SMOOTH NORMS ON CERTAIN \( C(K) \) SPACES

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Abstract. \( C(K) \) spaces admit an equivalent \( C^\infty \)-smooth renorming whenever \( K^{(\omega_1)} = \emptyset \).

In this note we consider the problem of finding on a given \( C(K) \) space an equivalent norm of the highest possible smoothness. It is a classical result (e.g. \[DGZ\]) that the existence of an equivalent \( C^1 \)-smooth norm on a Banach space implies that the space is Asplund. On the other hand, deep examples of Haydon (\[H\], see also \[DGZ\]) show that not every \( C(K) \) Asplund space admits an equivalent Gâteaux smooth renorming.

So far, an equivalent \( C^\infty \) renorming was constructed on \( C(K) \) spaces where \( K^{(\omega_0)} = \emptyset \) (\[GPWZ\]), and a \( C^1 \) norm is guaranteed when \( K^{(\omega_1)} = \emptyset \) (\[D\]). Haydon’s \( C^\infty \) renorming techniques work well for certain tree-like compact sets \( K \), which may have nonempty derived sets of arbitrary large ordinal number, but their disadvantage is that they put very strong structural restrictions on \( K \) (apart from the obvious and necessary scatteredness). This is not accidental, because the above-mentioned example of \( C(K) \) without a Gâteaux norm has \( K^{(\omega_1)} \) a singleton.

In our note we show the existence of \( C^\infty \) renormings whenever \( K^{(\omega_1)} = \emptyset \). This is the best possible result without additional structural assumptions on \( K \).

However, it is really only a small step towards a desired general theorem linking the existence of \( C^\infty \) renorming of \( C(K) \) to some other properties of the space, such as the existence of a dual LUR renorming of \( C(K) \). For background material and notation we refer to \[DGZ\].

Definition 1. Let \( S \subset \ell_\infty(\Gamma) \), \( \Phi : S \to \mathbb{R} \). We say that \( \Phi \) locally depends on finitely many coordinates (LDF) if for every \( f \in S \) there exist \( \varepsilon > 0 \), \( \gamma_1, \ldots, \gamma_n \in \Gamma \) and \( F : \mathbb{R}^n \to \mathbb{R} \) such that:

\[
\Phi(g) = F(g(\gamma_1), \ldots, g(\gamma_n)) \quad \text{whenever } g \in B(f, \varepsilon) \cap S.
\]

Given \( 1 > \delta > 0 \), find \( \phi_\delta : \mathbb{R} \to \mathbb{R} \) such that \( \phi_\delta \) is \( C^\infty \)-smooth, even and convex, and \( \phi_\delta([0, 1-\delta]) = 0 \), \( \phi_\delta(1) = 1 \).

Definition 2. Let \( f \in \ell_\infty(\Gamma) \). Put \( f^\infty = \inf \{ t, \text{card}\{ \gamma, |f(\gamma)| > t \} \text{ is finite} \} \).
Lemma 3. Let $1 > \delta > 0$, and let $\Phi : \ell_\infty(\Gamma) \to \mathbb{R} \cup \{+\infty\}$ be a convex function defined by

$$\Phi(f) = \sum_{\gamma \in \Gamma} \phi_\delta(f(\gamma)).$$

Then $\Phi$ restricted to $\{f \in \ell_\infty(\Gamma), f^\infty < 1 - \delta\}$ is finite, LDF and $C^\infty$-smooth.

Proof. Given $f, f^\infty < 1 - \delta$, the set $\Theta = \{\gamma \in \Gamma, |f(\gamma)| > f^\infty + \frac{1 - \delta}{2}\}$ is finite. Thus for $g \in B(f, \frac{1 - \delta - f^\infty}{2})$ we have $\Phi(g) = \sum_{\gamma \in \Theta} \phi_\delta(g(\gamma))$, which is a finite sum of $C^\infty$ smooth convex functions. $\square$

Theorem 4. Let $K$ be a scattered compact, $K^{(\omega_1)} = \emptyset$. Then $C(K)$ admits an equivalent LDF and $C^\infty$-smooth norm.

Proof. There is $\Lambda < \omega_1$ such that $K^{(\Lambda)} \neq \emptyset$ is finite and $K^{(\Lambda+1)} = \emptyset$. The space $C_0(K) = \{f \in C(K), f(K^{(\Lambda)}) = 0\}$ is isomorphic to $C(K)$.

Put $L_\alpha = K^{(\alpha)} \setminus K^{(\alpha+1)}$, $\alpha \leq \Lambda$, and fix $\{\delta_\alpha\}_{\alpha \leq \Lambda}$ such that $\delta_\alpha > 0$ and

$$\prod_{\alpha=0}^\Lambda (1 + \delta_\alpha).$$

Put $\psi_\alpha = \phi_{\delta_{\alpha+1}}$, $D_\alpha = \prod_{\beta=0}^\alpha (1 + \delta_\beta)$. Let us define a convex function $\Psi : C_0(K) \to \mathbb{R} \cup \{+\infty\}$ by $\Psi(f) = \sum_{0 \leq \alpha \leq \Lambda} \sum_{\gamma \in \Lambda} \phi_\delta(D_\alpha \cdot f(\gamma))$.

Our aim will be to show that $\Psi^{-1}(\{0, \frac{1}{2}\})$ is the unit ball of an equivalent LDF (canonically, $C_0(K) \subset \ell_\infty(K)$) and $C^\infty$ smooth norm on $C_0(K)$

For every $f \in C_\infty(K)$, $0 \leq \alpha \leq \Lambda$ put

$$a_\alpha^f = (f|_{L_\alpha})^\infty, \quad b_\alpha^f = \|f|_{L_\alpha}\|_\infty.$$

Note that $\alpha \to D_\alpha$ is an increasing and thus upper semicontinuous (usc) function on $[0, \Lambda]$ (which is a compact space when considered with its natural interval topology). We have:

$$b_{\alpha+1}^f = a_\alpha^f \quad \text{and} \quad \alpha \to b_\alpha^f \in C[0, \Lambda],$$

(*)

$$b_\alpha^f = \lim_{\tau/\alpha, \tau < \alpha} a_\alpha^f.$$

Therefore, for a fixed $f \neq 0$, $\alpha \to D_\alpha \cdot b_\alpha^f$ is usc on $[0, \Lambda]$, so it attains its maximum $M_f$ at some $\beta \in [0, \Lambda]$. Clearly, $M_f \geq D_{\alpha+1}b_{\alpha+1}^f = (1 + \delta_{\alpha+1})D_\alpha a_\alpha^f$, and so $D_\alpha a_\alpha^f \leq \frac{M_f}{1 + \delta_{\alpha+1}} \leq M_f (1 - \frac{\delta_{\alpha+1}}{2})$, for all $\alpha \in [0, \Lambda]$.

First note that $D_\beta b_\beta^f > (D_\alpha a_\alpha^f)^\infty$. Indeed, otherwise there exist increasing sequences $\alpha_n \not\to \gamma$, $\alpha_n \in [0, \Lambda)$, $D_\alpha a_\alpha^f \not\to D_\beta b_\beta^f$. However, by (*), $a_\alpha^f \to b_\gamma^f$, $D_\gamma > (1 + \delta_\gamma)D_{\alpha_n}$ for $n \in \mathbb{N}$, and consequently $D_\gamma b_\gamma^f \geq (1 + \delta_\gamma)D_\beta b_\beta^f$, a contradiction. So there exists $\varepsilon_f > 0$, such that $\text{card}\{\alpha, D_\alpha a_\alpha^f > D_\beta b_\beta^f - \varepsilon_f\}$ is finite. Next we claim that $\text{card}\{\alpha, D_\alpha a_\alpha^f > D_\beta b_\beta^f - \varepsilon_f\}$ is also finite.

Again, otherwise there exists $\alpha_n \not\to \gamma$, $D_{\alpha_n} b_{\alpha_n}^f > D_\beta b_\beta^f - \varepsilon_f$. Using (*), and passing to a suitable subsequence of $\{\alpha_n\}$ we find $\beta_n \not\to \gamma$, $\alpha_n \leq \beta_n < \alpha_{n+1}$ such that

$$D_{\beta_n} b_{\beta_n}^f \geq D_{\alpha_{n+1}} b_{\alpha_{n+1}}^f - \frac{\varepsilon_f}{2}.$$

This is a contradiction with the definition of $\varepsilon_f$, since

$$\{\beta_n\} \subset \{\alpha, D_\alpha a_\alpha^f > D_\beta a_\beta^f - 2\varepsilon_f\}.$$
Put $O = \{ f \in C_0(K), M_f < 1 \} \subset 2B_{C_0(K)}$. We claim that $\Psi|_O$ is finite, $C^\infty$-smooth and LDF. Moreover, $\Psi^{-1}([0, \frac{1}{2}]) \subset \text{int } O$, which by the implicit function theorem \([D3]\) finishes the proof.

Choose any $f \in O$. Consider the finite set $A = \{ \alpha \in [0, A], \text{ either } D_\alpha b_{\alpha}^f > M_f - \varepsilon_f \text{ or } \delta_{\alpha+1} \geq \frac{\varepsilon_f}{4} \}$, and put $\delta_f = \min \{ \frac{\delta_{\alpha+1}}{16}, \alpha \in A \}$. Let us check that $\Psi|_{B(f, \delta_f)}$ depends on finitely many coordinates (therefore it is necessarily $C^\infty$-smooth). If $\gamma \in L_\alpha$, $\alpha \notin A$, then $4\delta_{\alpha+1} < \varepsilon_f$ and $D_\alpha b_{\alpha}^f \leq M_f - \varepsilon_f \leq 1 - \varepsilon_f \leq 1 - 4\delta_{\alpha+1}$. For $g \in B(f, \delta_f)$, $|g(\gamma)| \leq |f(\gamma)| + \delta_f$, so

$$|D_\alpha g(\gamma)| \leq D_\alpha b_{\alpha}^f + 2\delta_f \leq 1 - \varepsilon_f + 2\delta_f \leq 1 - 2\delta_{\alpha+1}.$$ Consequently, 

$$\psi_\alpha(D_\alpha g(\gamma)) = 0 \quad \text{and} \quad \Psi(g) = \sum_{\alpha \in A} \sum_{\gamma \in L_\alpha} \psi_\alpha(D_\alpha g(\gamma)).$$ If $\alpha \in A$, $\gamma \in L_\alpha$, and $|f(\gamma)| < a_\alpha^f (1 + \frac{\delta_{\alpha+1}}{8})$, then

$$|D_\alpha f(\gamma)| \leq D_\alpha a_{\alpha}^f \left( 1 + \frac{\delta_{\alpha+1}}{8} \right) \leq M_f \left( 1 - \frac{\delta_{\alpha+1}}{2} \right) \left( 1 + \frac{\delta_{\alpha+1}}{8} \right) \leq 1 - \frac{\delta_{\alpha+1}}{4}.$$ If $g \in B(f, \delta_f)$, we then have

$$|D_\alpha g(\gamma)| \leq 1 - \frac{\delta_{\alpha+1}}{4} + 2\delta_f \leq 1 - \frac{\delta_{\alpha+1}}{8}.$$ Consequently, in this case also $\psi_\alpha(D_\alpha g(\gamma)) = 0$. The remaining set $S = \{ \gamma, \gamma \in L_\alpha \text{ for } \alpha \in A \}$ and $f(\gamma) \geq a_{\alpha}^f (1 + \frac{\delta_{\alpha+1}}{8})$ is clearly finite, and we have

$$\Psi(g) = \sum_{\alpha \in A} \sum_{\gamma \in S \setminus L_\alpha} (D_\alpha g(\gamma))$$ whenever $g \in B(f, \delta_f)$. This proves (Lemma 3) that $\Psi|_O$ is $C^\infty$-smooth and LDF. It is obvious that $M_f \leq \frac{1}{2}$ implies $\Psi(f) = 0$ and $M_f = 1$ implies $\Psi(f) \geq 1$. Thus $B = \Psi^{-1}([0, \frac{1}{2}])$ is an equivalent unit ball of $C_0(K)$. By the implicit function theorem, its Minkowski functional is $C^\infty$-smooth and LDF. \hfill $\square$

**References**


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