ON QUASINILPOTENT OPERATORS

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Abstract. In this note we modify a new technique of Enflo for producing hyperinvariant subspaces to obtain a much improved version of his “two sequences” theorem with a somewhat simpler proof. As a corollary we get a proof of the “best” theorem (due to V. Lomonosov) known about hyperinvariant subspaces for quasinilpotent operators that uses neither the Schauder-Tychonoff fixed point theorem nor the more recent techniques of Lomonosov.

1. Introduction

Let \( \mathcal{H} \) be a separable, infinite dimensional, complex Hilbert space, and denote by \( \mathcal{L}(\mathcal{H}) \) the algebra of all bounded linear operators on \( \mathcal{H} \) and by \( \mathbf{K} = \mathbf{K}(\mathcal{H}) \) the ideal of compact operators in \( \mathcal{L}(\mathcal{H}) \). Perhaps the first invariant-subspace theorem for operators in \( \mathcal{L}(\mathcal{H}) \), other than those provided by the spectral theorem for normal operators, was that every operator in \( \mathbf{K}(\mathcal{H}) \) has a nontrivial invariant subspace.

According to Aronszajn-Smith [3], this was proved by John von Neumann (unpublished) about 1935. Thus there has now been over a half-century of work devoted to establishing that operators in \( \mathcal{L}(\mathcal{H}) \) that have a nice enough relation to some compact operator have nontrivial invariant subspaces. Without attempting to be exhaustive we mention the papers of Bernstein-Robinson [4], Halmos [9], [10], Arveson-Feldman [1], Deckard-Douglas-Pearcy [7], Pearcy-Salinas [14], Lomonosov [11], [12], [13], Pearcy-Shields [15], Scott Brown [5], and, more recently, Chevreau-Li-Pearcy [6], Simonic [16], Ansari-Enflo [2], and Enflo-Lomonosov [8]. Several of these works took something from previous ones, but many also added new techniques, some dramatically new (for example, the use by Lomonosov in [11] of the Schauder-Tychonoff fixed point theorem for nonlinear mappings).

In [2], a very recent new technique was introduced (and ascribed there to Enflo) for producing invariant subspaces for compact-related operators in \( \mathcal{L}(\mathcal{H}) \). The following old theorem of Lomonosov ([11]; cf. also [15]) was thus given in [2] a completely different proof (neither utilizing the Schauder-Tychonoff fixed point theorem nor the ideas of [12]).

Theorem 1.1. Every nonzero compact operator in \( \mathcal{L}(\mathcal{H}) \) has a nontrivial hyperinvariant subspace.

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As another consequence of this technique, Enflo in [8] obtained the following interesting “two sequences” theorem.

**Theorem 1.2.** Let $A \subset \mathcal{L}(\mathcal{H})$ be any commutative algebra that contains a nonzero quasinilpotent operator. Then there exist sequences $\{s_k\}_{k=1}^{\infty}$ and $\{t_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ that converge weakly to nonzero vectors $s_0$ and $t_0$, respectively, such that for every bounded sequence $\{A_k\}_{k=1}^{\infty} \subset A$,

$$\lim_{k} (A_k s_k, t_k) = 0.$$ 

This technique of proof (from [2] and [8]) uses some “extremal vectors” in a very clever way, and, as was mentioned in [8], is so new that most likely it will be some time before one knows whether the technique (or modifications thereof) will yield all the stronger theorems from [11] and [12] as well as perhaps some completely new results in the same direction.

The purpose of this note is to show that by modifying Enflo’s new technique, a considerably better version of Theorem 1.2, with a somewhat simpler proof, can be obtained as follows.

**Theorem 1.3.** Suppose $Q \neq 0$ is a quasinilpotent operator in $\mathcal{L}(\mathcal{H})$ and $\{Q\}'$ denotes the commutant of $Q$, i.e., $\{Q\}' = \{A \in \mathcal{L}(\mathcal{H}) : AQ = QA\}$. Let $B_0$ be an arbitrary nonzero operator in $\{Q\}'$ such that $B_0Q \neq 0$. Then there exist sequences $\{s_k\}_{k=1}^{\infty}$ and $\{t_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ that converge weakly to nonzero vectors $s_0$ and $t_0$, respectively, with $B_0 s_0 \neq 0$, and a sequence $\{\beta_k\}$ of positive numbers converging to zero, such that for every doubly indexed sequence $\{A_{m,k}\}_{m,k \in \mathbb{N}}$ in the unit ball of $\{Q\}'$, we have

$$|(A_{m,k} s_k, t_k)| < \beta_k, \quad m, k \in \mathbb{N}.$$ 

Also as a corollary of Theorem 1.3, the following better (than Theorem 1.1) but not so old theorem of Lomonosov [12] can be deduced.

**Corollary 1.4** ([12]). Suppose that $Q \neq 0$ is a quasinilpotent operator in $\mathcal{L}(\mathcal{H})$ and there exist a sequence $\{D_m\}_{m \in \mathbb{N}} \subset \{Q\}'$ converging in the weak operator topology to a nonzero $(C$ in $\{Q\}')$ and a sequence $\{K_m\}_{m \in \mathbb{N}}$ of compact operators such that

$$\lim_{m} \|D_m - K_m\| = 0.$$ 

(In other words, in the language of [12], we suppose that $\{Q\}'$ has the Pearcy-Salinas property.) Then $Q$ has a nontrivial hyperinvariant subspace.

In other words, this note may be considered as a first step in the direction of determining what are the best theorems that can be obtained by (modifications of) this new Enflo technique from [2] and [8]. We remark that Corollary 1.4 is the “strongest” theorem known which produces hyperinvariant subspaces for a quasinilpotent operator, so, at least in this direction, Enflo’s new technique produces the “best” theorem known.

2. Some Lemmas

Our proof of Theorem 1.3 depends on several lemmas (essentially) from [2].

**Lemma 2.1.** Suppose $u$ and $v$ are nonzero vectors in $\mathcal{H}$ such that for every $z \in \mathcal{H}$, $\text{Re}(u, z) < 0$ implies that $\text{Re}(v, z) \geq 0$. Then there exists a negative number $r_0$ such that $v = r_0 u$. 

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Thus suppose that \( z \) is a nonzero vector in \( \mathcal{H} \), then for \( \gamma \) a nonzero vector in \( \mathcal{H} \), we must have
\[
\text{Re}(v, z) = \text{Re}(\gamma u + x) = \|u\|^2 \text{Re}(\gamma) < 0.
\]
Thus, for all \( x \) orthogonal to \( u \) and for all \( \gamma \) with \( \text{Re}\gamma < 0 \), we have, by hypothesis,
\[
\text{Re}(v, z(\gamma, x)) = \text{Re}(\alpha_0 u + w, \gamma u + x) = \|u\|^2 \text{Re}(\alpha_0 \gamma) + \text{Re}(w, x) \geq 0.
\]
Upon fixing \( \gamma \) and taking \( x \) to be a large enough negative scalar multiple of \( w \), we see that necessarily \( w = 0 \) and that \( \text{Re}(\alpha_0 \gamma) \geq 0 \). Upon writing \( \alpha_0 = r_0 + i s_0 \) and \( \gamma = t + iq \) where \( r_0, s_0, t, q \) are real, we get that \( r_0 t - s_0 q \geq 0 \) for all \( q \in \mathbb{R} \) and all \( t < 0 \). Fixing \( t \) and letting \( q \) run we get \( s_0 = 0 \) and then \( r_0 \leq 0 \). Since \( v = r_0 u \) and \( v \neq 0 \), we must have \( r_0 < 0 \), so the proof is complete.

Lemma 2.2. Suppose \( T \in \mathcal{L}(\mathcal{H}) \) has dense range. Suppose also that \( x_0 \) is a nonzero vector in \( \mathcal{H} \) and that \( \varepsilon \) satisfies \( 0 < \varepsilon < \|x_0\| \). Then there exists a unique nonzero vector \( y_0 = y_0(x_0, \varepsilon) \) such that
\[
a) \|y_0\| = \inf \{|y| : \|Ty - x_0\| \leq \varepsilon\} \quad \text{and} \quad \text{b)} \quad \|Ty_0 - x_0\| = \varepsilon.
\]

Proof. Let \( \mathcal{F} = \{y \in \mathcal{H} : \|Ty - x_0\| \leq \varepsilon\} \). Since \( T \) has dense range, clearly \( \mathcal{F} \) is nonempty, and since \( T \) is continuous and \( \mathcal{F} \) is the inverse image under \( T \) of the norm-closed ball centered at \( x_0 \) with radius \( \varepsilon \), \( \mathcal{F} \) is a norm-closed set. Moreover an easy calculation shows that \( \mathcal{F} \) is a convex set. But, as is well-known, such a set has a unique vector \( y_0 \) of minimal norm. Thus a) is satisfied, and if \( \|Ty_0 - x_0\| < \varepsilon \), then for \( \delta > 0 \) sufficiently small, \( (1 - \delta)y_0 \) would belong to \( \mathcal{F} \) and have smaller norm, so b) is satisfied.

Lemma 2.3. Suppose \( T, x_0, y_0, \) and \( \varepsilon \) are as in Lemma 2.2. Then there exists a negative number \( r \) such that \( T^*(Ty_0 - x_0) = ry_0 \).

Proof. We apply Lemma 2.1 to \( u = T^*(Ty_0 - x_0) \) and \( v = y_0 \). Clearly it suffices to show that \( u \) and \( v \) satisfy the hypotheses of Lemma 2.1 (and then set \( r = 1/r_0 \)). Thus suppose that \( z_0 \) is any vector in \( \mathcal{H} \) satisfying
\[
\text{Re}(u, z) = \text{Re}(T^*(Ty_0 - x_0), z_0) = \text{Re}(Ty_0 - x_0, Tz_0) < 0.
\]
It follows easily that there exists a sufficiently small interval \( [0, t_0] \), on which the function \( t \mapsto \|T(y_0 + tz_0) - x_0\|^2 \) is strictly monotone decreasing (its derivative is continuous and negative at the origin). Thus for \( t \in (0, t_0] \) we have
\[
\|T(y_0 + tz_0) - x_0\| < \|Ty_0 - x_0\| = \varepsilon.
\]
Thus for \( t \in (0, t_0] \), \( y_0 + tz_0 \in \mathcal{F} \), and by the minimality of \( y_0 \), we must have
\[
|y_0 + tz_0|^2 \geq \|y_0\|^2, \quad t \in (0, t_0].
\]
But the derivative of the function
\[
t \mapsto |y_0 + tz_0|^2
\]
is continuous and its value at the origin is \( 2\text{Re}(y_0, z_0) \), which must therefore satisfy \( \text{Re}(y_0, z_0) \geq 0 \), and the lemma is proved.

Lemma 2.4. Suppose \( T \in \mathcal{L}(\mathcal{H}) \) is quasinilpotent with dense range, let \( x_0 \) be a nonzero vector in \( \mathcal{H} \), let \( \varepsilon \) satisfy \( 0 < \varepsilon < \|x_0\| \), and (via Lemma 2.2) let, for each \( n \in \mathbb{N} \), \( y_n = y_n(\varepsilon, x_0) \) be a (nonzero) vector satisfying
\[
a) \|y_n\| = \inf \{|y| : \|T^ny - x_0\| \leq \varepsilon\} \quad \text{and} \quad \text{b)} \quad |y_n + Tz_0|^2 \geq |y_n|^2, \quad t \in (0, t_0].
\]
b) \(\|T^n y_n - x_0\| = \varepsilon.\)

Then there exists a subsequence \(\{y_{n_k}\}_{k=1}^\infty\) of the sequence \(\{y_n\}\) satisfying

\[
\lim_k \frac{\|y_{n_k}\|}{\|y_{n_k+1}\|} = 0.
\]

Proof. Suppose, to the contrary, that there exist \(t > 0\) and \(N_t \in \mathbb{N}\) such that

\[
\inf_{n \geq N_t} \frac{\|y_n\|}{\|y_{n+1}\|} = t.
\]

Then

\[
\|y_{N_t}\| \geq t \|y_{N_t+1}\| \geq t^2 \|y_{N_t+2}\| \geq \cdots \geq t^n \|y_{N_t+n}\|, \quad n \in \mathbb{N}.
\]

By the minimality of \(\|y_{N_t}\|\) from a), we have (since \(\|T^{N_t+n} y_{N_t+n} - x_0\| = \varepsilon\))

\[
\|T^n y_{N_t+n}\| \geq \|y_{N_t}\|,
\]

\(n \in \mathbb{N}.
\]

Thus

\[
\|T^n\| \geq \|y_{N_t+n}\| \geq t^n \|y_{N_t+n}\|, \quad n \in \mathbb{N},
\]

and hence \(\|T^n\| \geq t^n\), \(n \in \mathbb{N}\), which contradicts the fact that \(\sigma(T) = \{0\}\). The result follows.

3. Proofs of the results

On the basis of these lemmas, we now prove Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3. Let \(B_0 \in \{Q\}^\perp\) be such that \(B_0 Q \neq 0\). If \(\mathcal{M} = \text{(range } Q)^\perp \neq \mathcal{H}\), then \(\mathcal{M}\) is a nontrivial hyperinvariant subspace for \(Q\) and the result follows by choosing nonzero vectors \(s_0 \in \mathcal{M}\) and \(t_0 \in \mathcal{M}^\perp\) such that \(B_0 s_0 \neq 0\) and defining \(s_k = s_0, t_k = t_0, k \in \mathbb{N}\), and \(\beta_k\) to be an arbitrary sequence of positive numbers tending to zero. Thus we may suppose that \(Q\) has dense range (which implies that each \(Q^n\) also has dense range). Let \(x_0\) be a nonzero vector in \(\mathcal{H}\) such that \(B_0 Q x_0 = Q B_0 x_0 \neq 0\), and let \(\varepsilon\) satisfy

\[
0 < \varepsilon < \min\{\|x_0\|, \|Q x_0\|, (1/\|B_0\|) \|B_0 x_0\|\}.
\]

For each \(n \in \mathbb{N}\), let \(y_n = y_n(\varepsilon, x_0\) satisfy a) and b) of Lemma 2.4 (with \(T = Q\)). By Lemma 2.4, we can choose a subsequence \(\{y_{n_k}\}\) of \(\{y_n\}\) such that

\[
\lim_k \frac{\|y_{n_k}\|}{\|y_{n_k+1}\|} = 0.
\]

By dropping down to successive subsequences of \(\{y_{n_k}\}\) we may suppose (without changing the notation accordingly), since all of the vectors \(Q^n y_n, n \in \mathbb{N}\), belong to the norm-closed ball of radius \(\varepsilon\) centered at \(x_0\), that the sequence \(\{Q^n y_{n_k}\}\) converges weakly to a vector \(z_0\) and, similarly, that the sequence \(\{Q^{n_k+1} y_{n_k+1}\}\) converges weakly to a vector \(v_0\). Since norm-closed balls in \(\mathcal{H}\) are weakly closed, we have \(\|v_0 - x_0\| \leq \varepsilon, \|z_0 - x_0\| \leq \varepsilon\) (so, in particular, \(v_0 \neq 0 \neq z_0\), \(\|B_0 z_0 - B_0 x_0\| \leq \|B_0\| \varepsilon,\) and \(B_0 z_0 \neq 0\).

We next show that \(v_0 - x_0 \neq 0\), which shows also (since \(\mathcal{M} = \mathcal{H}\)) that \(Q^t (v_0 - x_0) \neq 0\). By the definition of the \(y_n\) and Lemma 2.3, we have

\[
\varepsilon^2 = \|Q^{n_k+1} y_{n_k+1} - x_0\|^2
= \|Q^{n_k+1} y_{n_k+1} - (x_0, Q^{n_k+1} y_{n_k+1} - x_0)\| - \|Q^{n_k+1} y_{n_k+1} - x_0\|
= r_{n_k+1} \|y_{n_k+1}\|^2 - \|Q^{n_k+1} y_{n_k+1} - x_0\|,
\]

\(k \in \mathbb{N},\)

\(\|Q^{n_k+1} y_{n_k+1} - x_0\| \geq \varepsilon\).
where \( r_{n+1} < 0 \) for all \( k \in \mathbb{N} \). Thus
\[
-\varepsilon^2 \geq (x_0, Q^{n+1}y_{n+1} - x_0), \quad k \in \mathbb{N},
\]
and, taking limits as \( k \to \infty \), we get \(-\varepsilon^2 \geq (x_0, v_0 - x_0)\), so \( v_0 - x_0 \neq 0 \) and \( Q^* (v_0 - x_0) \neq 0 \).

Now define
\[
s_k = Q^{n_k} y_{n_k},
\]
\[
t_k = Q^* (Q^{n_k+1} y_{n_k+1} - x_0),
\]
and note that the sequences \( s_k \) and \( t_k \) converge weakly to the nonzero vectors \( s_0 := z_0 \) and \( t_0 := Q^* (v_0 - x_0) \), respectively, and that the sequence \( \{\beta_k\} \) converges to zero. Next we let \( \{A_{m, k}\}_{m, k \in \mathbb{N}} \) be an arbitrary doubly indexed sequence in the unit ball of \( \{Q^\dagger\} \), and we write
\[
A_{m, k} y_{n_k} = d_{n_k}^{(m)} y_{n_k+1} + w_{n_k+1}^{(m)}, \quad m, k \in \mathbb{N},
\]
where \( d_{n_k}^{(m)} \in \mathbb{C} \) and \( w_{n_k+1}^{(m)} \) is orthogonal to \( y_{n_k+1} \) for all \( m, k \in \mathbb{N} \). Note that
\[
\|y_{n_k}\|^2 + \|A_{m, k} y_{n_k}\|^2 
\geq |\alpha_{n_k}^{(m)}|^2 \|y_{n_k+1}\|^2 + \|w_{n_k+1}^{(m)}\|^2,
\]
and thus
\[
|\alpha_{n_k}^{(m)}|^2 \geq \frac{\|y_{n_k}\|^2}{\|y_{n_k+1}\|^2}, \quad m, k \in \mathbb{N}.
\]
An application of \( Q^{n_k+1} \) to each side of (2) gives
\[
Q A_{m, k} Q^{n_k} y_{n_k} = Q^{n_k+1} A_{m, k} y_{n_k}
\]

(4)
\[
= d_{n_k}^{(m)} Q^{n_k+1} y_{n_k+1} + Q^{n_k+1} w_{n_k+1}^{(m)}, \quad m, k \in \mathbb{N}.
\]

Upon taking the inner product of each side of (4) with \( Q^{n_k+1} y_{n_k+1} - x_0 \), we obtain
\[
(A_{m, k} s_k, t_k) = d_{n_k}^{(m)} (Q^{n_k+1} y_{n_k+1}, Q^{n_k+1} y_{n_k+1} - x_0), \quad m, k \in \mathbb{N},
\]
since, by Lemma 2.3,
\[
(w_{n_k+1}^{(m)}, Q^{n_k+1} (Q^{n_k+1} y_{n_k+1} - x_0)) = (w_{n_k+1}^{(m)}, r_{n_k+1} y_{n_k+1}) = 0, \quad m, k \in \mathbb{N}.
\]
Moreover, since
\[
|(Q^{n_k+1} y_{n_k+1}, Q^{n_k+1} y_{n_k+1} - x_0)| \leq (\|x_0\| + \varepsilon) \varepsilon, \quad k \in \mathbb{N},
\]
we have from the definition of \( \beta_k \), (3), and (5) that
\[
|(A_{m, k} s_k, t_k)| \leq \beta_k, \quad m, k \in \mathbb{N}.
\]
Since we saw earlier that \( B_0 s_0 = B_0 t_0 \neq 0 \), the theorem is proved. \( \square \)

Proof of Corollary 1.4. We may suppose, without loss of generality, that \( Q \) is a quasi-infinity (otherwise ker \( Q \) or \( \text{range } Q \)) is a nontrivial hyperinvariant subspace for \( Q \). Thus \( CQ \neq 0 \), and we set \( B_0 \) of Theorem 1.3 equal to \( C \). Now let the sequences \( \{s_k\}, \{t_k\}, \) and \( \{\beta_k\} \) be as in Theorem 1.3, with \( \{s_k\} \) and \( \{t_k\} \) having nonzero weak limits \( s_0 \) and \( t_0 \), respectively. Also let \( A_0 \) be an arbitrary operator in the unit ball of \( \{Q^\dagger\} \) such that \( A_0 C \neq 0 \). We will show that \( (A_0 C s_0, t_0) = 0 \), and therefore that \( \mathcal{M} = (\{Q^\dagger\} C s_0)^- \) is the desired nontrivial hyperinvariant subspace.
Define the doubly indexed sequence 
\( \{d_{m,k}\}_{m,k \in \mathbb{N}} \) by \( A_{m,k} = A_0 D_m, m, k \in \mathbb{N} \).

Then, from Theorem 1.3, we know that
\begin{equation}
\|D_m - K_m\| < \eta /\left\{2 \|A_0\| \left(\sup_k \|s_k\| \|t_k\|\right)\right\}, \quad m \geq M_\eta.
\end{equation}

Now let \( \eta > 0 \) be given and note that (since \( \{K_m\} \) tends to \( C \) in the weak operator topology) it suffices to find \( M_\eta > 0 \) such that
\begin{equation}
|(A_0 K_m s_0, t_0)| \leq \eta, \quad m \geq M_\eta.
\end{equation}

Choose \( K > 0 \) such that for \( k \geq K \), \( \beta_k < \eta /2 \), and, by (1), choose \( M_\eta > 0 \) such that
\begin{equation}
\|D_m - K_m\| < \eta /\left\{2 \|A_0\| \left(\sup_k \|s_k\| \|t_k\|\right)\right\}, \quad m \geq M_\eta.
\end{equation}

Then, via (6) and (8),
\begin{equation}
|(A_0 K_m s_0, t_0)| \leq \left| (A_0 D_m s_0, t_0) \right| + \left| (A_0 (K_m - D_m) s_0, t_0) \right| < \eta, \quad m \geq M_\eta, \quad k \geq K.
\end{equation}

Fix an arbitrary \( m_0 \geq M_\eta \), and note that since \( \{s_k\} \) tends weakly to \( s_0 \) and \( A_0 K_{m_0} \) is compact, we obtain
\begin{equation}
\lim_k \|A_0 K_{m_0} s_k - A_0 K_{m_0} s_0\| = 0.
\end{equation}

Moreover, since \( \{t_k\} \) tends weakly to \( t_0 \), we get from (9), (10), and a short calculation, that
\begin{equation}
|(A_0 K_{m_0} s_0, t_0)| = \lim_k \left| (A_0 K_{m_0} s_k, t_k) \right| \leq \eta, \quad m_0 \geq M_\eta,
\end{equation}
which establishes (7) and completes the proof. \( \square \)

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