SMOOTH NORMS ON CERTAIN $C(K)$ SPACES

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Abstract. $C(K)$ spaces admit an equivalent $C^\infty$-smooth renorming whenever $K^{(\omega_1)} = \emptyset$.

In this note we consider the problem of finding on a given $C(K)$ space an equivalent norm of the highest possible smoothness. It is a classical result (e.g. [DGZ]) that the existence of an equivalent $C^1$-smooth norm on a Banach space implies that the space is Asplund. On the other hand, deep examples of Haydon ([H], see also [DGZ]) show that not every $C(K)$ Asplund space admits an equivalent Gâteaux smooth renorming.

So far, an equivalent $C^\infty$ renorming was constructed on $C(K)$ spaces where $K^{(\omega)} = \emptyset$ ([GPWZ]), and a $C^1$ norm is guaranteed when $K^{(\omega_1)} = \emptyset$ ([D]). Haydon’s $C^\infty$ renorming techniques work well for certain tree-like compact sets $K$, which may have nonempty derived sets of arbitrary large ordinal number, but their disadvantage is that they put very strong structural restrictions on $K$ (apart from the obvious and necessary scatteredness). This is not accidental, because the above-mentioned example of $C(K)$ without a Gâteaux norm has $K^{(\omega_1)}$ a singleton.

In our note we show the existence of $C^\infty$ renormings whenever $K^{(\omega_1)} = \emptyset$. This is the best possible result without additional structural assumptions on $K$.

However, it is really only a small step towards a desired general theorem linking the existence of $C^\infty$ renorming of $C(K)$ to some other properties of the space, such as the existence of a dual LUR renorming of $C(K)$. For background material and notation we refer to [DGZ].

Definition 1. Let $S \subset \ell_\infty(\Gamma)$, $\Phi : S \to \mathbb{R}$. We say that $\Phi$ locally depends on finitely many coordinates (LDF) if for every $f \in S$ there exist $\varepsilon > 0$, $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $F : \mathbb{R}^n \to \mathbb{R}$ such that:

$$\Phi(g) = F(g(\gamma_1), \ldots, g(\gamma_n)) \text{ whenever } g \in B(f, \varepsilon) \cap S.$$ 

Given $1 > \delta > 0$, find $\phi_\delta : \mathbb{R} \to \mathbb{R}$ such that $\phi_\delta$ is $C^\infty$-smooth, even and convex, and $\phi_\delta([0, 1 - \delta]) = 0$, $\phi_\delta(1) = 1$.

Definition 2. Let $f \in \ell_\infty(\Gamma)$. Put $f^\infty = \inf\{t, \text{card}\{\gamma, |f(\gamma)| > t\} \text{ is finite}\}$. 

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Lemma 3. Let $1 > \delta > 0$, and let $\Phi : \ell_{\infty}(\Gamma) \to \mathbb{R} \cup \{+\infty\}$ be a convex function defined by
\[
\Phi(f) = \sum_{\gamma \in \Gamma} \phi_\delta(f(\gamma)).
\]

Then $\Phi$ restricted to $\{f \in \ell_{\infty}(\Gamma), f^\infty < 1 - \delta\}$ is finite, LDF and $C^\infty$-smooth.

Proof. Given $f, f^\infty < 1 - \delta$, the set $\Theta = \{\gamma \in \Gamma, |f(\gamma)| > f^\infty + \frac{1 - \delta - f^\infty}{2}\}$ is finite. Thus for $g \in B(f, \frac{1 - \delta - f^\infty}{2})$ we have $\Phi(g) = \sum_{\gamma \in \Theta} \phi_\delta(g(\gamma))$, which is a finite sum of $C^\infty$ smooth convex functions. \[\Box\]

Theorem 4. Let $K$ be a scattered compact, $K^{(\omega_1)} = \emptyset$. Then $C(K)$ admits an equivalent LDF and $C^\infty$-smooth norm.

Proof. There is $\Lambda < \omega_1$ such that $K^{(\Lambda)} \neq \emptyset$ is finite and $K^{(\Lambda+1)} = \emptyset$. The space $C_0(K) = \{f \in C(K), f(K^{(\Lambda)}) = 0\}$ is isomorphic to $C(K)$.

Put $L_\alpha = K^{(\alpha)} \setminus K^{(\alpha+1)}$, $\alpha \leq \Lambda$, and fix $\{\delta_\alpha\}_{\alpha \leq \Lambda}$, such that $\delta_\alpha > 0$ and $\prod_{\alpha=0}^\Lambda (1 + \delta_\alpha)$. Let us define a convex function $\Psi : C_0(K) \to \mathbb{R} \cup \{+\infty\}$ by $\Psi(f) = \sum_{\alpha < \Lambda} \sum_{\gamma \in L_\alpha} \phi_\delta(f(\gamma))$.

Our aim will be to show that $\Psi^{-1}([0, \frac12])$ is the unit ball of an equivalent LDF (canonically, $C_0(K) \subset \ell_{\infty}(K)$) and $C^\infty$ smooth norm on $C_0(K)$.

For every $f \in C_{\infty}(K)$, $0 \leq \alpha \leq \Lambda$ put
\[
a_\alpha^f = (f|_{L_\alpha})^\infty, \quad b_\alpha^f = \|f|_{L_\alpha}\|_{\infty}.
\]

Note that $\alpha \to D_\alpha$ is an increasing and thus upper semicontinuous (usc) function on $[0, \Lambda]$ (which is a compact space when considered with its natural interval topology). We have:

\[
(b^f_{\alpha+1} = a_\alpha^f \text{ and } \alpha \to b^f_\alpha \in C[0, \Lambda],
\]

\[
(b^f_\alpha = \lim_{\tau \to \alpha, \tau < \alpha} a_\tau^f, \quad (\ast)
\]

Therefore, for a fixed $f \neq 0$, $\alpha \to D_\alpha \cdot b^f_\alpha$ is usc on $[0, \Lambda]$, so it attains its maximum $M_f$ at some $\beta \in [0, \Lambda]$. Clearly, $M_f \geq D_{\alpha+1} b^f_{\alpha+1} = (1 + \delta_{\alpha+1}) D_\alpha a_\alpha^f$, and so $D_\alpha a_\alpha^f \leq \frac{M_f}{1 + \delta_{\alpha+1}} \leq M_f (1 - \delta_{\alpha+1})$, for all $\alpha \in [0, \Lambda]$.

First note that $D_\beta b^f_\beta > (D_\alpha a_\alpha^f)^\infty$. Indeed, otherwise there exist increasing sequences $\alpha_n \nearrow \gamma$, $\alpha_n \in [0, \Lambda)$, $D_\alpha a_\alpha^f \nearrow D_\beta b^f_\beta$. However, by (\ast), $a_\alpha^f \to b^f_\gamma$, $D_\gamma > (1 + \delta_\gamma) D_\alpha$ for $n \in \mathbb{N}$, and consequently $D_\alpha b^f_\alpha \geq (1 + \delta_\gamma) D_\beta b^f_\beta$, a contradiction. So there exists $\varepsilon_f > 0$, such that $\text{card}\{\alpha, D_\alpha a_\alpha^f > D_\beta b^f_\beta - \varepsilon_f\}$ is finite. Next we claim that $\text{card}\{\alpha, D_\alpha a_\alpha^f > D_\beta b^f_\beta - \varepsilon_f\}$ is also finite.

Again, otherwise there exists $\alpha_n \nearrow \gamma$, $D_\alpha a_\alpha^f \nearrow D_\beta b^f_\beta - \varepsilon_f$. Using (\ast), and passing to a suitable subsequence of $\{\alpha_n\}$ we find $\beta_n \nearrow \gamma$, $\alpha_n \leq \beta_n < \alpha_{n+1}$ such that
\[
D_{\beta_n} a_{\beta_n}^f \geq D_{\alpha_{n+1}} b_{\alpha_{n+1}}^f - \frac{\varepsilon_f}{2}.
\]

This is a contradiction with the definition of $\varepsilon_f$, since
\[
\{\beta_n\} \subset \{\alpha, D_\alpha a_\alpha^f > D_\beta b^f_\beta - 2\varepsilon_f\}.
\]
Put $O = \{ f \in C_0(K), M_f < 1 \} \subset 2B_{C_0(K)}$. We claim that $\Psi|_O$ is finite, $C^\infty$-smooth and LDF. Moreover, $\Psi^{-1}([0, \frac{1}{2}]) \subset \text{int} \ O$, which by the implicit function theorem (D) finishes the proof.

Choose any $f \in O$. Consider the finite set $A = \{ \alpha \in [0, A], \text{either} \ D_\alpha b_{\alpha}^f > M_f - \varepsilon_f \text{ or } \delta_{\alpha + 1} \geq \frac{\varepsilon_f}{4} \}$, and put $\delta_f = \min\{\frac{\delta_{\alpha + 1}}{16}, \alpha \in A\}$. Let us check that $|\Psi|_{B(f, \delta_f)}$ depends on finitely many coordinates (therefore it is necessarily $C^\infty$-smooth). If $\gamma \in L_\alpha$, $\alpha \notin A$, then $4\delta_{\alpha + 1} < \varepsilon_f$ and $D_\alpha b_{\alpha}^f \leq M_f - \varepsilon_f \leq 1 - \varepsilon_f \leq 1 - 4\delta_{\alpha + 1}$. For $g \in B(f, \delta_f)$, $|g(\gamma)| \leq |f(\gamma)| + \delta_f$, so

$$|D_{\alpha}g(\gamma)| \leq D_{\alpha} b_{\alpha}^f + 2\delta_f \leq 1 - \varepsilon_f + 2\delta_f \leq 1 - 2\delta_{\alpha + 1}.$$  

Consequently,

$$\psi_\alpha(D_{\alpha}g(\gamma)) = 0 \quad \text{and} \quad \Psi(g) = \sum_{\alpha \in A} \sum_{\gamma \in L_\alpha} \psi_\alpha(D_{\alpha}g(\gamma)).$$

If $\alpha \in A$, $\gamma \in L_\alpha$, and $|f(\gamma)| < a_{\alpha}^f (1 + \frac{\delta_{\alpha + 1}}{8})$, then

$$|D_{\alpha}f(\gamma)| \leq D_{\alpha} a_{\alpha}^f \left(1 + \frac{\delta_{\alpha + 1}}{8}\right) \leq M_f \left(1 - \frac{\delta_{\alpha + 1}}{2}\right) \left(1 + \frac{\delta_{\alpha + 1}}{8}\right) \leq 1 - \frac{\delta_{\alpha + 1}}{4}. $$

If $g \in B(f, \delta_f)$, we then have

$$|D_{\alpha}g(\gamma)| \leq 1 - \frac{\delta_{\alpha + 1}}{4} + 2\delta_f \leq 1 - \frac{\delta_{\alpha + 1}}{8}.$$  

Consequently, in this case also $\psi_\alpha(D_{\alpha}g(\gamma)) = 0$. The remaining set $S = \{ \gamma \in L_\alpha \text{ for } \alpha \in A \text{ and } f(\gamma) \geq a_{\alpha}^f (1 + \frac{\delta_{\alpha + 1}}{8}) \}$ is clearly finite, and we have

$$\Psi(g) = \sum_{\alpha \in A} \sum_{\gamma \in S \setminus L_\alpha} (D_{\alpha}g(\gamma))$$

whenever $g \in B(f, \delta_f)$. This proves (Lemma 3) that $\Psi|_O$ is $C^\infty$-smooth and LDF. It is obvious that $M_f \leq \frac{1}{4}$ implies $\Psi(f) = 0$ and $M_f = 1$ implies $\Psi(f) \geq 1$. Thus $B = \Psi^{-1}([0, \frac{1}{2}])$ is an equivalent unit ball of $C_0(K)$. By the implicit function theorem, its Minkowski functional is $C^\infty$-smooth and LDF. 

\[ \square \]

REFERENCES


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