

VECTOR BUNDLES WITH INFINITELY MANY SOULS

IGOR BELEGRADEK

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ABSTRACT. We construct the first examples of manifolds, the simplest one being $S^3 \times S^4 \times \mathbb{R}^5$, which admit infinitely many complete nonnegatively curved metrics with pairwise nonhomeomorphic souls.

According to the soul theorem of J. Cheeger and D. Gromoll [CG72], a complete open manifold of nonnegative sectional curvature is diffeomorphic to the total space of the normal bundle of a compact totally geodesic submanifold, called a soul. The soul is not unique but any two souls are mapped to each other by an ambient diffeomorphism inducing an isometry on the souls [Sha79]. In this note we show that the homeomorphism type of the soul generally depends on the metric; namely the following is true.

Theorem 1. *There exist infinitely many complete Riemannian metrics on $S^3 \times S^4 \times \mathbb{R}^5$ with $\text{sec} \geq 0$ and pairwise nonhomeomorphic souls.*

The proof applies some classical techniques of geometric topology to recent examples of nonnegatively curved manifolds due to K. Grove and W. Ziller [GZ00]. As we explain below, it is much easier to produce a manifold with *finitely many* nonnegatively curved metrics having nonhomeomorphic souls, however the full power of [GZ00] is needed to get infinitely many such metrics.

Grove and Ziller [GZ00] showed that any principal $S^3 \times S^3$ -bundle over S^4 admits an $S^3 \times S^3$ -invariant metric with $\text{sec} \geq 0$. By O’Neill’s formula, all associated bundles admit metrics with $\text{sec} \geq 0$ which gives rise to a rich class of examples, including all sphere bundles over S^4 with structure group $SO(4)$. Note that the souls in Theorem 1 are the total spaces of S^3 -bundles over S^4 with structure group $SO(3)$. Theorem 1 is a particular case of the following.

Theorem 2. *Let ξ be a rank n vector bundle over S^4 with structure group $SO(3)$, let $q: S \rightarrow S^4$ be a smooth S^{m-1} -bundle with structure group $SO(3)$, and let η be the q -pullback of ξ . If $m = 4$, $n > 4$, or if $m > 4$, $n > m + 3$, then the total space of η admits infinitely many complete Riemannian metrics with $\text{sec} \geq 0$ and pairwise nonhomeomorphic souls.*

The main topological tool used in this paper is a result of L. Siebenmann [Sie69] that generalizes the famous Masur’s theorem: any tangential homotopy equivalence of closed smooth n -manifolds is homotopic to a diffeomorphism after taking the

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product with the identity map of \mathbb{R}^{n+1} . Exotic 7-spheres are stably parallelizable, so they all become diffeomorphic after taking the product with \mathbb{R}^8 , and in fact it suffices to take product with \mathbb{R}^5 . A homotopy 7-sphere is called a *Milnor sphere* if it is diffeomorphic to the total space of an S^3 -bundle over S^4 ; it is known that 10 out of 14 homotopy 7-spheres are Milnor. According to [GZ00], any Milnor sphere carries a metric with $\text{sec} \geq 0$, which leads to the following.

Proposition 3. *For every Milnor sphere Σ , the manifold $S^7 \times \mathbb{R}^5$ has a complete Riemannian metric of $\text{sec} \geq 0$ with soul diffeomorphic to Σ .*

Gromoll and Tapp recently classified [GT01] all nonnegatively curved metrics on $S^2 \times \mathbb{R}^2$, and asked for a similar classification on $S^n \times \mathbb{R}^k$. The above proposition indicates that such a classification would be rather involved.

It would be interesting to classify the total spaces of S^3 -bundles over S^4 with nonzero Euler class up to tangential homotopy equivalence. Indeed, by Masur's theorem such a classification should lead to an analog of Proposition 3, with S^7 replaced by any S^3 -bundle S over S^4 with nonzero Euler class. It is worth mentioning that nonvanishing of the Euler class implies that the homotopy type of S contains at most finitely many nondiffeomorphic S^3 -bundles over S^4 [Tam58] (cf. [CE00]).

It is easy to construct manifolds admitting two nonnegatively curved metrics with pairwise nonhomeomorphic souls. Here we describe two such situations where souls are lens spaces, or simply-connected homogeneous manifolds.

Example 4. It is well known [Coh73, pp. 96, 100] that the lens spaces $L(7, 1)$ and $L(7, 2)$ are homotopy equivalent, but not simply-homotopy equivalent (hence nonhomeomorphic). Orientable 3-manifolds are parallelizable, so any homotopy equivalence of $L(7, 1)$, $L(7, 2)$ is tangential, and therefore [Sie69, Theorem 2.3] we get a diffeomorphism of manifolds $L(7, 1) \times \mathbb{R}^4$ and $L(7, 2) \times \mathbb{R}^4$ admitting obvious product metrics of $\text{sec} \geq 0$ with souls $L(7, 1) \times \{0\}$, $L(7, 2) \times \{0\}$. Since every homotopy type contains at most finitely many nonhomeomorphic lens spaces, this procedure yields only finitely many pairwise nonhomeomorphic souls on a given manifold.

Proposition 5. *If i is a positive integer divisible by 24, then there is a compact homogeneous space G/H homotopy equivalent but not homeomorphic to $S^{4i-1} \times S^4$, such that for $n > 4i + 3$, the manifold $G/H \times \mathbb{R}^n$ carries two complete nonnegatively curved metrics with souls diffeomorphic to G/H and $S^{4i-1} \times S^4$.*

The above-mentioned result of Siebenmann [Sie69, Theorem 2.2] states that if ξ is a rank > 2 vector bundle over a compact manifold, and the total space $E(\xi)$ of ξ contains a smooth compact submanifold S such that the inclusion $S \rightarrow E(\xi)$ is a homotopy equivalence, then $E(\xi)$ has a vector bundle structure with zero section S .

Then it easily follows [Sie69, Theorem 2.3] that if t is tangential homotopy equivalence of n -manifolds, each being the total space of a vector bundle over a compact smooth manifold of dimension less than the rank of the bundle, then t is homotopic to a diffeomorphism. The upshot is that the total space of vector bundle of sufficiently large rank often has many other vector bundle structures, perhaps with nonhomeomorphic base spaces, and this fact plays a crucial role in this note.

Proof of Theorem 2. In what follows we denote the total space of a vector bundle ζ by $E(\zeta)$, and the associated sphere bundle by $S(\zeta)$. Principal $SO(3)$ -bundles

over S^4 are in one-to-one correspondence with $\pi_4(BSO(3)) \cong \mathbb{Z}$. Let P_k be the principal $SO(3)$ -bundle over S^4 corresponding to $k \in \pi_4(BSO(3))$. Let ξ_k^n be the rank n vector bundle over S^4 associated with P_k via the standard inclusion $SO(3) \rightarrow SO(n)$.

Let $P_{k,l}$ be the principal $SO(3) \times SO(3)$ -bundle over S^4 which is the pullback of $P_k \times P_l$ via the diagonal map $\Delta: S^4 \rightarrow S^4 \times S^4$. According to [GZ00], $P_{k,l}$ admits an $SO(3) \times SO(3)$ -invariant metric with $\text{sec} \geq 0$. Consider the $S^{m-1} \times \mathbb{R}^n$ -bundle

$$\Delta^\#(S(\xi_k^m) \times \xi_l^n) = S^{m-1} \times_{SO(3) \times 1} P_{k,l} \times_{1 \times SO(3)} \mathbb{R}^n$$

associated with $P_{k,l}$. The total space of $\Delta^\#(S(\xi_k^m) \times \xi_l^n)$ is also the total space of a rank n vector bundle over $S(\xi_k^m)$ which we denote by $\eta_{k,l}^{m,n}$. Note that $\eta_{k,l}^{m,n}$ is the pullback of ξ_l^n via the projection $q_k^m: S(\xi_k^m) \rightarrow S^4$. By O'Neill's formula for submersions, $E(\eta_{k,l}^{m,n})$ carries a complete metric with $\text{sec} \geq 0$ with zero section being a soul.

First, note that $S(\xi_k^m)$ is fiber homotopy equivalent to $S(\xi_i^m)$ if $k \equiv i \pmod{12}$. Indeed, S^{m-1} -fibrations over S^4 are classified, up to fiber homotopy equivalence, by $\pi_3(SG_m)$ where SG_m is the space of orientation-preserving self-homotopy equivalences of S^{m-1} . The fibrations $S(\xi_k^m)$ are classified by the image of

$$\phi: \pi_3(SO(3)) \rightarrow \pi_3(SO(m)) \rightarrow \pi_3(SG_m).$$

Since $m \geq 4$, ϕ factors as $\pi_3(SO(3)) \rightarrow \pi_3(SF_3) \rightarrow \pi_3(SG_4) \rightarrow \pi_3(SG_m)$ where SF_3 is the space of base-point-preserving elements of SG_4 . It is well known that SF_3 is the identity component of the loop space $\Omega^3 S^3$ [MM79, Chapter 3], hence $\pi_3(SF_3) \cong \pi_6(S^3)$. Thus, ϕ factors through the J -homomorphism $\mathbb{Z} \cong \pi_3(SO(3)) \rightarrow \pi_6(S^3) \cong \mathbb{Z}_{12}$, and the result follows.

Second, show that $S(\xi_k^m)$ is homeomorphic to $S(\xi_i^m)$ iff $k = \pm i$ iff $S(\xi_k^m)$ is diffeomorphic to $S(\xi_l^m)$. Indeed, the tangent bundle of $S(\xi_k^m)$ is stably isomorphic to the q_k^m -pullback of ξ_k^m , because $S(\xi_k^m)$ is a two-sided hypersurface in $E(\xi_k^m)$. Since ξ_k^m is an $SO(3)$ -bundle, the first Pontrjagin class of ξ_k^m is the $\pm 4k$ -multiple of a generator of $H^4(S^4, \mathbb{Z})$ (cf. [Mil56]). Also $S(\xi_k^m)$ has a section so that q_k^m induces an isomorphism on the 4th cohomology, hence $p_1(TS(\xi_k^m))$ is the $\pm 4k$ -multiple of a generator. By the topological invariance of rational Pontrjagin classes, $S(\xi_k^m)$ is not homeomorphic to $S(\xi_i^m)$ unless $k = \pm i$. Finally, P_k is the pullback of P_{-k} via an orientation-reversing self-diffeomorphism of S^4 , so $S(\xi_k^m)$ and $S(\xi_{-k}^m)$ are diffeomorphic.

Third, note that $E(\eta_{k,l}^{m,n})$ and $E(\eta_{i,j}^{m,n})$ are tangentially homotopy equivalent provided $k \equiv i \pmod{12}$, and $k+l = i+j$. Indeed, fix k, l, i, j with these properties. The tangent bundle to $E(\eta_{k,l}^{m,n})$ is determined by its restriction to the zero section. This restriction is the q_k^m -pullback of $\xi_k^m \oplus \xi_l^n$ for any k, l . The images of ξ_k^m, ξ_l^n under the homomorphism

$$\pi_4(BSO(3)) \rightarrow \pi_4(BSO)$$

add up to $k+l$, and the addition in $\pi_4(BSO)$ is given by the Whitney sum \oplus . Thus $k+l = i+j$ implies that $\xi_k^m \oplus \xi_l^n$ and $\xi_i^m \oplus \xi_j^n$ are stably isomorphic. Since $k \equiv i \pmod{12}$ there is a fiber homotopy equivalence $g: S(\xi_k^m) \rightarrow S(\xi_i^m)$, so that $g \circ q_i^m$ is homotopic to q_k^m . Therefore, g induces a tangential homotopy equivalence $t: E(\eta_{k,l}^{m,n}) \rightarrow E(\eta_{i,j}^{m,n})$, which is the composition of the projection of $\eta_{k,l}^{m,n}$, followed by g , and then by the zero section of $\eta_{i,j}^{m,n}$.

Next we show that $E(\eta_{k,l}^{m,n})$ and $E(\eta_{i,j}^{m,n})$ are diffeomorphic if $k \equiv i \pmod{12}$, and $k+l = i+j$. By Haefliger's embedding theorem [Hae61], the restriction of t to the zero section of $\eta_{k,l}^{m,n}$ is homotopic to a smooth embedding f because $2n \geq m+6$, which means we are in metastable range. Since $n \geq 3$, [Sie69, Theorem 2.2] implies that $E(\eta_{i,j}^{m,n})$ has a vector bundle structure with zero section f . Since t is tangential, $\eta_{k,l}^{m,n}$ is stably isomorphic to ν_f which is the normal bundle to f .

In fact, our assumptions on n, m imply that $\eta_{k,l}^{m,n}$ and ν_f are isomorphic. Indeed, if $m > 4$, $n > m+3 = \dim(S(\xi_k^m))$, then we are in stable range, hence $\eta_{k,l}^{m,n} \cong \nu_f$. If $m = 4$, $n > 4$, we apply obstruction theory comparing ν_f , $\eta_{k,l}^{m,n}$, which are thought of as classifying maps from $S(\xi_k^m)$ to $BSO(n)$. Since $S(\xi_k^m)$ has a section, it can be obtained by attaching a 7-cell to $S^3 \vee S^4$. The bundles ν_f , $\eta_{k,l}^{m,n}$ are isomorphic on the 6-skeleton $S^3 \vee S^4$, because they are stably isomorphic and $n > 4 = \dim(S^3 \vee S^4)$. Comparing ν_f , $\eta_{k,l}^{m,n}$ on the 7-cell, we get a map $S^7 \rightarrow BSO(n)$ which is nullhomotopic since $\pi_7(BSO(n)) = 0$ if $n > 4$ [Mim95, page 970]). Thus, ν_f , $\eta_{k,l}^{m,n}$ are isomorphic, and hence $E(\eta_{k,l}^{m,n})$ and $E(\eta_{i,j}^{m,n})$ are diffeomorphic.

To summarize, each manifold $E(\eta_{k,l}^{m,n})$ has infinitely many vector bundle structures $\eta_{i,j}^{m,n}$ with base manifolds $S(\xi_i^m)$ for any i, j satisfying $k \equiv i \pmod{12}$, and $k+l = i+j$, and the proof is complete. \square

Proof of Proposition 3. Let Σ be a homotopy 7-sphere of $\text{sec} \geq 0$ [GM74, GZ00]. The product metric on $\Sigma \times \mathbb{R}^n$ has $\text{sec} \geq 0$ with a soul $\Sigma \times \{0\}$. By [Hae61] any homotopy equivalence $S^7 \rightarrow \Sigma \times \mathbb{R}^n$ is homotopic to a smooth embedding if $n \geq 5$, so $\Sigma \times \mathbb{R}^n$ gets a structure of a vector bundle over S^7 which is necessarily trivial since $\pi_7(BSO(n)) = 0$ for $n \geq 5$. Thus, $\Sigma \times \mathbb{R}^5$ is diffeomorphic to $S^7 \times \mathbb{R}^5$, as promised. \square

Proof of Proposition 5. Kamerich [Kam77, page 116] (cf. [Oni94, page 275]) showed that if i is a positive integer divisible by 24, then there is a compact homogeneous space G/H homotopy equivalent but not homeomorphic to $S^{4i-1} \times S^4$. Here $G = Sp(i) \times Sp(2)$ and $H = Sp(i-1) \times Sp(1) \times Sp(1)$, where $Sp(i-1)$ is embedded in $Sp(i)$ is the standard way, the first $Sp(1)$ is embedded into $Sp(i) \times Sp(2)$ diagonally so that a quaternion goes to the last diagonal entry of the matrix in $Sp(i)$, and also to the first diagonal entry in $Sp(2)$, while the second $Sp(1)$ goes into the last diagonal entry of $Sp(2)$.

Since $n > 4i+3$, the homotopy equivalence $S^{4i-1} \times S^4 \rightarrow G/H \times \mathbb{R}^n$ is homotopic to a smooth embedding f . By [Sie69, Theorem 2.2], $G/H \times \mathbb{R}^n$ admits an \mathbb{R}^n -bundle structure with zero section f .

Note that any \mathbb{R}^n -bundle ξ over $S^{4i-1} \times S^4$ is the pullback of a \mathbb{R}^n -bundle over S^4 via the projection $S^{4i-1} \times S^4 \rightarrow S^4$. This is proved by obstruction theory for maps $S^{4i-1} \times S^4 \rightarrow BSO$ by comparing ξ with the bundle ξ_4 obtained by pullbacking ξ to S^4 via an inclusion $S^4 \rightarrow S^{4i-1} \times S^4$, and then pullbacking it back to $S^{4i-1} \times S^4$ via the projection $S^{4i-1} \times S^4 \rightarrow S^4$. Since $\pi_{4i-1}(BSO) = 0$, ξ and ξ_4 agree on $S^{4i-1} \vee S^4$, and they are homotopic on the top $4i+3$ -cell as $\pi_{4i+3}(BSO) = 0$.

Since each vector bundle over S^4 carries $\text{sec} \geq 0$ with zero section being a soul [GZ00], so does the product of the bundle and S^{4i-1} . Thus $G/H \times \mathbb{R}^n$ gets a metric with $\text{sec} \geq 0$ and soul $S^{4i-1} \times S^4$. On the other hand, $G/H \times \mathbb{R}^n$ has the product metric with soul G/H . \square

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DEPARTMENT OF MATHEMATICS, 253-37, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CALIFORNIA 91125

E-mail address: `ibeleg@its.caltech.edu`