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LIMITS OF RESIDUALLY IRREDUCIBLE p-ADIC GALOIS REPRESENTATIONS

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Abstract. In this paper we produce examples of converging sequences of Galois representations, and study some of their properties.

1. Introduction

Consider a continuous representation

$$\rho : G_L \to GL_m(K)$$

of the absolute Galois group G_L of a number field L, with K a finite extension of $\mathbf{Q}_p$, with $\mathcal{O}$ its ring of integers, $|\ \ |$ its norm, and k its residue field. Then ρ has an integral model taking values in $GL_m(\mathcal{O})$, and the semisimplification of its reduction modulo the maximal ideal m of $\mathcal{O}$, denoted by $\overline{\rho}$, is independent of the choice of integral model. We assume that $\overline{\rho}$ is absolutely irreducible and, in fact, we assume that all the p-adic representations considered in this paper are *residually absolutely irreducible.*

Definition 1. An infinite sequence of (residually absolutely irreducible) continuous representations $\rho_i : G_L \to GL_m(K)$ tends to $\rho : G_L \to GL_m(K)$, if $|\mathrm{tr}(\rho_i(g)) - \mathrm{tr}(\rho(g))| \to 0$ uniformly for all $g \in G_L$. We also say that the ρ_i's converge to ρ, or ρ is their limit point.

By Theorem 1 of [Ca], which we can apply because of our blanket assumption of residual absolute irreducibility, this is equivalent to saying that given any integer n, for all $i >> 0$, the reduction mod m^n, $\rho_{i,n}$, of (an integral model of) ρ_i is isomorphic to the reduction mod m^n, ρ_n, of (an integral model of) ρ. Note that we are not assuming that the ρ_i's (or ρ) are *finitely ramified*, though we do know by the main theorem of [KhRa] that the density of primes which ramify in a given ρ_i is 0.

In this paper we study the limiting behavior of the lifts produced in [R1] and completely characterize the limit points of these lifts (see Theorem 1 below). This suggests another approach to certain special cases of the modularity lifting theorems of Wiles, Taylor-Wiles, et al. In the process we construct many sequences of converging p-adic Galois representations (of fixed determinant and fixed ramification behaviour at p). This raises many questions that can be posed far more easily than answered.

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Consider $\overline{\rho} : G_{\mathbf{Q}} \to GL_2(k)$ that satisfies the conditions of [R1], namely:

- $\overline{\rho}$ and $\mathrm{Ad}^0(\overline{\rho})$ are absolutely irreducible Galois representations, and the finite field k of characteristic p is the minimal field of definition of $\overline{\rho}$.
- The (prime to p) Artin conductor $N(\overline{\rho})$ of $\overline{\rho}$ is minimal amongst its twists. Denote by S the set of primes given by the union of the places where $\overline{\rho}$ is ramified and $\{p, \infty\}$.
- If $\overline{\rho}$ is even, then for the decomposition group G_p above p we assume that $\overline{\rho}\mid_{G_p}$ is not twist equivalent to $\left(\begin{smallmatrix} \chi & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or twist equivalent to the indecomposable representation $\begin{pmatrix} \chi^{p-2} & * \\ 0 & 1 \end{pmatrix}$ where χ is the mod p cyclotomic character.
- If $\overline{\rho}$ is odd, we assume $\overline{\rho}\mid_{G_p}$ is not twist equivalent to the trivial representation or the indecomposable unramified representation given by $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$.
- $p \geq 7$ and the order of the projective image of $\overline{\rho}$ is a multiple of p.

Let $Q = \{q_1, \ldots, q_n\}$ be a finite set of primes such that $q_i \neq \pm 1 \bmod p$, unramified in $\overline{\rho}$, and the ratio of the eigenvalues of $\overline{\rho}(\mathrm{Frob}_{q_i})$ equal to $q_i^{\pm 1}$. We will call the primes as in Q above *Ramakrishna primes for* $\overline{\rho}$ or *R-primes* for short (suppressing the $\overline{\rho}$ which is fixed). We consider the deformation ring $R_{S\cup Q}^{Q-new}$ of [KR] (see Definition 1 of loc. cit.). To orient the reader we recall the definition of $R_{S\cup Q}^{Q-new}$. For this we need:

Definition 2. If q is a prime, $G_{\mathbf{Q}_q}$ the absolute Galois group of $\mathbf{Q}_q$ and R a complete Noetherian local ring with residue field k, a continuous representation $\rho : G_{\mathbf{Q}_q} \to GL_2(R)$ is said to be special if up to conjugacy it is of the form $\begin{pmatrix} \varepsilon\chi' & * \\ 0 & \chi' \end{pmatrix}$ for ε the p-adic cyclotomic character, and $\chi' : G_{\mathbf{Q}_q} \to R^*$ a continuous character. A continuous representation $\tilde{\rho} : G_{\mathbf{Q}} \to GL_2(R)$, is said to be special at a prime q if $\tilde{\rho}|_{D_q}$, with D_q a decomposition group at q, is special.

Then $R_{S\cup Q}^{Q-new}$ is the universal ring that parametrizes deformations of $\overline{\rho}$ that are minimally ramified at S (in the sense of [R1]: thus when $\overline{\rho}$ is even there is no condition at p) and such that at primes $q \in Q$ these deformations are *special*. The ring $R_{S\cup Q}^{Q-new}$ is a complete Noetherian local $W(k)$-algebra, with $W(k)$ the Witt vectors of k. The deformation rings considered here are for the deformation problem with a certain fixed (arithmetic) determinant character, and all the deformations of $\overline{\rho}$ we consider will have this fixed determinant character.

Definition 3. A finite set of R-primes Q is said to be auxiliary if $R_{S\cup Q}^{Q-new} \simeq W(k)$.

In [R1] auxiliary sets Q of the above type were proven to exist. The representation corresponding to $R_{S\cup Q}^{Q-new} \simeq W(k)$ is denoted by $\rho_{S\cup Q}^{Q-new}$. We will call these lifts *Ramakrishna lifts of* $\overline{\rho}$ or *R-lifts* for short (suppressing the $\overline{\rho}$ which is fixed).

Theorem 1. *A continuous representation $\rho : G_{\mathbf{Q}} \to GL_2(W(k))$ that is a deformation of $\overline{\rho}$, is a limit point of distinct R-lifts, if and only if ρ is unramified outside S and the set of all R-primes, and minimally ramified at primes of S.*

Remark. Thus, we have a complete description of the "p-adic closure" of *R-lifts*. Note that, in particular, each R-lift is a limit point of other R-lifts. Note also that any deformation $G_{\mathbf{Q}} \to GL_2(K)$ of $\overline{\rho}$ that is a limit point of R-lifts has a model that takes values in $GL_2(W(k))$. The above theorem can be viewed in a sense as producing an "infinite fern" structure (in the sense of Mazur) in the set of all R-lifts

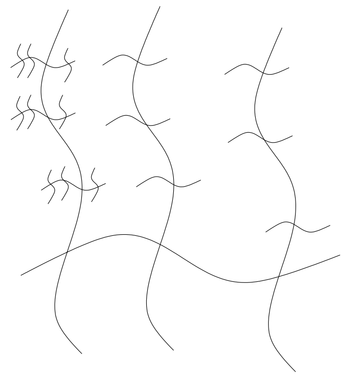

FIGURE 1. An infinite fern

of a given $\overline{\rho}$ as above (see the picture below). From the proof of Theorem 1 above, we in fact can deduce that each R-lift gives rise to infinitely many "splines" passing through it, where a "spline" consists of a sequence of R-lifts converging to it, and each element in a spline gives rise to its own infinitely many splines. Missing from the picture are the limit points of R-lifts which themselves are not R-lifts and which the theorem above characterizes completely.

In Section 2 we prove Theorem 1 which is a simple consequence of the methods of [R1] and [T1]. In Section 3 we prove a result about converging sequences of representations arising from newforms, and point out a possible approach to the lifting theorems of Wiles, et al. that is suggested by the work here. In Section 4 we raise questions about rationality and motivic properties of converging sequences of p-adic Galois representations.

2. Converging sequences of Galois representations

We now prove Theorem 1, which follows from the methods of [R1] and [T1]. For the proof we need the following lemma which follows from the methods of [R1] (see also Lemma 1.2 of [T1]) and Lemma 8 of [KR].

Lemma 1. *Let $\rho_n : G_{\mathbf{Q}} \to GL_2(W(k)/(p^n))$ be a lift of $\overline{\rho}$ that is unramified outside S and the set of all R-primes, minimally ramified at primes of S, and special at all the primes outside S at which it is ramified. Let Q'_n be any finite set of primes that includes the primes of ramification of ρ_n such that $Q'_n \backslash S$ contains only R-primes and such that $\rho_n|_{D_q}$ is special for $q \in Q'_n \backslash S$. Then there exists a finite set of primes Q_n that contains Q'_n, such that $\rho_n|_{D_q}$ is special for $q \in Q_n \backslash S$, $Q_n \backslash S$ contains only R-primes and $Q_n \backslash S$ is auxiliary.*

Proof. We use [R1] and Lemma 8 of [KR] to construct an auxiliary set of primes T_n such that $\rho_n|_{D_q}$ is special for $q \in T_n$. Then as $Q'_n \backslash S$ contains only R-primes, it follows (using notation of [R1]) from Proposition 1.6 of [W] that the kernel and cokernel of the map

$$H^1(G_{S \cup T_n \cup Q'_n},\ \mathrm{Ad}^0(\overline{\rho})) \to \bigoplus_{v \in S \cup T_n \cup Q'_n} H^1(G_v,\ \mathrm{Ad}^0(\overline{\rho}))/\mathcal{N}_v$$

have the same cardinality. Then using Proposition 10 of [R1], or Lemma 1.2 of [T1], and Lemma 8 of [KR], we can augment the set $S \cup T_n \cup Q'_n$ to get a set Q_n as in the statement of the lemma.

We are now ready to prove Theorem 1. If $\rho : G_{\mathbf{Q}} \to GL_2(W(k))$ is a limit point of R-lifts, then it is clear that ρ is unramified outside S and the set of all R-primes, and minimally ramified at primes of S. We prove the converse. So let ρ satisfy the conditions of Theorem 1, and recall that we denote by ρ_n the reduction modulo p^n of ρ. It is easily checked that if q is an R-prime, then any deformation of $\overline{\rho}|_{D_q}$ to a *ramified* p-adic representation is *special*; this follows from the structure of tame inertia and the fact that $q^2 \neq 1$ mod p. Furthermore, from the method of proof of Proposition 1 of [KhRa], we easily deduce that the set of primes q for which $\rho|_{D_q}$ is special is of density 0. Thus using Cebotarev and the assumptions on $\overline{\rho}$ in the introduction, we choose a finite set of primes Q'_n such that

- $Q'_n \backslash S$ consists of R-primes and $\rho_n|_{D_q}$ is special for $q \in Q'_n \backslash S$,
- Q'_n contains all the ramified primes of ρ_n,
- for some prime $q \in Q'_n \backslash S$, $\rho|_{D_q}$ is not special.

Using Lemma 1 we complete Q'_n to a set Q_n such that $Q_n \backslash S$ is auxiliary and $\rho_n|_{D_q}$ is special for $q \in Q_n \backslash S$. Then we claim $\rho_{S \cup Q_n}^{Q_n \backslash S-new} \equiv \rho$ mod p^n. The claim is true as there is a unique representation $G_{\mathbf{Q}} \to GL_2(W(k)/(p^n))$ (with the determinant that we have fixed) that is unramified outside $S \cup Q_n$, minimal at S and special at primes of $Q_n \backslash S$ (as $R_{S \cup Q_n}^{Q_n \backslash S-new} \simeq W(k)$). By construction the sets Q_n contain at least one prime at which ρ is not special. Thus, we see that we can pick a subsequence of mutually distinct representations ρ_i from the $\rho_{S \cup Q_n}^{Q_n \backslash S-new}$'s such that $\rho_i \to \rho$. □

Remark. It is of vital importance that ρ is $GL_2(W(k))$-valued since, otherwise, we would not be able to invoke the disjointness results that are used in the proof of Lemma 1 (Lemma 8 of [KR]).

Remark. Theorem 1 can be applied in practice to give many examples of converging sequences of p-adic representations: for a non-CM elliptic curve $E_{/\mathbf{Q}}$ for most primes p the mod p representation satisfies the conditions given in the introduction, and the corresponding p-adic representation is minimally ramified and $GL_2(\mathbf{Z}_p)$ valued.

We end this section with a result that refines the main result of [KhRa].

Proposition 1. *If $\rho_i : G_L \to GL_m(K)$ is a sequence of (residually absolutely irreducible) continuous representations that converges to ρ, then the set of primes where any of the ρ_i's is ramified (i.e., $\bigcup \mathrm{Ram}(\rho_i)$ where $\mathrm{Ram}(\rho_i)$ is the set of primes at which ρ_i is ramified) is of density zero.*

Proof. Denote by $\rho_{i,n}$ (resp., ρ_n) the reduction mod m^n of an integral model of ρ_i (resp., ρ). The proof consists of applying Theorem 1 of [KhRa] twice; more precisely, first its statement, and then its proof. By an application of its statement we conclude that the density of $\bigcup_{i=1}^n \mathrm{Ram}(\rho_i)$ is 0 for any n. Now applying the proof of Theorem 1 of [KhRa], we define $c_{\rho,n}$ to be the upper density of the set $S_{\rho,n}$ of primes q of L that

- lie above primes which split in $L/\mathbf{Q}$,
- are unramified in ρ_1 and $\neq p$,

- $\rho_n|_{D_q}$ is unramified, but there exists a "lift" of $\rho_n|_{D_q}$, with D_q the decomposition group at q, to a representation $\tilde{\rho}_q$ of D_q to $GL_m(K)$ that is ramified at q; by a lift we mean some conjugate of $\tilde{\rho}_q$ reduces mod m^n to $\rho_n|_{D_q}$.

We have from [KhRa] (see Proposition 1 of loc. cit. which was stated in greater generality than needed there with the present application in mind):

Lemma 2. *Given any $\varepsilon > 0$, there is an integer N_ε such that $c_{\rho,n} < \varepsilon$ for $n > N_\varepsilon$.*

To prove Proposition 1 it is enough to show that given any $\varepsilon > 0$, the upper density of the set $\bigcup \mathrm{Ram}(\rho_i)$ is $< \varepsilon$. Since $\bigcup_{i=1}^{n} \mathrm{Ram}(\rho_i)$ has density 0 for (the finite) n that is the supremum of the i's such that ρ_{i,N_ε} is not isomorphic to ρ_{N_ε}, and ρ_{N_ε} is finitely ramified, it follows from the lemma above that the upper density of $\bigcup \mathrm{Ram}(\rho_i)$ is $< \varepsilon$. Hence Proposition 1. $\square$

Remark. One can ask for more refined information about the asymptotics of ramified primes in (limits of) residually absolutely irreducible p-adic Galois representations. For instance, in Theorem 1 of [KhRa] one can ask (clued by Theorem 10 of [S1]) if the order of growth of ramified primes can be proved to be bounded by $O(x^{1-\frac{1}{2N}+\epsilon})$, where N is the p-adic analytic dimension of $\mathrm{im}(\rho)$, for any $\epsilon > 0$. Such quantitative refinements asked for by Serre in an e-mail message to the author are difficult and will require a new idea (that goes beyond [KhRa]) and a strong use of effective versions of the Cebotarev density theorem.

3. Finite and infinite ramification

Let L be a number field and K a finite extension of $\mathbf{Q}_p$ as before.

Definition 4. We say that a residually absolutely irreducible continuous representation $\rho : G_L \to GL_n(K)$ is motivic if ρ arises as a subquotient of the i^{th} étale cohomology $H^i(X \times_L \overline{L}, K)$ of a smooth projective variety X defined over a number field L.

A *motivic* representation is *finitely ramified*. In [R] examples of residually irreducible representations $\rho : G_{\mathbf{Q}} \to GL_2(K)$ were constructed that were infinitely ramified (see also the last section of [KR]). Infinitely ramified p-adic representations cannot be *motivic*. But they can arise as limits of p-adic representations that are *motivic*. Fix an embedding $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}_p}$. Then as in [R] (and the last section of [KR]), there is a sequence of eigenforms $f_i \in S_2(\Gamma_0(N_i))$, for a sequence of squarefree integers N_i such that $N_i \to \infty$ and $(p, N_i) = 1$, new of level N_i such that the corresponding p-adic representations $\rho_{f_i} : G_{\mathbf{Q}} \to GL_2(\mathbf{Z}_p)$ have a p-adic limit ρ, with ρ *infinitely ramified*. Such a ρ is *non-motivic*, but is the limit of *motivic* p-adic representations. Such limits of eigenforms (in the works of Serre and Katz; for instance, cf., [Ka]) have been considered when varying weights or varying the p-power level, while fixing the prime-to-p part of the level.

Proposition 2. *Let $f_i \in S_2(\Gamma_0(N_i))$ be a sequence of eigenforms with coefficients in a finite extension K of $\mathbf{Q}_p$ with $(N_i, p) = 1$ and $p \geq 3$, that in the p-adic q-expansion topology tend to an element $f \in K[[q]]$, such that the corresponding residual representation $\overline{\rho}$ satisfies the conditions in the introduction. The element f, that gives rise naturally to a Galois representation $\rho_f : G_{\mathbf{Q}} \to GL_2(K)$, is the q-expansion of a classical eigenform (of weight 2) if and only if ρ_f is finitely ramified.*

Proof. The only if part is clear. The if part follows from the methods of Wiles (see Chapter 3 of [W] and also [TW]) and their refinements: note that ρ_f is finite flat at p. $\square$

Remark. Applying Theorem 1 when $\overline{\rho}$ is odd and finite flat at p, in which case the R-lifts are modular by Theorem 1 of [K], we can construct systematically many examples of sequences of eigenforms $f_i \to f$ $(f \in K[[q]])$, with the levels of f_i unbounded and such that ρ_f is finitely ramified (f, in fact, is then a classical eigenform as above). On the other hand, as recalled above in [R] (see also last section of [KR]), we have examples of situations as above with ρ_f infinitely ramified.

It will be of interest to see if Proposition 2 could be proved in a more self-contained manner. The proof above does not use seriously the fact that one does know that f arises as a limit of the classical forms f_i. If such a proof could be devised, in conjunction with Theorem 1 above and Theorem 1 of [K] (which is due to Ravi Ramakrishna) it would give, in special cases, a simpler approach using R-primes to the modularity lifting theorems of Wiles, et al. (see also [K]) that directly works with the p-adic Galois representation that needs to be proved modular, and if it could be implemented, would avoid (albeit in special cases) the sophisticated deformation theoretic approach of [W].

We elaborate on this: Assume that $\overline{\rho}$ is modular. In Theorem 1 we have characterized the limit points of R-lifts. By Theorem 1 of [K] which proves that the representation corresponding to $R_{S\cup Q}^{Q-new} \simeq W(k)$ is modular as a consequence of the isomorphism $R_{S\cup Q}^{Q-new} \simeq \mathbf{T}_{S\cup Q}^{Q-new}$ (using notation of [K]), we know that R-lifts are modular. Hence, limits of R-lifts do arise as limits of p-adic representations arising from classical newforms. It only (!) remains to prove that a limit of a converging sequence of p-adic representations arising from newforms (say of weight 2 and level prime to p to avoid delicate considerations at p) that is finitely ramified itself arises from a newform (i.e., prove Proposition 2 without appealing directly to [W]). Note that for a semistable elliptic curve E, for all large enough primes p (bigger than 3 for the methods here to directly work unfortunately!), $T_p(E)$ is a limit point of R-lifts.

Note. In recent work we have indeed been able to give a self-contained approach to a result such as Proposition 2 above under some technical restrictions; see [K1].

4. Questions

Proposition 2 suggests that a representation that arises as a limit of motivic representations (of "bounded weights"; see Definition 6 below) is finitely ramified if and only if it is motivic. We first recall one of the main conjectures in [FM] in a form that is most pertinent for the considerations here.

Conjecture 1 (Fontaine-Mazur)**.** Consider a continuous residually absolutely irreducible representation $\rho : G_L \to GL_m(K)$ that is potentially semistable at places above p. Then the following are equivalent:

(1) ρ is motivic,
(2) ρ is finitely ramified.

From our earlier considerations it is natural to ask the following weaker question.

Question 1. Consider a continuous residually absolutely irreducible representation $\rho : G_L \to GL_m(K)$ that is potentially semistable at places above p and arises as the limit of motivic representations ρ_i. Then if ρ is finitely ramified, is ρ motivic?

It seems unlikely that the infinitely ramified representations produced in [R] are *algebraic* (see definition below). This motivates the following considerations.

Definition 5. A continuous (residually absolutely irreducible) representation $\rho : G_L \to GL_m(K)$ is said to be algebraic if there is a number field F such that the characteristic polynomial of $\rho(\mathrm{Frob}_q)$ has coefficients in the ring of integers of F for all primes q which are unramified in ρ. The minimal such field is the field of definition of ρ.

As by the main theorem of [KhRa], the set of primes at which ρ ramifies is of density 0, the definition above is a sensible one.

Definition 6. A continuous (residually absolutely irreducible) algebraic representation $\rho : G_L \to GL_m(K)$ is said to be of weight $\leq t$ ($t \in \mathbf{Z}$) if for primes q that are unramified in ρ, any root α of the characteristic polynomial of $\rho(\mathrm{Frob}_q)$ satisfies $|\iota(\alpha)| \leq |k_q|^{\frac{t-1}{2}}$ for any embedding $\iota : \overline{\mathbf{Q}} \to \mathbf{C}$, with k_q the residue field at q.

Question 2. If $\rho_i : G_L \to GL_m(K)$ is an infinite sequence of (residually absolutely irreducible) distinct algebraic representations, all of weight $\leq t$ for some fixed integer t, converging to $\rho : G_L \to GL_m(K)$, and K_i the field of definition of ρ_i, does $[K_i : \mathbf{Q}] \to \infty$ as $i \to \infty$?

Remark.

- It is observed in [R] (this is a remark of Fred Diamond) that in the situation of Question 2 only finitely many of the ρ_i's can arise from elliptic curves; this is a consequence of the Mordell conjecture which gives that suitable twists of the classical modular curves $X(p^n)$ for $n >> 0$ have finitely many L-valued points for a given number field L.
- If Question 2 has a negative answer, using Proposition 1, we deduce that for a set of primes $\{r\}$ of density one, the characteristic polynomials of $\rho_i(\mathrm{Frob}_r)$ are eventually constant. Hence, we deduce that the characteristic polynomials of $\rho(\mathrm{Frob}_r)$ are defined and integral over a fixed number field F, i.e., ρ is algebraic (in the case when ρ is infinitely ramified this is linked to the questions below).

Question 3. Let $\rho : G_L \to GL_m(K)$ be a continuous, residually absolutely irreducible representation that is potentially semistable at places above p. Then are the following equivalent:

(1) ρ is motivic,
(2) ρ is finitely ramified,
(3) ρ is algebraic?

In the question above, the equivalence of 1 and 2 is the Fontaine-Mazur conjecture recalled above: the possible equivalence of 3 to 1 and 2 is the main thrust of the question. One might even ask the stronger question: If $\rho : G_L \to GL_m(K)$, a continuous, residually absolutely irreducible representation, is algebraic, then is ρ *forced* to be both finitely ramified *and* potentially semistable at places above p? All the questions of this section have a positive answer when $m = 1$.

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