PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 7, Pages 2121–2127
S 0002-9939(03)06895-3

ON QUASINILPOTENT OPERATORS

IL BONG JUNG, EUNGIL KO, AND CARL PEARCY

(Communicated by David R. Larson)

Abstract. In this note we modify a new technique of Enflo for producing hyperinvariant subspaces to obtain a much improved version of his "two sequences" theorem with a somewhat simpler proof. As a corollary we get a proof of the "best" theorem (due to V. Lomonosov) known about hyperinvariant subspaces for quasinilpotent operators that uses neither the Schauder-Tychonoff fixed point theorem nor the more recent techniques of Lomonosov.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and by $\mathbf{K} = \mathbf{K}(\mathcal{H})$ the ideal of compact operators in $\mathcal{L}(\mathcal{H})$. Perhaps the first invariant-subspace theorem for operators in $\mathcal{L}(\mathcal{H})$, other than those provided by the spectral theorem for normal operators, was that every operator in $\mathbf{K}(\mathcal{H})$ has a nontrivial invariant subspace.

According to Aronszajn-Smith [3], this was proved by John von Neumann (unpublished) about 1935. Thus there has now been over a half-century of work devoted to establishing that operators in $\mathcal{L}(\mathcal{H})$ that have a nice enough relation to some compact operator have nontrivial invariant subspaces. Without attempting to be exhaustive we mention the papers of Bernstein-Robinson [4], Halmos [9], [10], Arveson-Feldman [1], Deckard-Douglas-Pearcy [7], Pearcy-Salinas [14], Lomonosov [11], [12], [13], Pearcy-Shields [15], Scott Brown [5], and, more recently, Chevreau-Li-Pearcy [6], Simonic [16], Ansari-Enflo [2], and Enflo-Lomonosov [8]. Several of these works took something from previous ones, but many also added new techniques, some dramatically new (for example, the use by Lomonosov in [11] of the Schauder-Tychonoff fixed point theorem for nonlinear mappings).

In [2], a very recent new technique was introduced (and ascribed there to Enflo) for producing invariant subspaces for compact-related operators in $\mathcal{L}(\mathcal{H})$. The following old theorem of Lomonosov ([11]; cf. also [15]) was thus given in [2] a completely different proof (neither utilizing the Schauder-Tychonoff fixed point theorem nor the ideas of [12]).

Theorem 1.1. *Every nonzero compact operator in $\mathcal{L}(\mathcal{H})$ has a nontrivial hyperinvariant subspace.*

Received by the editors February 6, 2002.
2000 *Mathematics Subject Classification.* Primary 47A15.
Key words and phrases. Invariant subspaces, Enflo technique, extremal vectors.

As another consequence of this technique, Enflo in [8] obtained the following interesting "two sequences" theorem.

Theorem 1.2. *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be any commutative algebra that contains a nonzero quasinilpotent operator. Then there exist sequences $\{s_k\}_{k=1}^{\infty}$ and $\{t_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ that converge weakly to nonzero vectors s_0 and t_0, respectively, such that for every bounded sequence $\{A_k\}_{k=1}^{\infty} \subset \mathcal{A}$,*

$$\lim_k (A_k s_k, t_k) = 0.$$

This technique of proof (from [2] and [8]) uses some "extremal vectors" in a very clever way, and, as was mentioned in [8], is so new that most likely it will be some time before one knows whether the technique (or modifications thereof) will yield all the stronger theorems from [11] and [12] as well as perhaps some completely new results in the same direction.

The purpose of this note is to show that by modifying Enflo's new technique, a considerably better version of Theorem 1.2, with a somewhat simpler proof, can be obtained as follows.

Theorem 1.3. *Suppose $Q \neq 0$ is a quasinilpotent operator in $\mathcal{L}(\mathcal{H})$ and $\{Q\}'$ denotes the commutant of Q, i.e., $\{Q\}' = \{A \in \mathcal{L}(\mathcal{H}) : AQ = QA\}$. Let B_0 be an arbitrary nonzero operator in $\{Q\}'$ such that $B_0 Q \neq 0$. Then there exist sequences $\{s_k\}_{k=1}^{\infty}$ and $\{t_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ that converge weakly to nonzero vectors s_0 and t_0, respectively, with $B_0 s_0 \neq 0$, and a sequence $\{\beta_k\}$ of positive numbers converging to zero, such that for every doubly indexed sequence $\{A_{m,k}\}_{m,k\in\mathbb{N}}$ in the unit ball of $\{Q\}'$, we have*

$$|(A_{m,k} s_k, t_k)| < \beta_k, \qquad m, k \in \mathbb{N}.$$

Also as a corollary of Theorem 1.3, the following better (than Theorem 1.1) but not so old theorem of Lomonosov [12] can be deduced.

Corollary 1.4 ([12])**.** *Suppose that $Q \neq 0$ is a quasinilpotent operator in $\mathcal{L}(\mathcal{H})$ and there exist a sequence $\{D_m\}_{m\in\mathbb{N}} \subset \{Q\}'$ converging in the weak operator topology to a nonzero (C in $\{Q\}'$) and a sequence $\{K_m\}_{m\in\mathbb{N}}$ of compact operators such that*

$$\lim_m \|D_m - K_m\| = 0. \tag{1}$$

(*In other words, in the language of* [12], *we suppose that $\{Q\}'$ has the Pearcy-Salinas property.*) *Then Q has a nontrivial hyperinvariant subspace.*

In other words, this note may be considered as a first step in the direction of determining what are the best theorems that can be obtained by (modifications of) this new Enflo technique from [2] and [8]. We remark that Corollary 1.4 is the "strongest" theorem known which produces hyperinvariant subspaces for a quasinilpotent operator, so, at least in this direction, Enflo's new technique produces the "best" theorem known.

2. Some lemmas

Our proof of Theorem 1.3 depends on several lemmas (essentially) from [2].

Lemma 2.1. *Suppose u and v are nonzero vectors in $\mathcal{H}$ such that for every $z \in \mathcal{H}$, $\mathrm{Re}(u, z) < 0$ implies that $\mathrm{Re}(v, z) \geq 0$. Then there exists a negative number r_0 such that $v = r_0 u$.*

Proof. Write $v = \alpha_0 u + w$ where $\alpha_0 \in \mathbb{C}$ and w is orthogonal to u. Note that if we set $z = z(\gamma, x) = \gamma u + x$, where x is orthogonal to u and $\mathrm{Re}\gamma < 0$, then

$$\mathrm{Re}(u, z) = \mathrm{Re}(u, \gamma u + x) = \|u\|^2 \mathrm{Re}\gamma < 0.$$

Thus, for all x orthogonal to u and for all γ with $\mathrm{Re}\gamma < 0$, we have, by hypothesis,

$$\mathrm{Re}(v, z(\gamma, x)) = \mathrm{Re}(\alpha_0 u + w, \gamma u + x) = \|u\|^2 \mathrm{Re}(\alpha_0 \gamma) + \mathrm{Re}(w, x) \geq 0.$$

Upon fixing γ and taking x to be a large enough negative scalar multiple of w, we see that necessarily $w = 0$ and that $\mathrm{Re}(\alpha_0 \gamma) \geq 0$. Upon writing $\alpha_0 = r_0 + is_0$ and $\gamma = t + iq$ where r_0, s_0, t, q are real, we get that $r_0 t - s_0 q \geq 0$ for all $q \in \mathbb{R}$ and all $t < 0$. Fixing t and letting q run we get $s_0 = 0$ and then $r_0 \leq 0$. Since $v = r_0 u$ and $v \neq 0$, we must have $r_0 < 0$, so the proof is complete. ☐

Lemma 2.2. *Suppose $T \in \mathcal{L}(\mathcal{H})$ and has dense range. Suppose also that x_0 is a nonzero vector in $\mathcal{H}$ and that ε satisfies $0 < \varepsilon < \|x_0\|$. Then there exists a unique nonzero vector $y_0 = y_0(x_0, \varepsilon)$ such that*

a) $\|y_0\| = \inf\{\|y\| : \|Ty - x_0\| \leq \varepsilon\}$ *and*

b) $\|Ty_0 - x_0\| = \varepsilon$.

Proof. Let $\mathcal{F} = \{y \in \mathcal{H} : \|Ty - x_0\| \leq \varepsilon\}$. Since T has dense range, clearly $\mathcal{F}$ is nonempty, and since T is continuous and $\mathcal{F}$ is the inverse image under T of the norm-closed ball centered at x_0 with radius ε, $\mathcal{F}$ is a norm-closed set. Moreover an easy calculation shows that $\mathcal{F}$ is a convex set. But, as is well-known, such a set has a unique vector y_0 of minimal norm. Thus a) is satisfied, and if $\|Ty_0 - x_0\| < \varepsilon$, then for $\delta > 0$ sufficiently small, $(1 - \delta)y_0$ would belong to $\mathcal{F}$ and have smaller norm, so b) is satisfied. ☐

Lemma 2.3. *Suppose T, x_0, y_0, and ε are as in Lemma 2.2. Then there exists a negative number r such that $T^*(Ty_0 - x_0) = ry_0$.*

Proof. We apply Lemma 2.1 to $u = T^*(Ty_0 - x_0)$ and $v = y_0$. Clearly it suffices to show that u and v satisfy the hypotheses of Lemma 2.1 (and then set $r = 1/r_0$). Thus suppose that z_0 is any vector in $\mathcal{H}$ satisfying

$$\mathrm{Re}(u, z) = \mathrm{Re}(T^*(Ty_0 - x_0), z_0) = \mathrm{Re}(Ty_0 - x_0, Tz_0) < 0.$$

It follows easily that there exists a sufficiently small interval $[0, t_0]$, on which the function $t \to \|(Ty_0 - x_0) + tTz_0\|^2$ is strictly monotone decreasing (its derivative is continuous and negative at the origin). Thus for $t \in (0, t_0]$ we have

$$\|T(y_0 + tz_0) - x_0\| < \|Ty_0 - x_0\| = \varepsilon.$$

Thus for $t \in (0, t_0]$, $y_0 + tz_0 \in \mathcal{F}$, and by the minimality of y_0, we must have

$$\|y_0 + tz_0\|^2 \geq \|y_0\|^2, \qquad t \in (0, t_0].$$

But the derivative of the function

$$t \to \|y_0 + tz_0\|^2$$

is continuous and its value at the origin is $2\mathrm{Re}(y_0, z_0)$, which must therefore satisfy $\mathrm{Re}(y_0, z_0) \geq 0$, and the lemma is proved. ☐

Lemma 2.4. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is quasinilpotent with dense range, let x_0 be a nonzero vector in $\mathcal{H}$, let ε satisfy $0 < \varepsilon < \|x_0\|$, and (via Lemma 2.2) let, for each $n \in \mathbb{N}$, $y_n = y_n(\varepsilon, x_0)$ be a (nonzero) vector satisfying*

a) $\|y_n\| = \inf\{\|y\| : \|T^n y - x_0\| \leq \varepsilon\}$ *and*

b) $\|T^n y_n - x_0\| = \varepsilon$.

Then there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ *of the sequence* $\{y_n\}$ *satisfying*

$$\lim_k \frac{\|y_{n_k}\|}{\|y_{n_k+1}\|} = 0.$$

Proof. Suppose, to the contrary, that there exist $t > 0$ and $N_t \in \mathbb{N}$ such that

$$\inf_{n \geq N_t} \frac{\|y_n\|}{\|y_{n+1}\|} = t.$$

Then

$$\|y_{N_t}\| \geq t\|y_{N_t+1}\| \geq t^2\|y_{N_t+2}\| \geq \cdots \geq t^n\|y_{N_t+n}\|, \qquad n \in \mathbb{N}.$$

By the minimality of $\|y_{N_t}\|$ from a), we have (since $\|T^{N_t+n}y_{N_t+n} - x_0\| = \varepsilon$)

$$\|T^n y_{N_t+n}\| \geq \|y_{N_t}\|, \qquad n \in \mathbb{N}.$$

Thus

$$\|T^n\|\|y_{N_t+n}\| \geq \|y_{N_t}\| \geq t^n\|y_{N_t+n}\|, \qquad n \in \mathbb{N},$$

and hence $\|T^n\| \geq t^n$, $n \in \mathbb{N}$, which contradicts the fact that $\sigma(T) = \{0\}$. The result follows. □

3. PROOFS OF THE RESULTS

On the basis of these lemmas, we now prove Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3. Let $B_0 \in \{Q\}'$ be such that $B_0 Q \neq 0$. If $\mathcal{M} = (\text{range } Q)^- \neq \mathcal{H}$, then $\mathcal{M}$ is a nontrivial hyperinvariant subspace for Q and the result follows by choosing nonzero vectors $s_0 \in \mathcal{M}$ and $t_0 \in \mathcal{M}^\perp$ such that $B_0 s_0 \neq 0$ and defining $s_k = s_0$, $t_k = t_0$, $k \in \mathbb{N}$, and $\{\beta_k\}$ to be an arbitrary sequence of positive numbers tending to zero. Thus we may suppose that Q has dense range (which implies that each Q^n also has dense range). Let x_0 be a nonzero vector in $\mathcal{H}$ such that $B_0 Q x_0 \,(= Q B_0 x_0) \neq 0$, and let ε satisfy

$$0 < \varepsilon < \min\{\|x_0\|, \|Qx_0\|, (1/\|B_0\|)\|B_0 x_0\|\}.$$

For each $n \in \mathbb{N}$, let $y_n = y_n(\varepsilon, x_0)$ satisfy a) and b) of Lemma 2.4 (with $T = Q$). By Lemma 2.4, we can choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\lim_k \frac{\|y_{n_k}\|}{\|y_{n_k+1}\|} = 0.$$

By dropping down to successive subsequences of $\{y_{n_k}\}$ we may suppose (without changing the notation accordingly), since all of the vectors $Q^n y_n$, $n \in \mathbb{N}$, belong to the norm-closed ball of radius ε centered at x_0, that the sequence $\{Q^{n_k} y_{n_k}\}$ converges weakly to a vector z_0 and, similarly, that the sequence $\{Q^{n_k+1} y_{n_k+1}\}$ converges weakly to a vector v_0. Since norm-closed balls in $\mathcal{H}$ are weakly closed, we have $\|v_0 - x_0\| \leq \varepsilon$, $\|z_0 - x_0\| \leq \varepsilon$ (so, in particular, $v_0 \neq 0 \neq z_0$), $\|B_0 z_0 - B_0 x_0\| \leq \|B_0\|\,\varepsilon$, and $B_0 z_0 \neq 0$.

We next show that $v_0 - x_0 \neq 0$, which shows also (since $\mathcal{M} = \mathcal{H}$) that $Q^*(v_0 - x_0) \neq 0$. By the definition of the y_n and Lemma 2.3, we have

$$\begin{aligned}
\varepsilon^2 &= \left\|Q^{n_k+1} y_{n_k+1} - x_0\right\|^2 \\
&= (y_{n_k+1}, (Q^{n_k+1})^*(Q^{n_k+1} y_{n_k+1} - x_0)) - (x_0, Q^{n_k+1} y_{n_k+1} - x_0) \\
&= r_{n_k+1}\|y_{n_k+1}\|^2 - (x_0, Q^{n_k+1} y_{n_k+1} - x_0), \qquad k \in \mathbb{N},
\end{aligned}$$

where $r_{n_k+1} < 0$ for all $k \in \mathbb{N}$. Thus

$$-\varepsilon^2 \geq (x_0,\, Q^{n_k+1}y_{n_k+1} - x_0), \qquad k \in \mathbb{N},$$

and, taking limits as $k \to \infty$, we get $-\varepsilon^2 \geq (x_0,\, v_0 - x_0)$, so $v_0 - x_0 \neq 0$ and $Q^*(v_0 - x_0) \neq 0$.

Now define

$$\begin{aligned} s_k &= Q^{n_k}y_{n_k}, \\ t_k &= Q^*(Q^{n_k+1}y_{n_k+1} - x_0), \\ \beta_k &= \|y_{n_k}\|\,\varepsilon\,(\|x_0\| + \varepsilon)\,/\,\|y_{n_k+1}\|\,, \quad k \in \mathbb{N}, \end{aligned}$$

and note that the sequences $\{s_k\}_{k=1}^{\infty}$ and $\{t_k\}_{k=1}^{\infty}$ converge weakly to the nonzero vectors $s_0 := z_0$ and $t_0 := Q^*(v_0 - x_0)$, respectively, and that the sequence $\{\beta_k\}$ converges to zero. Next we let $\{A_{m,k}\}_{m,k\in\mathbb{N}}$ be an arbitrary doubly indexed sequence in the unit ball of $\{Q\}'$, and we write

$$A_{m,k}y_{n_k} = \alpha_{n_k}^{(m)}y_{n_k+1} + w_{n_k+1}^{(m)}, \qquad m,\, k \in \mathbb{N}, \tag{2}$$

where $\alpha_{n_k}^{(m)} \in \mathbb{C}$ and $w_{n_k+1}^{(m)}$ is orthogonal to y_{n_k+1} for all $m,\, k \in \mathbb{N}$. Note that

$$\begin{aligned} \|y_{n_k}\|^2 &\geq \|A_{m,k}y_{n_k}\|^2 \\ &= |\alpha_{n_k}^{(m)}|^2\,\|y_{n_k+1}\|^2 + \|w_{n_k+1}^{(m)}\|^2, \qquad m,\, k \in \mathbb{N}, \end{aligned}$$

and thus

$$\frac{\|y_{n_k}\|}{\|y_{n_k+1}\|} \geq |\alpha_{n_k}^{(m)}|, \qquad m,\, k \in \mathbb{N}. \tag{3}$$

An application of Q^{n_k+1} to each side of (2) gives

$$\begin{aligned} QA_{m,k}Q^{n_k}y_{n_k} &= Q^{n_k+1}A_{m,k}y_{n_k} \\ &= \alpha_{n_k}^{(m)}Q^{n_k+1}y_{n_k+1} + Q^{n_k+1}w_{n_k+1}^{(m)}, \quad m,\, k \in \mathbb{N}. \end{aligned} \tag{4}$$

Upon taking the inner product of each side of (4) with $Q^{n_k+1}y_{n_k+1} - x_0$, we obtain

$$(A_{m,k}s_k,\, t_k) = \alpha_{n_k}^{(m)}(Q^{n_k+1}y_{n_k+1},\, Q^{n_k+1}y_{n_k+1} - x_0), \quad m,\, k \in \mathbb{N}, \tag{5}$$

since, by Lemma 2.3,

$$(w_{n_k+1}^{(m)},\, (Q^{n_k+1})^*(Q^{n_k+1}y_{n_k+1} - x_0)) = (w_{n_k+1}^{(m)},\, r_{n_k+1}y_{n_k+1}) = 0, \quad m,\, k \in \mathbb{N}.$$

Moreover, since

$$\big|(Q^{n_k+1}y_{n_k+1},\, Q^{n_k+1}y_{n_k+1} - x_0)\big| \leq (\|x_0\| + \varepsilon)\varepsilon, \qquad k \in \mathbb{N},$$

we have from the definition of β_k, (3), and (5) that

$$|(A_{m,k}s_k,\, t_k)| \leq \beta_k, \qquad m,\, k \in \mathbb{N}.$$

Since we saw earlier that $B_0s_0 = B_0z_0 \neq 0$, the theorem is proved. □

Proof of Corollary 1.4. We may suppose, without loss of generality, that Q is a quasiaffinity (otherwise ker Q or (range $Q)^-$ is a nontrivial hyperinvariant subspace for Q). Thus $CQ \neq 0$, and we set B_0 of Theorem 1.3 equal to C. Now let the sequences $\{s_k\}$, $\{t_k\}$, and $\{\beta_k\}$ be as in Theorem 1.3, with $\{s_k\}$ and $\{t_k\}$ having nonzero weak limits s_0 and t_0, respectively. Also let A_0 be an arbitrary operator in the unit ball of $\{Q\}'$ such that $A_0C \neq 0$. We will show that $(A_0Cs_0,\, t_0) = 0$, and therefore that $\mathcal{M} = (\{Q\}'Cs_0)^-$ is the desired nontrivial hyperinvariant subspace for Q. (Note that $0 \neq Cs_0 \in \mathcal{M}$ and that t_0 is orthogonal to $\mathcal{M}$.) We may suppose, without loss of generality, that the sequence $\{D_m\}$ lies in the unit ball of $\{Q\}'$. Define the doubly indexed sequence $\{A_{m,k}\}_{m,k\in\mathbb{N}}$ by $A_{m,k} = A_0 D_m$, $m, k \in \mathbb{N}$. Then, from Theorem 1.3, we know that

$$|(A_0 D_m s_k, t_k)| < \beta_k, \qquad m, k \in \mathbb{N}. \tag{6}$$

Now let $\eta > 0$ be given and note that (since $\{K_m\}$ tends to C in the weak operator topology) it suffices to find $M_\eta > 0$ such that

$$|(A_0 K_m s_0, t_0)| \leq \eta, \qquad m \geq M_\eta. \tag{7}$$

Choose $K > 0$ such that for $k \geq K$, $\beta_k < \eta\ /2$, and, by (1), choose $M_\eta > 0$ such that

$$\|D_m - K_m\| < \eta/\{2\ \|A_0\|\,(\sup_k \|s_k\|\,\|t_k\|)\}, \qquad m \geq M_\eta. \tag{8}$$

Then, via (6) and (8),

$$\begin{aligned} |(A_0 K_m s_k, t_k)| &\leq |(A_0 D_m s_k, t_k)| \\ &\quad + |(A_0(K_m - D_m)s_k, t_k)| < \eta, \quad m \geq M_\eta,\ k \geq K. \end{aligned} \tag{9}$$

Fix an arbitrary $m_0 \geq M_\eta$, and note that since $\{s_k\}$ tends weakly to s_0 and $A_0 K_{m_0}$ is compact, we obtain

$$\lim_k \|A_0 K_{m_0} s_k - A_0 K_{m_0} s_0\| = 0. \tag{10}$$

Moreover, since $\{t_k\}$ tends weakly to t_0, we get from (9), (10), and a short calculation, that

$$|(A_0 K_{m_0} s_0, t_0)| = \lim_k |(A_0 K_{m_0} s_k, t_k)| \leq \eta, \qquad m_0 \geq M_\eta,$$

which establishes (7) and completes the proof. □

ACKNOWLEDGEMENT

The first and second authors were supported by KOSEF 2000-1-10100-002-3. The third author gratefully acknowledges the support of the National Science Foundation.

REFERENCES

[1] W. Arveson and J. Feldman, *A note on invariant subspaces*, Michigan Math. J. **15**(1968), 61-64. MR **36:**6969

[2] S. Ansari and P. Enflo, *Extremal vectors and invariant subspaces*, Trans. Amer. Math. Soc. **350**(1998), 539-558. MR **98d:**47019

[3] N. Aronszajn and K. Smith, *Invariant subspaces of completely continuous operators*, Ann. of Math. **60**(1954), 345-350. MR **16:**488b

[4] A. Bernstein and A. Robinson, *Solution of an invariant subspace problem of K.T. Smith and P.R. Halmos*, Pacific J. Math. **16**(1966), 421-431. MR **33:**1724

[5] S. Brown, *Hyponormal operators with thick spectrum have invariant subspaces*, Ann. of Math. **125**(1987), 93-103. MR **88c:**47010

[6] B. Chevreau, W. Li, and C. Pearcy, *A new Lomonosov lemma*, J. Operator Theory **40**(1998), 409-417. MR **2000b:**47014

[7] D. Deckard, R. Douglas, and C. Pearcy, *On invariant subspaces of quasitriangular operators*, Amer. J. Math. **9**(1969), 637-647. MR **41:**859

[8] P. Enflo and V. Lomonosov, *Some aspects of the invariant subspace problem*, preprint.

[9] P. Halmos, *Invariant subspaces of polynomially compact operators*, Pacific J. Math. **16**(1966), 433-437. MR **33:**1725

[10] ———, *Quasitriangular operators*, Acta Sci. Math.(Szeged) **29**(1968), 283-293. MR **38:**2627

[11] V. Lomonosov, *On invariant subspaces of families of operators commuting with a completely continuous operator* (in Russian), Funkcional Anal. i Prilozen **7**(1973), 55-56. MR **54:**8319

[12] ______, *An extension of Burnside's theorem to infinite dimensional spaces*, Israel J. Math. **75**(1991), 329-339. MR **93h:**47007

[13] ______, *On real invariant subspaces of bounded operators with compact imaginary part*, Proc. Amer. Math. Soc. **115**(1992), 775-777. MR **92i:**47003

[14] C. Pearcy and N. Salinas, *An invariant subspace theorem*, Michigan Math. J. **20**(1973), 21-31. MR **47:**5623

[15] C. Pearcy and A. Shields, *A survey of the Lomonosov technique in the theory of invariant subspaces*, Topics in Operator Theory, Amer. Math. Soc. Surveys, No. **13**(1974), 219-229. MR **50:**8113

[16] A. Simonic, *An extension of Lomonosov's techniques to non-compact operators*, Trans. Amer. Math. Soc. **348**(1996), 975-995. MR **96j:**47005

DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA
E-mail address: `ibjung@kyungpook.ac.kr`

DEPARTMENT OF MATHEMATICS, EWHA WOMEN'S UNIVERSITY, SEOUL 120-750, KOREA
E-mail address: `eiko@mm.ewha.ac.kr`

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TEXAS 77843
E-mail address: `pearcy@math.tamu.edu`