g^{-1}$.

Proposition 1.7. *Let $\Gamma := N \rtimes G$ be a semidirect product group and let Γ_0 be a subgroup of N . Suppose that S is a subsemigroup of G such that $S^{-1}S = G$ and $s^{-1}\Gamma_0 s \subset \Gamma_0$ for every $s \in S$. Then (Γ, Γ_0) is a Hecke pair if and only if (N, Γ_0) is a Hecke pair and $|s^{-1}\Gamma_0 s \setminus \Gamma_0| < \infty$ for every $s \in S$. Furthermore, $R(s) = |s^{-1}\Gamma_0 s \setminus \Gamma_0|$ for every $s \in S$, and the identity map gives an embedding $\mathcal{H}(N, \Gamma_0) \hookrightarrow \mathcal{H}(\Gamma, \Gamma_0)$.*

Proof. The assumption $s^{-1}\Gamma_0 s \subset \Gamma_0$ says that the action of S^{-1} leaves Γ_0 invariant. This is also equivalent to saying that $S \subset \Sigma$, because $s^{-1}\Gamma_0 s \subset \Gamma_0$ implies $\Gamma_0 s \Gamma_0 = s \Gamma_0$.

Assume first that Γ_0 is a Hecke subgroup of Γ . Then it is clearly also a Hecke subgroup of N , and for each $s \in S$ one has $|s^{-1}\Gamma_0s \setminus \Gamma_0| = |\Gamma_0 \setminus s\Gamma_0s^{-1}| = |\Gamma_0 \setminus \Gamma_0s\Gamma_0s^{-1}| = |\Gamma_0 \setminus (\Gamma_0s\Gamma_0)| = R(s) < \infty$.

Assume now that Γ_0 is a Hecke subgroup of N and that $|s^{-1}\Gamma_0s \setminus \Gamma_0| < \infty$ for every $s \in S$. Let $\gamma \in \Gamma$ and write $\gamma = t^{-1}ns$ with $t, s \in S$ and $n \in N$ (an easy argument shows this is always possible because $S^{-1}S = G$). In order to show that $|\Gamma_0 \setminus \Gamma_0t^{-1}ns\Gamma_0|$ is finite, we write

$$\begin{aligned} \Gamma_0t^{-1}ns\Gamma_0 &= \Gamma_0t^{-1}\Gamma_0n\Gamma_0s\Gamma_0 \\ &= \bigcup_j \Gamma_0t^{-1}\Gamma_0n\Gamma_0\gamma_j && \text{because } s\Gamma_0 = \Gamma_0s\Gamma_0 = \bigsqcup_{j=1}^{R(s)} \Gamma_0\gamma_j \\ &= \bigcup_j \bigcup_i \Gamma_0t^{-1}\Gamma_0n_i\gamma_j && \text{because } \Gamma_0n\Gamma_0 = \bigsqcup_{i=1}^{R(n)} \Gamma_0n_i \\ &= \bigcup_j \bigcup_i \Gamma_0t^{-1}n_i\gamma_j, \end{aligned}$$

which is a union of at most $R(s)R(n)$ right cosets. This proves that (Γ, Γ_0) is a Hecke pair, and hence that $R(s) = |s^{-1}\Gamma_0s \setminus \Gamma_0|$, as shown above. Finally, the operations defined on $\mathcal{H}(\Gamma, \Gamma_0)$ restrict to those on $\mathcal{H}(N, \Gamma_0)$, so the identity map gives a unital embedding of $*$ -algebras $\mathcal{H}(N, \Gamma_0) \hookrightarrow \mathcal{H}(\Gamma, \Gamma_0)$. \square

Remark 1.8. The set $S_0 := \{s \in G : |s^{-1}\Gamma_0s \setminus \Gamma_0| < \infty\}$ is itself always a semigroup, in fact, the largest possible subsemigroup of G that can satisfy the conditions of hypothesis. If $S_0^{-1}S_0 = G$, the proposition certainly applies with $S = S_0$, but in some situations one might be better off with a proper subsemigroup S of S_0 that happens to generate G by left quotients, too.

Theorem 1.9. *Let $\Gamma := N \rtimes G$ be a semidirect product group and let Γ_0 be a subgroup of N . Suppose that S is a subsemigroup of G such that $S^{-1}S = G$ and $|s^{-1}\Gamma_0s \setminus \Gamma_0| < \infty$ for every $s \in S$. If (N, Γ_0) is a Hecke pair, then (Γ, Γ_0) is a Hecke pair, with $R(s) = |s^{-1}\Gamma_0s \setminus \Gamma_0|$, and*

- (i) *the elements of $\mathcal{H}(\Gamma, \Gamma_0)$ given by $\mu_s := R(s)^{-1/2}[s]$ for $s \in S$ and $e(x) := [x]$ for $x \in N$ satisfy the relations:*
 - (h₁) $\mu_s^* \mu_s = I$ and $\mu_s \mu_t = \mu_{st}$, for $s, t \in S$;
 - (h₂) $e(1) = I$, $e(x)^* = e(x^{-1})$, and $e(x)e(y) = [x][y]$ (as in $\mathcal{H}(N, \Gamma_0)$); and
 - (h₃) $\mu_s e(x) \mu_s^* = R(s)^{-1} \sum_{\{[y] : [s^{-1}ys] = [x]\}} e(y)$, for $s \in S$ and $x \in N$;*and the set $\{\mu_t^* e(x) \mu_s : s, t \in S \text{ and } x \in N\}$ is a linear basis of $\mathcal{H}(\Gamma, \Gamma_0)$;*
- (ii) *there is an action α of S by injective corner endomorphisms of the Hecke $*$ -algebra $\mathcal{H}(N, \Gamma_0)$ defined by*

$$\alpha_s(e(x)) := R(s)^{-1} \sum_{\{[y] : [s^{-1}ys] = [x]\}} e(y),$$

and the Hecke $$ -algebra $\mathcal{H}(\Gamma, \Gamma_0)$ is canonically isomorphic to the $*$ -algebraic semigroup crossed product $\mathcal{H}(N, \Gamma_0) \rtimes_{\alpha} S$, and to the universal unital $*$ -algebra with presentation $(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)$.*

We point out that the given labelling of the elements of the linear basis of part (i) is not one-to-one, and Lemma 1.10 below deals precisely with this redundancy.

Proof. The first assertion is from Proposition 1.7. A somewhat tedious but straightforward computation of convolution products in $\mathcal{H}(\Gamma, \Gamma_0)$ shows that the μ_s and $e(x)$ satisfy the relations $(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)$. Notice that relation (\mathfrak{h}_1) simply says that μ is an isometric representation of S (see Theorem 1.4 of [5]) and relation (\mathfrak{h}_2) is just a restatement of the embedding of $\mathcal{H}(N, \Gamma_0)$ in $\mathcal{H}(\Gamma, \Gamma_0)$. In order to see that the set $\{\mu_t^*e(x)\mu_s : s, t \in S \text{ and } x \in N\}$ is a linear basis of $\mathcal{H}(\Gamma, \Gamma_0)$, recall that $\mu_t^*e(x)\mu_s$ is a function on Γ and notice that its support is the set $\Gamma_0t^{-1}\Gamma_0x\Gamma_0s\Gamma_0 = \Gamma_0t^{-1}xs\Gamma_0$. Since $\Gamma = S^{-1}NS$, all double cosets arise as supports. Thus the products $\mu_t^*e(x)\mu_s$ linearly span $\mathcal{H}(\Gamma, \Gamma_0)$, and we only have to prove that $\{\mu_t^*e(x)\mu_s : s, t \in S \text{ and } x \in N\}$ is a linearly independent set. Different supports are disjoint, so to prove linear independence it suffices to show that if two such products have the same support, then they coincide, which we do in the following lemma.

Lemma 1.10. *For $t, s, \tau, \sigma \in S$ and $m, n \in N$, we have that $[\tau^{-1}m\sigma] = [t^{-1}ns]$ if and only if $\mu_\tau^*[m]\mu_\sigma = \mu_t^*[n]\mu_s$.*

Proof. The ‘if’ part is immediate on considering supports. To prove the ‘only if’ part, begin by taking quotients modulo N in $[\tau^{-1}m\sigma] = [t^{-1}ns]$ to see that $\tau^{-1}\sigma = t^{-1}s$. Since S is directed there exist r and γ in S such that $rt = \gamma\tau$ and $rs = \gamma\sigma$. Since $S \subset \Sigma$, Lemma 1.3 implies that the function R is invariant under multiplication by elements of S^{-1} on the left, so

$$R(\sigma) = R(\tau^{-1}\sigma) = R(t^{-1}s) = R(s).$$

Since R is multiplicative on S and $R(\gamma\sigma) = R(rs)$, we also have $R(r) = R(\gamma)$. From the assumption $\Gamma_0\tau^{-1}m\sigma\Gamma_0 = \Gamma_0t^{-1}ns\Gamma_0$ it follows at once that

$$(1.2) \quad \Gamma_0\tau^{-1}\Gamma_0m\Gamma_0\sigma\Gamma_0 = \Gamma_0t^{-1}\Gamma_0n\Gamma_0s\Gamma_0,$$

at which point it becomes convenient to let $x := rt = \gamma\tau$ and $y := rs = \gamma\sigma$, and to write τ, σ, t and s in terms of x, y, r and γ . Multiplication of (1.2) on the left by x and on the right by y^{-1} yields

$$(x\Gamma_0x^{-1})(\gamma\Gamma_0m\Gamma_0\gamma^{-1})(y\Gamma_0y^{-1}) = (x\Gamma_0x^{-1})(r\Gamma_0n\Gamma_0r^{-1})(y\Gamma_0y^{-1}).$$

Using the multiplication rules in $\mathcal{H}(\Gamma, \Gamma_0)$ we see that

- $(x\Gamma_0x^{-1})$ is the support of $\mu_x\mu_x^*$,
- $(y\Gamma_0y^{-1})$ is the support of $\mu_y\mu_y^*$,
- $(\gamma\Gamma_0m\Gamma_0\gamma^{-1})$ is the support of $\mu_\gamma[m]\mu_\gamma^*$, and
- $(r\Gamma_0n\Gamma_0r^{-1})$ is the support of $\mu_r[n]\mu_r^*$.

It follows that

$$(1.3) \quad \mu_x\mu_x^*\mu_\gamma[m]\mu_\gamma^*\mu_y\mu_y^* = \mu_x\mu_x^*\mu_r[n]\mu_r^*\mu_y\mu_y^*,$$

because the supports match and since $R(\gamma) = R(r)$ the coefficients also coincide. Now simply multiply by μ_x^* on the left, by μ_y on the right, and simplify $\mu_x^*\mu_x\mu_x^*\mu_\gamma = \mu_\tau^*$ and other similar products to see that $\mu_\tau^*[m]\mu_\sigma = \mu_t^*[n]\mu_s$. This finishes the proof of the lemma and thus of part (i). \square

It is possible to give a direct computational proof of part (ii) along the lines of [9, 1, 5], but it is more efficient to use the Hecke algebra as in Remark 1.6. To this effect notice that, because of relation (\mathfrak{h}_3) with $x \in \Gamma_0$, the projection $\mu_s\mu_s^*$ is contained in $\mathcal{H}(N, \Gamma_0)$ for every $s \in S$. Since μ_s is a semigroup of isometries by relation (\mathfrak{h}_1) , the left-hand side of (\mathfrak{h}_3) defines a semigroup of injective corner

endomorphisms: the image of α_s is the corner $\mu_s \mu_s^* \mathcal{H}(N, \Gamma_0) \mu_s \mu_s^*$, on which α_s has an inverse defined by $X \mapsto \mu_s^* X \mu_s$.

A standard reinterpretation of the relations $(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)$ shows that they are in fact a presentation of the *-algebraic semigroup crossed product $\mathcal{H}(N, \Gamma_0) \rtimes_{\alpha} S$. Since the μ_s 's and $e(x)$'s in the Hecke algebra satisfy these relations, there is a canonical homomorphism of $\mathcal{H}(N, \Gamma_0) \rtimes_{\alpha} S$ to $\mathcal{H}(\Gamma, \Gamma_0)$, which we must show is an isomorphism. In order to avoid confusion we temporarily denote the generators of the universal *-algebra with the given presentation by $\tilde{\mu}_s$ and $\tilde{e}(x)$ while maintaining μ_s and $e(x)$ for the generators of the Hecke algebra. Thus, the canonical homomorphism maps $\tilde{\mu}_t^* \tilde{e}(n) \tilde{\mu}_s$ into $\mu_t^* e(n) \mu_s$, respectively, and we will show that it is an isomorphism by showing that this map is one-to-one from a spanning set onto the linear basis from part (i). Indeed, the linear span of products of the form $\tilde{\mu}_t^* \tilde{e}(x) \tilde{\mu}_s$ is multiplicatively closed, so it must be all of $\mathcal{H}(N, \Gamma_0) \rtimes_{\alpha} S$; see for example Remark 1.6 of [11]. Suppose now that $\mu_r^* e(m) \mu_{\sigma} = \mu_t^* e(n) \mu_s$. Then (1.3) holds and, by (\mathfrak{h}_3) , it corresponds to a relation satisfied in the subalgebra $\mathcal{H}(N, \Gamma_0)$. But the elements $\tilde{e}(x)$ satisfy all the relations that are satisfied in $\mathcal{H}(N, \Gamma_0)$, because (\mathfrak{h}_2) is a presentation of $\mathcal{H}(N, \Gamma_0)$, so it follows, again using (\mathfrak{h}_3) , that (1.3) also holds for the corresponding universal elements $\tilde{\mu}$ and \tilde{e} . On multiplying this universal version of (1.3) on the left by $\tilde{\mu}_x^*$ and on the right by $\tilde{\mu}_y$, and simplifying the result using (\mathfrak{h}_1) , as in the end of the proof of Lemma 1.10, we see that $\tilde{\mu}_r^* \tilde{e}(m) \tilde{\mu}_{\sigma} = \tilde{\mu}_t^* \tilde{e}(n) \tilde{\mu}_s$. This finishes the proof of (ii) and of the theorem. \square

An immediate application of Theorem 1.9 and the corresponding universal properties gives the following result about C*-algebras.

Theorem 1.11. *Under the assumptions of the preceding theorem, the universal Hecke C*-algebra $C_u^*(\Gamma, \Gamma_0)$ exists if and only if $C_u^*(N, \Gamma_0)$ exists, in which case the isomorphisms of part (ii) extend to C*-algebra isomorphisms:*

$$C_u^*(\Gamma, \Gamma_0) \cong C_u^*(N, \Gamma_0) \rtimes_{\alpha} S \cong C_u^*(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3).$$

Notice that, by [11, Theorem 2.1.1], the first isomorphism implies that $C_u^*(N, \Gamma_0)$ embeds in $C_u^*(\Gamma, \Gamma_0)$.

Corollary 1.12. *If Γ_0 is normal in N , then relation (\mathfrak{h}_2) defines the group algebra of N/Γ_0 so one has*

$$\mathcal{H}(\Gamma, \Gamma_0) \cong \mathbb{C}(N/\Gamma_0) \rtimes_{\alpha} S \quad \text{and} \quad C_u^*(\Gamma, \Gamma_0) \cong C^*(N/\Gamma_0) \rtimes_{\alpha} S.$$

Remark 1.13. By the dilation/extension theorems of [11] the representation theory of $\mathcal{H}(\Gamma, \Gamma_0)$ is equivalent to that of a C*-dynamical system (A, G, β) in which A is the inductive limit $\lim_S(\mathcal{H}(N, \Gamma_0), \alpha_s)$ and β is an action of G by automorphisms of A (we refer to the appendix of [14] for the categorical statement of this equivalence). This is particularly useful when (N, Γ_0) is a Gelfand pair, that is to say, when $\mathcal{H}(N, \Gamma_0)$ is commutative, since it then allows one to apply results about transformation group C*-algebras to Hecke C*-algebras. See for instance [10] and [14].

2. EXAMPLES

Example 2.1. Theorem 1.9 applies to the Bost-Connes Hecke C*-algebra from [4] and to the Hecke C*-algebras of the inclusions

$$\begin{pmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{pmatrix} \subset \begin{pmatrix} 1 & \mathcal{K} \\ 0 & \mathcal{K}^* \end{pmatrix}$$

where \mathcal{K} is a number field with ring of integers \mathcal{O} . These have already been shown to be semigroup crossed products in [9] and [1], and also in [5] under the additional assumption that \mathcal{K} is of class number 1. For the case when \mathcal{O} is one of the principal subrings of a global field \mathcal{K} , see [13].

Example 2.2. Now we briefly recall the situation discussed in Example 4.3 of [5]. Let $G = GL_d^+(\mathbb{Q})$ be the group of rational $d \times d$ matrices with positive determinant, and let $S = G \cap M_d(\mathbb{Z})$ be the subsemigroup of matrices having integer entries. Then G acts on $N = \mathbb{Q}^d$ by $\psi_g(n) = (g^t)^{-1}n$, where g^t indicates the transpose of g (we choose this action instead of the conventional action used in [5] in order to avoid having to switch from S to S^{-1} later on). Let $\Gamma := N \rtimes G = \mathbb{Q}^d \rtimes GL_d^+(\mathbb{Q})$, and consider the subgroup $\Gamma_0 = \mathbb{Z}^d \subset \mathbb{Q}^d = N$. It is clear that if $s \in S$, then $\psi_{s^{-1}}(\Gamma_0) = s^t \mathbb{Z}^d \subset \mathbb{Z}^d$ and since $|\Gamma_0 / \psi_{s^{-1}}(\Gamma_0)| = \det(s^t) < \infty$, we have a finite order inclusion for every $s \in S$.

From Theorem 1.9 we deduce that $(\mathbb{Q}^d \rtimes GL_d^+(\mathbb{Q}), \mathbb{Z}^d)$ is a Hecke pair and that its Hecke C*-algebra is isomorphic to the semigroup crossed product

$$C_u^*(\mathbb{Q}^d \rtimes GL_d^+(\mathbb{Q}), \mathbb{Z}^d) \cong C^*(\mathbb{Q}^d / \mathbb{Z}^d) \rtimes_\alpha S.$$

Remark 2.3. Theorem 2.11 of [5] suffices for the existence of this universal Hecke C*-algebra, but it does not give the semigroup crossed product structure, because in this case the normalizer of Γ_0 is not normal in Γ .

Next we shall use a multivariable version of the arguments of Section 3 of [11] to obtain an explicit realization of the minimal dilated system whose crossed product contains the Hecke C*-algebra as a full corner. Denote by \mathcal{A}_f the ring of finite adeles over \mathbb{Q} , with $\mathcal{R} = \prod_p \mathbb{Z}_p$ the maximal compact subring. Corresponding to the natural action $(g, x) \mapsto gx$ of G on \mathcal{A}_f^d there is an action β of G on $C_0(\mathcal{A}_f^d)$ given by $(\beta_g F)(x) = F(g^{-1}x)$ for $F \in C_0(\mathcal{A}_f^d)$.

Proposition 2.4. *The Hecke C*-algebra $C_u^*(\mathbb{Q}^d \rtimes GL_d^+(\mathbb{Q}), \mathbb{Z}^d)$ is (isomorphic to) the full corner in the crossed product $C_0(\mathcal{A}_f^d) \rtimes_\beta GL_d^+(\mathbb{Q})$ corresponding to the projection $1_{\mathcal{R}^d} \in \mathcal{A}_f^d$.*

Proof. Observe first that the duality pairing of $\mathbb{Q}^d / \mathbb{Z}^d$ with \mathcal{R}^d given by

$$\langle q, x \rangle = \exp(2\pi i \sum_{j=1}^d q_j x_j)$$

establishes a Gelfand-Fourier transform isomorphism of $C^*(\mathbb{Q}^d / \mathbb{Z}^d)$ to $C(\mathcal{R}^d)$. To see that the action β restricted to S is conjugate to the semigroup action of S on $C^*(\mathbb{Q}^d / \mathbb{Z}^d)$ from part (ii) of Theorem 1.9, it is enough to carry out the computation on the characters $e(x) = \langle x, \cdot \rangle$. Thus

$$\alpha_s(e(x)) = R(s)^{-1} \sum_{[s^t y = x]} e(y)$$

yields

$$\hat{\alpha}_s(\langle x, \cdot \rangle) = R(s)^{-1} \sum_{[s^t y = x]} \langle y, \cdot \rangle = R(s)^{-1} \sum_{[s^t y = x]} \langle (s^t)^{-1} s^t y, \cdot \rangle$$

which, when evaluated at $w \in \mathcal{R}^d$, equals $\langle x, s^{-1}w \rangle$ or zero, according to whether $w \in s\mathcal{R}^d$ or not. Hence $\beta_s \circ i = i \circ \alpha_s$, where $i : C(\mathcal{R}^d) \rightarrow C_0(\mathcal{A}_f^d)$ is the obvious inclusion.

Since for every $s \in S$ there is a positive integer $m \in \mathbb{N}^\times$ such that $ms^{-1} \in S$, the semigroup \mathbb{N}^\times of multiplicative positive integers is cofinal in S , and from this it follows easily that $\bigcup_S \beta_{s^{-1}} C(\mathcal{R}^d)$ is dense in $C_0(\mathcal{A}_f^d)$, because $\bigcup_{m \in \mathbb{N}^\times} m^{-1} \mathcal{R}^d$ is dense in \mathcal{A}_f^d . The result now follows from Theorems 2.1 and 2.4 of [11]. \square

Example 2.5. Suppose \mathcal{K} is a number field with ring of integers \mathcal{O} , and fix an integer $d \geq 1$. Define

$$\begin{aligned} G &:= GL_d(\mathcal{K}), \\ S &:= GL_d(\mathcal{K}) \cap \mathcal{M}_d(\mathcal{O}), \\ \Gamma_0 &:= \mathcal{O}^d, \\ N &:= \mathcal{K}^d, \end{aligned}$$

and let $\Gamma := N \rtimes G$ where the action of $g \in G$ on N is by $\psi_g(n) = (g^t)^{-1}n$.

Proposition 2.6. *With the above notation, (Γ, Γ_0) is a Hecke pair, whose Hecke C^* -algebra is isomorphic to the semigroup crossed product $C^*(\mathcal{K}^d/\mathcal{O}^d) \rtimes_\alpha S$, and to a corner, corresponding to the characteristic function of the maximal compact subring $\mathcal{R}_\mathcal{K}^d$ of $\mathcal{A}_\mathcal{K}^d$, in the crossed product $C_0(\mathcal{A}_\mathcal{K}^d) \rtimes_\beta GL_d(\mathcal{K})$, where β denotes the action of $GL_d(\mathcal{K})$ on $C_0(\mathcal{A}_\mathcal{K}^d)$ given by $\beta_g(f)(x) = f(g^{-1}x)$.*

The proof consists of an application of Theorem 1.9 and of Theorems 2.1 and 2.4 of [11], and is entirely analogous to that of the preceding example.

Example 2.7. Motivated by Example 4.6 of [5] we consider the following situation. Let R be a commutative unital ring and let Γ_0 be an R -module. Suppose $S \subset R$ is a multiplicatively closed set without zero-divisors, so that one has localizations $S^{-1}R$ of the ring R and $S^{-1}\Gamma_0$ of the module Γ_0 that contain, respectively, copies of R and Γ_0 .

Then $G = S^{-1}S$ is a subgroup of the unit group of $S^{-1}R$, there is an action ψ of G on $N := S^{-1}\Gamma_0$ given simply by $\psi_g(n) = g^{-1}n$, and we define $\Gamma := N \rtimes G$.

Proposition 2.8. *Assume $|s^{-1}\Gamma_0/\Gamma_0| < \infty$ for each $s \in S$ and let $\Gamma := S^{-1}\Gamma_0 \rtimes S^{-1}S$. Then (Γ, Γ_0) is a Hecke pair and its Hecke algebra (respectively C^* -algebra) is the semigroup crossed product $\mathbb{C}(S^{-1}\Gamma_0/\Gamma_0) \rtimes S$ (respectively $C^*(S^{-1}\Gamma_0/\Gamma_0) \rtimes S$).*

Example 2.9. In the above examples Γ_0 is normal in N , so Corollary 1.12 is all that is needed. Next we borrow from [12] an interesting example arising in number theory in which the inclusion $\Gamma_0 \subset N$ is not normal, so that the full generality of Theorems 1.9 and 1.11 is needed. Let \mathcal{K} be a number field, with ring of integers \mathcal{O} , and assume further that \mathcal{K} is quadratic imaginary, so that the unit group \mathcal{O}^* is finite. Consider the group inclusion $\Gamma_0 \subset \Gamma$ with $\Gamma_0 := \begin{pmatrix} 1 & \mathcal{O} \\ 0 & \mathcal{O}^* \end{pmatrix}$ and $\Gamma := \begin{pmatrix} 1 & \mathcal{K} \\ 0 & \mathcal{K}^* \end{pmatrix}$. The map $\begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix} \mapsto x\mathcal{O}^*$ is a homomorphism from Γ onto the group $G := \mathcal{K}^*/\mathcal{O}^*$

of principal (fractional) ideals of \mathcal{K} ; its kernel is $N := \begin{pmatrix} 1 & \mathcal{K} \\ 0 & \mathcal{O}^* \end{pmatrix}$. In general Γ_0 is not normal in N , but the double Γ_0 -coset of $\begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix} \in N$ is $\begin{pmatrix} 1 & y\mathcal{O}^* + \mathcal{O} \\ 0 & \mathcal{O}^* \end{pmatrix}$, which contains at most $|\mathcal{O}^*|$ right cosets, so (N, Γ_0) is a Hecke pair.

Let $S := \mathcal{O}^\times / \mathcal{O}^*$ be the semigroup of principal *integral* ideals, and let $\sigma : (\mathcal{O}^\times / \mathcal{O}^*) \rightarrow \mathcal{O}^\times$ be a cross section of $\mathcal{O}^\times \rightarrow (\mathcal{O}^\times / \mathcal{O}^*)$ (see for example [12, Lemma 2.5]). Then $(a\mathcal{O}^*) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \sigma(a\mathcal{O}^*) \end{pmatrix}$ determines a cross section $\tilde{\sigma}$ of the quotient $\Gamma \rightarrow \mathcal{K}^* / \mathcal{O}^*$, so we have that Γ is a semidirect product

$$\begin{pmatrix} 1 & \mathcal{K} \\ 0 & \mathcal{O}^* \end{pmatrix} \rtimes (\mathcal{K}^* / \mathcal{O}^*),$$

in which the action of $\mathcal{K}^* / \mathcal{O}^*$ is given by $\psi_g \begin{pmatrix} 1 & y \\ 0 & u \end{pmatrix} := \tilde{\sigma}(g) \begin{pmatrix} 1 & y \\ 0 & u \end{pmatrix} \tilde{\sigma}(g)^{-1}$.

For each $a\mathcal{O}^* \in S$ the subgroup $\tilde{\sigma}(a)^{-1}\Gamma_0\tilde{\sigma}(a) = \begin{pmatrix} 1 & a\mathcal{O} \\ 0 & \mathcal{O}^* \end{pmatrix}$ is of finite order in Γ_0 because $|\mathcal{O}/a\mathcal{O}| < \infty$. It follows from Theorem 1.9 that the pair $(\mathcal{K} \rtimes \mathcal{K}^*, \mathcal{O} \rtimes \mathcal{O}^*)$ is a Hecke pair and that

$$\mathcal{H}(\mathcal{K} \rtimes \mathcal{K}^*, \mathcal{O} \rtimes \mathcal{O}^*) \cong \mathcal{H}(\mathcal{K} \rtimes \mathcal{O}^*, \mathcal{O} \rtimes \mathcal{O}^*) \rtimes (\mathcal{O}^\times / \mathcal{O}^*).$$

Moreover, the pair $(\mathcal{K} \rtimes \mathcal{O}^*, \mathcal{O} \rtimes \mathcal{O}^*)$ has a universal Hecke C^* -algebra, and $C_u^*(\Gamma, \Gamma_0) \cong C_u^*(N, \Gamma_0) \rtimes (\mathcal{O}^\times / \mathcal{O}^*)$.

Remark 2.10. The assumption that \mathcal{K} be quadratic imaginary is not necessary for the result, but simplifies the argument; we refer to [12] for the proof of the general statement. Here we limit ourselves to point out that $C_u^*(N, \Gamma_0)$ exists because its generators have finite spectrum and hence uniformly bounded norm in any representation, and that the action of the principal integral ideals $\mathcal{O}^\times / \mathcal{O}^*$ by endomorphisms of $\mathcal{H}(\mathcal{K} \rtimes \mathcal{O}^*, \mathcal{O} \rtimes \mathcal{O}^*)$ is independent of the cross section σ , because the subgroup $\begin{pmatrix} 1 & 0 \\ 0 & \mathcal{O}^* \end{pmatrix}$ leaves Γ_0 invariant.

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