

ON THE IRRATIONALITY OF A CERTAIN MULTIVARIATE q SERIES

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ABSTRACT. We prove that for integers $q > 1, m \geq 1$ and positive rationals $r_1, r_2, \dots, r_m \neq q^j, j = 1, 2, \dots$, the series

$$\sum_{j=1}^{\infty} \frac{q^{-j}}{(1 - q^{-j}r_1)(1 - q^{-j}r_2) \cdots (1 - q^{-j}r_m)}$$

is irrational. Furthermore, if all the positive rationals r_1, r_2, \dots, r_m are less than q , then the series

$$\sum_{j_1, \dots, j_m=0}^{\infty} \frac{r_1^{j_1} \cdots r_m^{j_m}}{q^{j_1 + \cdots + j_m + 1} - 1}$$

is also irrational.

1. INTRODUCTION AND RESULTS

The main result of this paper is the following theorem:

Theorem 1.1. *If q is an integer greater than one, m is a positive integer, and r_1, r_2, \dots, r_m are any positive rationals such that $r_1, r_2, \dots, r_m \neq q^j, j = 1, 2, \dots$, then the series*

$$\sum_{j=1}^{\infty} \frac{q^{-j}}{(1 - q^{-j}r_1)(1 - q^{-j}r_2) \cdots (1 - q^{-j}r_m)}$$

is irrational. Furthermore, if all the positive rationals r_1, r_2, \dots, r_m are less than q , then the series

$$\sum_{j_1, \dots, j_m=0}^{\infty} \frac{r_1^{j_1} \cdots r_m^{j_m}}{q^{j_1 + \cdots + j_m + 1} - 1}$$

is also irrational.

This generalizes the irrationality results of the single variable case proved in Borwein [3], Erdős [6], and Erdős and Graham [7]. The approach is via Padé approximants. These provide, when appropriately specialized, rational approximations that are “too good” to allow for rationality. These methods are also used in Borwein and Zhou [4], Mahler [9], Chudnovsky and Chudnovsky [5], Wallisser [10], and Zhou and Lubinsky [11]. Unfortunately the methods are not sufficiently general

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to allow a unified treatment and each new class of functions requires considerable additional work.

As in [4] we use the standard q analogues of factorials and binomial coefficients. The q -factorial is

$$(1.1) \quad [n]_q! := [n]! := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q)}{(1-q)^n},$$

where $[0]_q! := 1$. The q -binomial coefficient is

$$(1.2) \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]! \cdot [n-k]!}.$$

As

$$q^i - 1 = (q-1)(q^{i-1} + q^{i-2} + \cdots + 1), \quad i \geq 1,$$

we have

$$(1.3) \quad \lim_{q \rightarrow 1} [n]_q! = n! \quad \text{and} \quad \lim_{q \rightarrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

Note that (see Borwein [3])

$$(1.4) \quad [n]_{q^{-1}}! = q^{-n(n-1)/2} [n]!,$$

$$(1.5) \quad \begin{bmatrix} n \\ k \end{bmatrix}_{-q} = q^{-k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix},$$

$$(1.6) \quad \prod_{\substack{h=0 \\ h \neq k}}^n (q^{-k} - q^{-h}) = (-1)^{n-k} q^{-k(k-1)/2 - n(n+1)/2} [n-k]! [k]! (1-q)^n,$$

and (see Gasper and Rahman [8]) for $|t| < q^{-n}$,

$$(1.7) \quad \frac{1}{\prod_{k=0}^n (t - q^{-k})} = (-1)^{n+1} q^{n(n+1)/2} \sum_{l=0}^{\infty} \begin{bmatrix} n+l \\ l \end{bmatrix} t^l.$$

We prove some properties of approximants to a related function in section 2, and use those properties to prove Theorem 1.1 in section 3.

2. SOME RESULTS ON A RELATED FUNCTION

Let $q > 1$, $|x_1|, \dots, |x_m| < q$, and integer $m \geq 1$, and let

$$(2.1) \quad L^*(x_1, \dots, x_m) := \sum_{j_1, \dots, j_m=0}^{\infty} \frac{x_1^{j_1} \cdots x_m^{j_m}}{q^{j_1 + \cdots + j_m + 1} - 1}.$$

For $m = 1$, and $|x| < 1$,

$$\begin{aligned} \lim_{q \rightarrow 1} (q-1)L^*(x) &= \lim_{q \rightarrow 1} \sum_{j=0}^{\infty} \frac{(q-1)x^j}{q^{j+1} - 1} \\ &= \sum_{j=0}^{\infty} \frac{x^j}{j+1} \\ &= \frac{1}{x} \ln(1-x). \end{aligned}$$

So we call $L^*(x_1, \dots, x_m)$ a multivariate q analogue of log. Now for $k \geq 1$ integer and $|x_1|, \dots, |x_m| < q$, as

$$\begin{aligned} L^*(q^{-1}x_1, \dots, q^{-1}x_m) &= \sum_{j_1, \dots, j_m=0}^{\infty} \frac{q^{-(j_1+\dots+j_m)} x_1^{j_1} \dots x_m^{j_m}}{q^{j_1+\dots+j_m+1} - 1} \\ &= \sum_{j_1, \dots, j_m=0}^{\infty} \frac{(1 - q^{j_1+\dots+j_m+1} + q^{j_1+\dots+j_m+1}) x_1^{j_1} \dots x_m^{j_m}}{q^{j_1+\dots+j_m} (q^{j_1+\dots+j_m+1} - 1)} \\ &= \sum_{j_1, \dots, j_m=0}^{\infty} \frac{q x_1^{j_1} \dots x_m^{j_m}}{q^{j_1+\dots+j_m+1} - 1} - \sum_{j_1, \dots, j_m=0}^{\infty} \frac{x_1^{j_1} \dots x_m^{j_m}}{q^{j_1+\dots+j_m}} \\ &= qL^*(x_1, \dots, x_m) - \frac{1}{(1 - q^{-1}x_1) \dots (1 - q^{-1}x_m)}, \end{aligned}$$

we have

$$\begin{aligned} L^*(q^{-k}x_1, \dots, q^{-k}x_m) &= q^k L^*(x_1, \dots, x_m) - \sum_{j=1}^k \frac{q^{k-j}}{(1 - q^{-j}x_1) \dots (1 - q^{-j}x_m)} \\ (2.2) \qquad \qquad \qquad &= : q^k L^*(x_1, \dots, x_m) - S_k(x_1, \dots, x_m), \end{aligned}$$

where

$$(2.3) \qquad S_k(x_1, \dots, x_m) := \sum_{j=1}^k \frac{q^{k-j}}{(1 - q^{-j}x_1) \dots (1 - q^{-j}x_m)}.$$

From (2.2), we have

$$L^*(x_1, \dots, x_m) = q^{-k} L^*(q^{-k}x_1, \dots, q^{-k}x_m) + \sum_{j=1}^k \frac{q^{-j}}{(1 - q^{-j}x_1) \dots (1 - q^{-j}x_m)},$$

and then

$$\begin{aligned} L^*(x_1, \dots, x_m) &= \lim_{k \rightarrow \infty} q^{-k} L^*(q^{-k}x_1, \dots, q^{-k}x_m) \\ &\quad + \lim_{k \rightarrow \infty} \sum_{j=1}^k \frac{q^{-j}}{(1 - q^{-j}x_1) \dots (1 - q^{-j}x_m)} \\ (2.4) \qquad \qquad \qquad &= \sum_{j=1}^{\infty} \frac{q^{-j}}{(1 - q^{-j}x_1) \dots (1 - q^{-j}x_m)}. \end{aligned}$$

Now let $q > 1$, $x_1, \dots, x_m \neq q^j$, $j = 1, 2, \dots$, and integer $m \geq 1$, and let

$$(2.5) \qquad L(x_1, \dots, x_m) := \sum_{j=1}^{\infty} \frac{q^{-j}}{(1 - q^{-j}x_1) \dots (1 - q^{-j}x_m)}.$$

Then $L(x_1, \dots, x_m)$ is an extension of $L^*(x_1, \dots, x_m)$, i.e.

$$(2.6) \qquad L(x_1, \dots, x_m) = L^*(x_1, \dots, x_m), \text{ for } |x_1|, \dots, |x_m| < q.$$

It is easy to see that we also have the following functional equation for $L(x_1, \dots, x_m)$:

$$(2.7) \qquad L(q^{-k}x_1, \dots, q^{-k}x_m) = q^k L(x_1, \dots, x_m) - S_k(x_1, \dots, x_m),$$

where $k \geq 1$ is an integer and $S_k(x_1, \dots, x_m)$ is defined by (2.3). Now we prove some properties of the function $L(x_1, \dots, x_m)$.

Theorem 2.1. *Let $n \geq 0$ be an integer, and let $L(x_1, \dots, x_m), S_k(x_1, \dots, x_m)$ be defined by (2.5) and (2.3) respectively. Let*

$$(2.8) \quad R_n(x_1, \dots, x_m) := \prod_{j=1}^n ((1 - q^{-j}x_1) \cdots (1 - q^{-j}x_m))$$

and

$$(2.9) \quad I(x_1, \dots, x_m) := \frac{R_n(x_1, \dots, x_m)}{2\pi i} \int_{\Gamma} \frac{L(tx_1, \dots, tx_m) dt}{(\prod_{k=0}^n (t - q^{-k})) t^{n+1}},$$

where Γ is a circular contour containing $0, q^{-n}, \dots, q^0$, and let

$$(2.10) \quad Q(x_1, \dots, x_m) := \frac{q^{n(n+1)/2}}{(1 - q)^n [n]!} \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} q^{nk+k(k+1)/2} R_n(x_1, \dots, x_m),$$

$$(2.11) \quad P(x_1, \dots, x_m) := \frac{q^{n(n+1)/2}}{(1 - q)^n [n]!} \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} q^{nk+k(k+1)/2} \cdot R_n(x_1, \dots, x_m) S_k(x_1, \dots, x_m) + \frac{R_n(x_1, \dots, x_m)}{n!} \frac{d^n}{dt^n} \left\{ \frac{L(tx_1, \dots, tx_m)}{\prod_{k=0}^n (t - q^k)} \right\}_{t=0}.$$

Then

(i)

$$(2.12) \quad I(x_1, \dots, x_m) = Q(x_1, \dots, x_m)L(x_1, \dots, x_m) + P(x_1, \dots, x_m);$$

(ii)

$$(2.13) \quad q^{(m-1)n(n+1)/2} \left(\prod_{j=1}^n (q^j - 1) \right) Q(x_1, \dots, x_m) \in \mathbb{Z}[q, x_1, \dots, x_m],$$

where $\mathbb{Z}[q, x_1, \dots, x_m]$ is the set of polynomials in q, x_1, \dots, x_m with integer coefficients;

(iii)

$$(2.14) \quad q^{(m-1)n(n+1)/2} \left(\prod_{j=1}^{n+1} (q^j - 1) \right) P(x_1, \dots, x_m) \in \mathbb{Z}[q, x_1, \dots, x_m];$$

(iv) for $n \in \mathbb{N}$ fixed, and $0 < |x_1|, \dots, |x_m| < q$,

$$(2.15) \quad |I(x_1, \dots, x_m)| \leq \frac{c_q}{q^{2mn(n+1)}},$$

where c_q is a constant depending only on q, m , and x_1, \dots, x_m .

Proof of Theorem 2.1. Proof of (i). We can see that the integrand in (2.9) has simple poles at $t = q^0, q^{-1}, \dots, q^{-n}$, and a pole of order $n + 1$ at $t = 0$, inside the contour Γ . By the residue theorem and the functional equation (2.7), and (1.6), we have

$$\begin{aligned} I(x_1, \dots, x_m) &= \frac{R_n(x_1, \dots, x_m)}{2\pi i} \int_{\Gamma} \frac{L(tx_1, \dots, tx_m) dt}{\left(\prod_{k=0}^n (t - q^{-k})\right) t^{n+1}} \\ &= R_n(x_1, \dots, x_m) \sum_{k=0}^n \frac{L(q^{-k}x_1, \dots, q^{-k}x_m)}{\left(\prod_{\substack{h=0 \\ h \neq k}}^n (q^{-k} - q^{-h})\right) q^{-k(n+1)}} \\ &\quad + \frac{R_n(x_1, \dots, x_m)}{n!} \frac{d^n}{dt^n} \left\{ \frac{L(tx_1, \dots, tx_m)}{\prod_{k=0}^n (t - q^{-k})} \right\}_{t=0} \\ &= \frac{q^{n(n+1)/2} R_n(x_1, \dots, x_m)}{(1 - q)^n [n]!} \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} q^{nk+k(k+1)/2} q^k L(x_1, \dots, x_m) \\ &\quad - \frac{q^{n(n+1)/2} R_n(x_1, \dots, x_m)}{(1 - q)^n [n]!} \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} q^{nk+k(k+1)/2} S_k(x_1, \dots, x_m) \\ &\quad + \frac{R_n(x_1, \dots, x_m)}{n!} \frac{d^n}{dt^n} \left\{ \frac{L(tx_1, \dots, tx_m)}{\prod_{k=0}^n (t - q^{-k})} \right\}_{t=0} \\ &= Q(x_1, \dots, x_m) L(x_1, \dots, x_m) + P(x_1, \dots, x_m). \end{aligned}$$

Proof of (ii). As $\begin{bmatrix} n \\ k \end{bmatrix}$ is a polynomial in q with integer coefficients, and

$$\begin{aligned} R_n(x_1, \dots, x_m) &= \prod_{j=1}^n ((1 - q^{-j}x_1) \cdots (1 - q^{-j}x_m)) \\ (2.16) \qquad \qquad &= q^{-mn(n+1)/2} \prod_{j=1}^n ((q^j - x_1) \cdots (q^j - x_m)), \end{aligned}$$

we have (2.13).

Proof of (iii). From (2.3) and (2.8), for $1 \leq k \leq n$,

$$R_n(x_1, \dots, x_m) S_k(x_1, \dots, x_m) = \sum_{h=1}^k q^{k-h} \prod_{\substack{j=1 \\ j \neq h}}^n ((1 - q^{-j}x_1) \cdots (1 - q^{-j}x_m)),$$

so from (2.16),

$$(2.17) \qquad q^{mn(n+1)/2} R_n(x_1, \dots, x_m) S_k(x_1, \dots, x_m) \in \mathbb{Z}[q, x_1, \dots, x_m].$$

Now for $t < q^{-\ell}$, where $\ell > 0$ is an integer such that $|q^{-\ell}x_i| < q$, for all $i = 1, \dots, m$,

$$(2.18) \qquad L(tx_1, \dots, tx_m) = L^*(tx_1, \dots, tx_m) = \sum_{j_1, \dots, j_m=0}^{\infty} \frac{x_1^{j_1} \cdots x_m^{j_m} t^{j_1+\dots+j_m}}{q^{j_1+\dots+j_m+1} - 1},$$

then from (1.7) and (2.18), for $t < \min\{q^{-n}, q^{-\ell}\}$,

$$\frac{L(tx_1, \dots, tx_m)}{\prod_{k=0}^n (t - q^{-k})} = (-1)^{n+1} q^{n(n+1)/2} \sum_{j_1, \dots, j_m, l=0}^{\infty} \begin{bmatrix} n+l \\ l \end{bmatrix} \frac{x_1^{j_1} \dots x_m^{j_m} t^{j_1+\dots+j_m+l}}{q^{j_1+\dots+j_m+1} - 1}.$$

So

$$\begin{aligned} & \frac{1}{n!} \frac{d^n}{dt^n} \left\{ \frac{L(tx_1, \dots, tx_m)}{\prod_{k=0}^n (t - q^{-k})} \right\}_{t=0} \\ (2.19) \quad &= (-1)^{n+1} q^{n(n+1)/2} \sum_{\substack{j_1+\dots+j_m+l=n \\ 0 \leq j_1, \dots, j_m, l \leq n}} \begin{bmatrix} n+l \\ l \end{bmatrix} \frac{x_1^{j_1} \dots x_m^{j_m}}{q^{j_1+\dots+j_m+1} - 1}, \end{aligned}$$

and (2.14) follows from (2.11), (2.17) and (2.19).

Proof of (iv). For $R > 1$ and $\Gamma := \{z : |z| = R\}$, we have from (2.9),

$$\begin{aligned} |I(x_1, \dots, x_m)| &\leq R \cdot \frac{|R_n(x_1, \dots, x_m)| \max_{|t|=R} |L(tx_1, \dots, tx_m)|}{R^{n+1} \prod_{k=0}^n (R - |q|^{-k})} \\ (2.20) \quad &\leq \frac{f_q \max_{|t|=R} |L(tx_1, \dots, tx_m)|}{R^n \prod_{k=0}^n (R - q^{-k})}. \end{aligned}$$

Now for $0 < |x_1|, \dots, |x_m| < q$,

$$\begin{aligned} |R_n(x_1, \dots, x_m)| &= \prod_{j=1}^n |(1 - q^{-j}x_1) \dots (1 - q^{-j}x_m)| \\ (2.21) \quad &\leq \prod_{j=0}^{\infty} (1 + q^{-j})^m := f_q, \end{aligned}$$

where f_q is a constant depending only on q and m .

Let $R = q^{mn}$. As

$$\max_{|t|=R} |1 - q^{-j}tx_i| \geq \max_{|t|=R} |1 - q^{-j}|t||x_i|| \geq |1 - q^{mn-j+1}|,$$

for $1 \leq i \leq m$, $j = 1, 2, \dots$, and

$$q^j - 1 = q^j (1 - q^{-j}) \geq \frac{1}{2}q^j,$$

as q is an integer greater than 1, then

$$\begin{aligned} (2.22) \quad & \max_{|t|=R} |L(tx_1, \dots, tx_m)| \\ &\leq \max_{|t|=R} \sum_{j=1}^{\infty} \left| \frac{q^{-j}}{(1 - q^{-j}x_1t) \dots (1 - q^{-j}x_mt)} \right| \\ &\leq \left(\sum_{j=1}^{mn} \frac{q^{j-mn-1}}{(q^j - 1)^m} \right) + \frac{q^{-mn-1}}{(1 - x_1/q) \dots (1 - x_m/q)} + \left(\sum_{j=1}^{\infty} \frac{q^{-j-mn-1}}{(1 - q^{-j})^m} \right) \end{aligned}$$

$$\begin{aligned} &\leq q^{-mn-1} \left(\sum_{j=1}^{mn-1} \frac{2^m}{q^{(m-1)j}} + \frac{1}{(1-x_1/q)\cdots(1-x_m/q)} + L(1, \dots, 1) \right) \\ &\leq q^{-mn-1} \left(2^m + \frac{1}{(1-x_1/q)\cdots(1-x_m/q)} + L(1, \dots, 1) \right) \\ &=: C_1 q^{-mn}, \end{aligned}$$

where $C_1 := 2^m q + \frac{q}{(1-x_1/q)\cdots(1-x_m/q)} + qL(1, \dots, 1)$ is a constant depending only on q, m , and x_1, \dots, x_m . Now

$$\begin{aligned} R^n \prod_{k=0}^n (R - q^{-k}) &= R^{2n+1} \prod_{k=0}^n (1 - q^{-n-k}) \\ &\geq R^{2n+1} \prod_{j=0}^{\infty} (1 - q^{-j}) \\ (2.23) \qquad \qquad \qquad &\geq C_2 q^{mn(2n+1)}, \end{aligned}$$

where $C_2 := \prod_{j=0}^{\infty} (1 - q^{-j})$ is a constant depending only on q . Putting (2.22) and (2.23) into (2.20), we have

$$|I(x_1, \dots, x_m)| \leq c_q q^{-2mn(n+1)},$$

where

$$c_q := f_q C_1 / C_2.$$

This completes the proof of Theorem 2.1.

3. PROOF OF THEOREM 1.1

We first prove that for $0 < x_1, \dots, x_m < q$, and $q > 1$,

$$(3.1) \qquad |I(x_1, \dots, x_m)| \neq 0,$$

where $I(x_1, \dots, x_m)$ is defined by (2.9). Note that if we choose the contour in (2.9) to be $\Gamma = \{z \in \mathbb{C} : |z| = 1 + \epsilon\}$, where $\epsilon > 0$ is small enough such that $0 < |tx_1|, \dots, |tx_m| < q$, for $t \in \Gamma$, then

$$L(tx_1, \dots, tx_m) = L^*(tx_1, \dots, tx_m), \quad t \in \Gamma.$$

Now

$$R_n(x_1, \dots, x_m) = \prod_{j=1}^n ((1 - q^{-j}x_1) \cdots (1 - q^{-j}x_m)) > 0$$

for $0 < x_1, \dots, x_m < q$, and

$$\begin{aligned}
 I(x_1, \dots, x_m) &= \frac{R_n(x_1, \dots, x_m)}{2\pi i} \int_{\Gamma} \frac{L(tx_1, \dots, tx_m) dt}{t^{2n+2} (\prod_{k=0}^n (1 - 1/(q^k t)))} \\
 &= \frac{R_n(x_1, \dots, x_m)}{2\pi i} \int_{\Gamma} \frac{L(tx_1, \dots, tx_m)}{t^{2n+2}} \left(\sum_{j_0, \dots, j_n \geq 0} \prod_{k=0}^n \left(\frac{1}{q^k t} \right)^{j_k} \right) dt \\
 &= R_n(x_1, \dots, x_m) \sum_{j_0, \dots, j_n \geq 0} q^{-\sum_{k=0}^n k j_k} \cdot \frac{1}{2\pi i} \int_{\Gamma} \left\{ \frac{1}{t^{2n+2+(j_0+\dots+j_n)}} \right. \\
 (3.2) \quad &\cdot \left. \sum_{i_1, \dots, i_m=0}^{\infty} \frac{x_1^{i_1} \dots x_m^{i_m} t^{i_1+\dots+i_m}}{q^{i_1+\dots+i_m+1} - 1} \right\} dt \\
 &= R_n(x_1, \dots, x_m) \sum_{j_0, \dots, j_n \geq 0} q^{-\sum_{k=0}^n k j_k} \\
 &\cdot \sum_{i_1+\dots+i_m-(2n+j_0+\dots+j_n+2)=-1} \frac{x_1^{i_1} \dots x_m^{i_m}}{q^{i_1+\dots+i_m+1} - 1} \\
 &= R_n(x_1, \dots, x_m) \sum_{\substack{i_1+\dots+i_m=2n+j_0+\dots+j_n+1 \\ q \text{ quad } j_0, \dots, j_n \geq 0}} q^{-\sum_{k=0}^n k j_k} \frac{x_1^{i_1} \dots x_m^{i_m}}{q^{i_1+\dots+i_m+1} - 1} \\
 &> 0,
 \end{aligned}$$

as $x_1, \dots, x_m \geq 0$, $q > 1$, and as infinitely many terms above are positive, so (3.1) holds.

Now let r_1, r_2, \dots, r_m be any fixed positive rational numbers such that $r_1, r_2, \dots, r_m \neq q^j$ for all $j = 1, 2, \dots$. From (2.7), we can see that the irrationality of $L(r_1, r_2, \dots, r_m)$ is equivalent to the irrationality of $L(q^{-k}r_1, q^{-k}r_2, \dots, q^{-k}r_m)$ for any integer $k \geq 1$, so we can assume that $0 < r_1, r_2, \dots, r_m < q$, and then

$$L(r_1, r_2, \dots, r_m) = \sum_{j_1, \dots, j_m=0}^{\infty} \frac{r_1^{j_1} \dots r_m^{j_m}}{q^{j_1+\dots+j_m+1} - 1} > 0.$$

Now let

$$(3.3) \quad H_{m,n}(q) := q^{(m-1)n(n+1)/2} \left(\prod_{j=1}^{n+1} (q^j - 1) \right).$$

Then

$$(3.4) \quad 0 < |H_{m,n}(q)| \leq q^{(mn+2)(n+1)/2}$$

and

$$(3.5) \quad H_{m,n}(q) \cdot \{Q(r_1, \dots, r_m), P(r_1, \dots, r_m)\} \subset \mathbb{Z}[q, r_1, \dots, r_m].$$

Now as

$$\begin{aligned}
 \Delta_{m,n} &:= |H_{m,n}(q)Q(r_1, \dots, r_m)L(r_1, \dots, r_m) + H_{m,n}(q)P(r_1, \dots, r_m)| \\
 &= |H_{m,n}(q)| |I(r_1, \dots, r_m)| \\
 (3.6) \quad &> 0,
 \end{aligned}$$

and from (2.15) and (3.3), we have

$$\begin{aligned}
 \Delta_{m,n} &\leq q^{(mn+2)(n+1)/2} \frac{c_q}{q^{2mn(n+1)}} \\
 &= \frac{c_q}{q^{3mn(n+1)/2-1}} \\
 (3.7) \qquad &\leq \frac{c_q}{q^{mn^2}}.
 \end{aligned}$$

Finally, if

$$(3.8) \qquad r_1 := \frac{i_1}{l_1}, r_2 := \frac{i_2}{l_2}, \dots, r_m := \frac{i_m}{l_m},$$

with i_1, \dots, i_m and l_1, \dots, l_m positive integers, then

$$(3.9) \qquad Q^*(r_1, \dots, r_m) := (l_1 \cdots l_m)^{2n} H_{m,n}(q) Q(r_1, \dots, r_m)$$

and

$$(3.10) \qquad P^*(r_1, \dots, r_m) := (l_1 \cdots l_m)^{2n} H_{m,n}(q) P(r_1, \dots, r_m)$$

are integers, and by (3.6) to (3.10),

$$\begin{aligned}
 0 &< |Q^*(r_1, \dots, r_m)L(r_1, \dots, r_m) + P^*(r_1, \dots, r_m)| \\
 &= (l_1 \cdots l_m)^{2n} |H_{m,n}(q)| |Q(r_1, \dots, r_m)L(r_1, \dots, r_m) + P(r_1, \dots, r_m)| \\
 &\leq (l_1 \cdots l_m)^{2n} \frac{c_q}{q^{mn^2}},
 \end{aligned}$$

which tends to zero as $n \rightarrow \infty$. This shows that $L(r_1, \dots, r_m)$ is irrational, that is,

$$\sum_{j_1, \dots, j_m=0}^{\infty} \frac{r_1^{j_1} \cdots r_m^{j_m}}{q^{j_1 + \cdots + j_m + 1} - 1}$$

is irrational for $q > 1$ integer, r_1, r_2, \dots, r_m positive rationals less than q and integer $m \geq 1$, and

$$\sum_{j=1}^{\infty} \frac{q^{-j}}{(1 - q^{-j}r_1)(1 - q^{-j}r_2) \cdots (1 - q^{-j}r_m)}$$

is irrational for $q > 1$ integer, r_1, r_2, \dots, r_m positive rationals such that $r_1, r_2, \dots, r_m \neq q^j$ for all $j = 1, 2, \dots$, and integer $m \geq 1$.

This completes the proof of Theorem 1.1.

Now by the standard methods (as in chapter 11 of Borwein and Borwein [1]), the estimates in the proof of Theorem 1.1 gives that, under the assumption of the theorem,

$$\left| L(r_1, \dots, r_m) - \frac{s}{t} \right| > \frac{1}{t^\alpha},$$

for some constant α and all integers s and t , and hence

$$\sum_{j_1, \dots, j_m=0}^{\infty} \frac{r_1^{j_1} \cdots r_m^{j_m}}{q^{j_1 + \cdots + j_m + 1} - 1}$$

is not a Liouville number.

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