

## LIMITS OF RESIDUALLY IRREDUCIBLE $p$ -ADIC GALOIS REPRESENTATIONS

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ABSTRACT. In this paper we produce examples of converging sequences of Galois representations, and study some of their properties.

### 1. INTRODUCTION

Consider a continuous representation

$$\rho : G_L \rightarrow GL_m(K)$$

of the absolute Galois group  $G_L$  of a number field  $L$ , with  $K$  a finite extension of  $\mathbf{Q}_p$ , with  $\mathcal{O}$  its ring of integers,  $|\cdot|$  its norm, and  $k$  its residue field. Then  $\rho$  has an integral model taking values in  $GL_m(\mathcal{O})$ , and the semisimplification of its reduction modulo the maximal ideal  $\mathfrak{m}$  of  $\mathcal{O}$ , denoted by  $\bar{\rho}$ , is independent of the choice of integral model. We assume that  $\bar{\rho}$  is absolutely irreducible and, in fact, we assume that all the  $p$ -adic representations considered in this paper are *residually absolutely irreducible*.

**Definition 1.** An infinite sequence of (residually absolutely irreducible) continuous representations  $\rho_i : G_L \rightarrow GL_m(K)$  tends to  $\rho : G_L \rightarrow GL_m(K)$ , if  $|\mathrm{tr}(\rho_i(g)) - \mathrm{tr}(\rho(g))| \rightarrow 0$  uniformly for all  $g \in G_L$ . We also say that the  $\rho_i$ 's converge to  $\rho$ , or  $\rho$  is their limit point.

By Theorem 1 of [Ca], which we can apply because of our blanket assumption of residual absolute irreducibility, this is equivalent to saying that given any integer  $n$ , for all  $i \gg 0$ , the reduction mod  $\mathfrak{m}^n$ ,  $\rho_{i,n}$ , of (an integral model of)  $\rho_i$  is isomorphic to the reduction mod  $\mathfrak{m}^n$ ,  $\rho_n$ , of (an integral model of)  $\rho$ . Note that we are not assuming that the  $\rho_i$ 's (or  $\rho$ ) are *finitely ramified*, though we do know by the main theorem of [KhRa] that the density of primes which ramify in a given  $\rho_i$  is 0.

In this paper we study the limiting behavior of the lifts produced in [R1] and completely characterize the limit points of these lifts (see Theorem 1 below). This suggests another approach to certain special cases of the modularity lifting theorems of Wiles, Taylor-Wiles, et al. In the process we construct many sequences of converging  $p$ -adic Galois representations (of fixed determinant and fixed ramification behaviour at  $p$ ). This raises many questions that can be posed far more easily than answered.

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Consider  $\bar{\rho} : G_{\mathbf{Q}} \rightarrow GL_2(k)$  that satisfies the conditions of [R1], namely:

- $\bar{\rho}$  and  $\text{Ad}^0(\bar{\rho})$  are absolutely irreducible Galois representations, and the finite field  $k$  of characteristic  $p$  is the minimal field of definition of  $\bar{\rho}$ .
- The (prime to  $p$ ) Artin conductor  $N(\bar{\rho})$  of  $\bar{\rho}$  is minimal amongst its twists. Denote by  $S$  the set of primes given by the union of the places where  $\bar{\rho}$  is ramified and  $\{p, \infty\}$ .
- If  $\bar{\rho}$  is even, then for the decomposition group  $G_p$  above  $p$  we assume that  $\bar{\rho}|_{G_p}$  is not twist equivalent to  $\begin{pmatrix} \chi & 0 \\ 0 & 1 \end{pmatrix}$  or twist equivalent to the indecomposable representation  $\begin{pmatrix} \chi^{p-2} & * \\ 0 & 1 \end{pmatrix}$  where  $\chi$  is the mod  $p$  cyclotomic character.
- If  $\bar{\rho}$  is odd, we assume  $\bar{\rho}|_{G_p}$  is not twist equivalent to the trivial representation or the indecomposable unramified representation given by  $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ .
- $p \geq 7$  and the order of the projective image of  $\bar{\rho}$  is a multiple of  $p$ .

Let  $Q = \{q_1, \dots, q_n\}$  be a finite set of primes such that  $q_i \not\equiv \pm 1 \pmod{p}$ , unramified in  $\bar{\rho}$ , and the ratio of the eigenvalues of  $\bar{\rho}(\text{Frob}_{q_i})$  equal to  $q_i^{\pm 1}$ . We will call the primes as in  $Q$  above *Ramakrishna primes for  $\bar{\rho}$*  or *R-primes* for short (suppressing the  $\bar{\rho}$  which is fixed). We consider the deformation ring  $R_{S \cup Q}^{Q\text{-new}}$  of [KR] (see Definition 1 of loc. cit.). To orient the reader we recall the definition of  $R_{S \cup Q}^{Q\text{-new}}$ . For this we need:

**Definition 2.** If  $q$  is a prime,  $G_{\mathbf{Q}_q}$  the absolute Galois group of  $\mathbf{Q}_q$  and  $R$  a complete Noetherian local ring with residue field  $k$ , a continuous representation  $\rho : G_{\mathbf{Q}_q} \rightarrow GL_2(R)$  is said to be special if up to conjugacy it is of the form  $\begin{pmatrix} \varepsilon \chi' & * \\ 0 & \chi' \end{pmatrix}$  for  $\varepsilon$  the  $p$ -adic cyclotomic character, and  $\chi' : G_{\mathbf{Q}_q} \rightarrow R^*$  a continuous character. A continuous representation  $\tilde{\rho} : G_{\mathbf{Q}} \rightarrow GL_2(R)$ , is said to be special at a prime  $q$  if  $\tilde{\rho}|_{D_q}$ , with  $D_q$  a decomposition group at  $q$ , is special.

Then  $R_{S \cup Q}^{Q\text{-new}}$  is the universal ring that parametrizes deformations of  $\bar{\rho}$  that are minimally ramified at  $S$  (in the sense of [R1]: thus when  $\bar{\rho}$  is even there is no condition at  $p$ ) and such that at primes  $q \in Q$  these deformations are *special*. The ring  $R_{S \cup Q}^{Q\text{-new}}$  is a complete Noetherian local  $W(k)$ -algebra, with  $W(k)$  the Witt vectors of  $k$ . The deformation rings considered here are for the deformation problem with a certain fixed (arithmetic) determinant character, and all the deformations of  $\bar{\rho}$  we consider will have this fixed determinant character.

**Definition 3.** A finite set of  $R$ -primes  $Q$  is said to be auxiliary if  $R_{S \cup Q}^{Q\text{-new}} \simeq W(k)$ .

In [R1] auxiliary sets  $Q$  of the above type were proven to exist. The representation corresponding to  $R_{S \cup Q}^{Q\text{-new}} \simeq W(k)$  is denoted by  $\rho_{S \cup Q}^{Q\text{-new}}$ . We will call these lifts *Ramakrishna lifts of  $\bar{\rho}$*  or *R-lifts* for short (suppressing the  $\bar{\rho}$  which is fixed).

**Theorem 1.** *A continuous representation  $\rho : G_{\mathbf{Q}} \rightarrow GL_2(W(k))$  that is a deformation of  $\bar{\rho}$ , is a limit point of distinct  $R$ -lifts, if and only if  $\rho$  is unramified outside  $S$  and the set of all  $R$ -primes, and minimally ramified at primes of  $S$ .*

*Remark.* Thus, we have a complete description of the “ $p$ -adic closure” of  $R$ -lifts. Note that, in particular, each  $R$ -lift is a limit point of other  $R$ -lifts. Note also that any deformation  $G_{\mathbf{Q}} \rightarrow GL_2(K)$  of  $\bar{\rho}$  that is a limit point of  $R$ -lifts has a model that takes values in  $GL_2(W(k))$ . The above theorem can be viewed in a sense as producing an “infinite fern” structure (in the sense of Mazur) in the set of all  $R$ -lifts

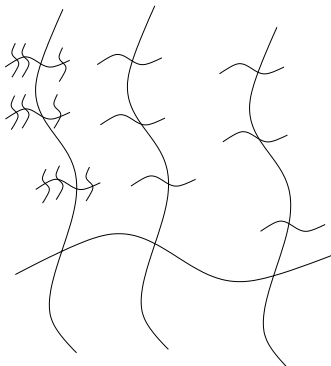


FIGURE 1. An infinite fern

of a given  $\bar{\rho}$  as above (see the picture below). From the proof of Theorem 1 above, we in fact can deduce that each  $R$ -lift gives rise to infinitely many “splines” passing through it, where a “spline” consists of a sequence of  $R$ -lifts converging to it, and each element in a spline gives rise to its own infinitely many splines. Missing from the picture are the limit points of  $R$ -lifts which themselves are not  $R$ -lifts and which the theorem above characterizes completely.

In Section 2 we prove Theorem 1 which is a simple consequence of the methods of [R1] and [T1]. In Section 3 we prove a result about converging sequences of representations arising from newforms, and point out a possible approach to the lifting theorems of Wiles, et al. that is suggested by the work here. In Section 4 we raise questions about rationality and motivic properties of converging sequences of  $p$ -adic Galois representations.

## 2. CONVERGING SEQUENCES OF GALOIS REPRESENTATIONS

We now prove Theorem 1, which follows from the methods of [R1] and [T1]. For the proof we need the following lemma which follows from the methods of [R1] (see also Lemma 1.2 of [T1]) and Lemma 8 of [KR].

**Lemma 1.** *Let  $\rho_n : G_{\mathbf{Q}} \rightarrow GL_2(W(k)/(p^n))$  be a lift of  $\bar{\rho}$  that is unramified outside  $S$  and the set of all  $R$ -primes, minimally ramified at primes of  $S$ , and special at all the primes outside  $S$  at which it is ramified. Let  $Q'_n$  be any finite set of primes that includes the primes of ramification of  $\rho_n$  such that  $Q'_n \setminus S$  contains only  $R$ -primes and such that  $\rho_n|_{D_q}$  is special for  $q \in Q'_n \setminus S$ . Then there exists a finite set of primes  $Q_n$  that contains  $Q'_n$ , such that  $\rho_n|_{D_q}$  is special for  $q \in Q_n \setminus S$ ,  $Q_n \setminus S$  contains only  $R$ -primes and  $Q_n \setminus S$  is auxiliary.*

*Proof.* We use [R1] and Lemma 8 of [KR] to construct an auxiliary set of primes  $T_n$  such that  $\rho_n|_{D_q}$  is special for  $q \in T_n$ . Then as  $Q'_n \setminus S$  contains only  $R$ -primes, it follows (using notation of [R1]) from Proposition 1.6 of [W] that the kernel and cokernel of the map

$$H^1(G_{S \cup T_n \cup Q'_n}, \text{Ad}^0(\bar{\rho})) \rightarrow \bigoplus_{v \in S \cup T_n \cup Q'_n} H^1(G_v, \text{Ad}^0(\bar{\rho}))/\mathcal{N}_v$$

have the same cardinality. Then using Proposition 10 of [R1], or Lemma 1.2 of [T1], and Lemma 8 of [KR], we can augment the set  $S \cup T_n \cup Q'_n$  to get a set  $Q_n$  as in the statement of the lemma.

We are now ready to prove Theorem 1. If  $\rho : G_{\mathbf{Q}} \rightarrow GL_2(W(k))$  is a limit point of  $R$ -lifts, then it is clear that  $\rho$  is unramified outside  $S$  and the set of all  $R$ -primes, and minimally ramified at primes of  $S$ . We prove the converse. So let  $\rho$  satisfy the conditions of Theorem 1, and recall that we denote by  $\rho_n$  the reduction modulo  $p^n$  of  $\rho$ . It is easily checked that if  $q$  is an  $R$ -prime, then any deformation of  $\bar{\rho}|_{D_q}$  to a ramified  $p$ -adic representation is special; this follows from the structure of tame inertia and the fact that  $q^2 \not\equiv 1 \pmod{p}$ . Furthermore, from the method of proof of Proposition 1 of [KhRa], we easily deduce that the set of primes  $q$  for which  $\rho|_{D_q}$  is special is of density 0. Thus using Chebotarev and the assumptions on  $\bar{\rho}$  in the introduction, we choose a finite set of primes  $Q'_n$  such that

- $Q'_n \setminus S$  consists of  $R$ -primes and  $\rho_n|_{D_q}$  is special for  $q \in Q'_n \setminus S$ ,
- $Q'_n$  contains all the ramified primes of  $\rho_n$ ,
- for some prime  $q \in Q'_n \setminus S$ ,  $\rho|_{D_q}$  is not special.

Using Lemma 1 we complete  $Q'_n$  to a set  $Q_n$  such that  $Q_n \setminus S$  is auxiliary and  $\rho_n|_{D_q}$  is special for  $q \in Q_n \setminus S$ . Then we claim  $\rho_{S \cup Q_n}^{Q_n \setminus S - new} \equiv \rho \pmod{p^n}$ . The claim is true as there is a unique representation  $G_{\mathbf{Q}} \rightarrow GL_2(W(k)/(p^n))$  (with the determinant that we have fixed) that is unramified outside  $S \cup Q_n$ , minimal at  $S$  and special at primes of  $Q_n \setminus S$  (as  $R_{S \cup Q_n}^{Q_n \setminus S - new} \simeq W(k)$ ). By construction the sets  $Q_n$  contain at least one prime at which  $\rho$  is not special. Thus, we see that we can pick a subsequence of mutually distinct representations  $\rho_i$  from the  $\rho_{S \cup Q_n}^{Q_n \setminus S - new}$ ,  $S$  such that  $\rho_i \rightarrow \rho$ .  $\square$

*Remark.* It is of vital importance that  $\rho$  is  $GL_2(W(k))$ -valued since, otherwise, we would not be able to invoke the disjointness results that are used in the proof of Lemma 1 (Lemma 8 of [KR]).

*Remark.* Theorem 1 can be applied in practice to give many examples of converging sequences of  $p$ -adic representations: for a non-CM elliptic curve  $E/\mathbf{Q}$  for most primes  $p$  the mod  $p$  representation satisfies the conditions given in the introduction, and the corresponding  $p$ -adic representation is minimally ramified and  $GL_2(\mathbf{Z}_p)$  valued.

We end this section with a result that refines the main result of [KhRa].

**Proposition 1.** *If  $\rho_i : G_L \rightarrow GL_m(K)$  is a sequence of (residually absolutely irreducible) continuous representations that converges to  $\rho$ , then the set of primes where any of the  $\rho_i$ 's is ramified (i.e.,  $\bigcup \text{Ram}(\rho_i)$  where  $\text{Ram}(\rho_i)$  is the set of primes at which  $\rho_i$  is ramified) is of density zero.*

*Proof.* Denote by  $\rho_{i,n}$  (resp.,  $\rho_n$ ) the reduction mod  $\mathfrak{m}^n$  of an integral model of  $\rho_i$  (resp.,  $\rho$ ). The proof consists of applying Theorem 1 of [KhRa] twice; more precisely, first its statement, and then its proof. By an application of its statement we conclude that the density of  $\bigcup_{i=1}^n \text{Ram}(\rho_i)$  is 0 for any  $n$ . Now applying the proof of Theorem 1 of [KhRa], we define  $c_{\rho,n}$  to be the upper density of the set  $S_{\rho,n}$  of primes  $q$  of  $L$  that

- lie above primes which split in  $L/\mathbf{Q}$ ,
- are unramified in  $\rho_1$  and  $\neq p$ ,

- $\rho_n|_{D_q}$  is unramified, but there exists a “lift” of  $\rho_n|_{D_q}$ , with  $D_q$  the decomposition group at  $q$ , to a representation  $\tilde{\rho}_q$  of  $D_q$  to  $GL_m(K)$  that is ramified at  $q$ ; by a lift we mean some conjugate of  $\tilde{\rho}_q$  reduces mod  $\mathfrak{m}^n$  to  $\rho_n|_{D_q}$ .

We have from [KhRa] (see Proposition 1 of loc. cit. which was stated in greater generality than needed there with the present application in mind):

**Lemma 2.** *Given any  $\varepsilon > 0$ , there is an integer  $N_\varepsilon$  such that  $c_{\rho,n} < \varepsilon$  for  $n > N_\varepsilon$ .*

To prove Proposition 1 it is enough to show that given any  $\varepsilon > 0$ , the upper density of the set  $\bigcup \text{Ram}(\rho_i)$  is  $< \varepsilon$ . Since  $\bigcup_{i=1}^n \text{Ram}(\rho_i)$  has density 0 for (the finite)  $n$  that is the supremum of the  $i$ 's such that  $\rho_{i,N_\varepsilon}$  is not isomorphic to  $\rho_{N_\varepsilon}$ , and  $\rho_{N_\varepsilon}$  is finitely ramified, it follows from the lemma above that the upper density of  $\bigcup \text{Ram}(\rho_i)$  is  $< \varepsilon$ . Hence Proposition 1.  $\square$

*Remark.* One can ask for more refined information about the asymptotics of ramified primes in (limits of) residually absolutely irreducible  $p$ -adic Galois representations. For instance, in Theorem 1 of [KhRa] one can ask (clued by Theorem 10 of [S1]) if the order of growth of ramified primes can be proved to be bounded by  $O(x^{1-\frac{1}{2N}+\epsilon})$ , where  $N$  is the  $p$ -adic analytic dimension of  $\text{im}(\rho)$ , for any  $\epsilon > 0$ . Such quantitative refinements asked for by Serre in an e-mail message to the author are difficult and will require a new idea (that goes beyond [KhRa]) and a strong use of effective versions of the Chebotarev density theorem.

### 3. FINITE AND INFINITE RAMIFICATION

Let  $L$  be a number field and  $K$  a finite extension of  $\mathbf{Q}_p$  as before.

**Definition 4.** We say that a residually absolutely irreducible continuous representation  $\rho : G_L \rightarrow GL_n(K)$  is motivic if  $\rho$  arises as a subquotient of the  $i^{\text{th}}$  étale cohomology  $H^i(X \times_L \overline{L}, K)$  of a smooth projective variety  $X$  defined over a number field  $L$ .

A motivic representation is *finitely ramified*. In [R] examples of residually irreducible representations  $\rho : G_{\mathbf{Q}} \rightarrow GL_2(K)$  were constructed that were infinitely ramified (see also the last section of [KR]). Infinitely ramified  $p$ -adic representations cannot be *motivic*. But they can arise as limits of  $p$ -adic representations that are *motivic*. Fix an embedding  $\overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}_p}$ . Then as in [R] (and the last section of [KR]), there is a sequence of eigenforms  $f_i \in S_2(\Gamma_0(N_i))$ , for a sequence of square-free integers  $N_i$  such that  $N_i \rightarrow \infty$  and  $(p, N_i) = 1$ , new of level  $N_i$  such that the corresponding  $p$ -adic representations  $\rho_{f_i} : G_{\mathbf{Q}} \rightarrow GL_2(\mathbf{Z}_p)$  have a  $p$ -adic limit  $\rho$ , with  $\rho$  *infinitely ramified*. Such a  $\rho$  is *non-motivic*, but is the limit of *motivic*  $p$ -adic representations. Such limits of eigenforms (in the works of Serre and Katz; for instance, cf., [Ka]) have been considered when varying weights or varying the  $p$ -power level, while fixing the prime-to- $p$  part of the level.

**Proposition 2.** *Let  $f_i \in S_2(\Gamma_0(N_i))$  be a sequence of eigenforms with coefficients in a finite extension  $K$  of  $\mathbf{Q}_p$  with  $(N_i, p) = 1$  and  $p \geq 3$ , that in the  $p$ -adic  $q$ -expansion topology tend to an element  $f \in K[[q]]$ , such that the corresponding residual representation  $\overline{\rho}$  satisfies the conditions in the introduction. The element  $f$ , that gives rise naturally to a Galois representation  $\rho_f : G_{\mathbf{Q}} \rightarrow GL_2(K)$ , is the  $q$ -expansion of a classical eigenform (of weight 2) if and only if  $\rho_f$  is finitely ramified.*

*Proof.* The only if part is clear. The if part follows from the methods of Wiles (see Chapter 3 of [W] and also [TW]) and their refinements: note that  $\rho_f$  is finite flat at  $p$ .  $\square$

*Remark.* Applying Theorem 1 when  $\bar{\rho}$  is odd and finite flat at  $p$ , in which case the  $R$ -lifts are modular by Theorem 1 of [K], we can construct systematically many examples of sequences of eigenforms  $f_i \rightarrow f$  ( $f \in K[[q]]$ ), with the levels of  $f_i$  unbounded and such that  $\rho_f$  is finitely ramified ( $f$ , in fact, is then a classical eigenform as above). On the other hand, as recalled above in [R] (see also last section of [KR]), we have examples of situations as above with  $\rho_f$  infinitely ramified.

It will be of interest to see if Proposition 2 could be proved in a more self-contained manner. The proof above does not use seriously the fact that one does know that  $f$  arises as a limit of the classical forms  $f_i$ . If such a proof could be devised, in conjunction with Theorem 1 above and Theorem 1 of [K] (which is due to Ravi Ramakrishna) it would give, in special cases, a simpler approach using  $R$ -primes to the modularity lifting theorems of Wiles, et al. (see also [K]) that directly works with the  $p$ -adic Galois representation that needs to be proved modular, and if it could be implemented, would avoid (albeit in special cases) the sophisticated deformation theoretic approach of [W].

We elaborate on this: Assume that  $\bar{\rho}$  is modular. In Theorem 1 we have characterized the limit points of  $R$ -lifts. By Theorem 1 of [K] which proves that the representation corresponding to  $R_{S \cup Q}^{Q-new} \simeq W(k)$  is modular as a consequence of the isomorphism  $R_{S \cup Q}^{Q-new} \simeq \mathbf{T}_{S \cup Q}^{Q-new}$  (using notation of [K]), we know that  $R$ -lifts are modular. Hence, limits of  $R$ -lifts do arise as limits of  $p$ -adic representations arising from classical newforms. It only (!) remains to prove that a limit of a converging sequence of  $p$ -adic representations arising from newforms (say of weight 2 and level prime to  $p$  to avoid delicate considerations at  $p$ ) that is finitely ramified itself arises from a newform (i.e., prove Proposition 2 without appealing directly to [W]). Note that for a semistable elliptic curve  $E$ , for all large enough primes  $p$  (bigger than 3 for the methods here to directly work unfortunately!),  $T_p(E)$  is a limit point of  $R$ -lifts.

*Note.* In recent work we have indeed been able to give a self-contained approach to a result such as Proposition 2 above under some technical restrictions; see [K1].

#### 4. QUESTIONS

Proposition 2 suggests that a representation that arises as a limit of motivic representations (of “bounded weights”; see Definition 6 below) is finitely ramified if and only if it is motivic. We first recall one of the main conjectures in [FM] in a form that is most pertinent for the considerations here.

**Conjecture 1** (Fontaine-Mazur). Consider a continuous residually absolutely irreducible representation  $\rho : G_L \rightarrow GL_m(K)$  that is potentially semistable at places above  $p$ . Then the following are equivalent:

- (1)  $\rho$  is motivic,
- (2)  $\rho$  is finitely ramified.

From our earlier considerations it is natural to ask the following weaker question.

**Question 1.** Consider a continuous residually absolutely irreducible representation  $\rho : G_L \rightarrow GL_m(K)$  that is potentially semistable at places above  $p$  and arises as the limit of motivic representations  $\rho_i$ . Then if  $\rho$  is finitely ramified, is  $\rho$  motivic?

It seems unlikely that the infinitely ramified representations produced in [R] are *algebraic* (see definition below). This motivates the following considerations.

**Definition 5.** A continuous (residually absolutely irreducible) representation  $\rho : G_L \rightarrow GL_m(K)$  is said to be algebraic if there is a number field  $F$  such that the characteristic polynomial of  $\rho(\text{Frob}_q)$  has coefficients in the ring of integers of  $F$  for all primes  $q$  which are unramified in  $\rho$ . The minimal such field is the field of definition of  $\rho$ .

As by the main theorem of [KhRa], the set of primes at which  $\rho$  ramifies is of density 0, the definition above is a sensible one.

**Definition 6.** A continuous (residually absolutely irreducible) algebraic representation  $\rho : G_L \rightarrow GL_m(K)$  is said to be of weight  $\leq t$  ( $t \in \mathbf{Z}$ ) if for primes  $q$  that are unramified in  $\rho$ , any root  $\alpha$  of the characteristic polynomial of  $\rho(\text{Frob}_q)$  satisfies  $|\iota(\alpha)| \leq |k_q|^{\frac{t-1}{2}}$  for any embedding  $\iota : \overline{\mathbf{Q}} \rightarrow \mathbf{C}$ , with  $k_q$  the residue field at  $q$ .

**Question 2.** If  $\rho_i : G_L \rightarrow GL_m(K)$  is an infinite sequence of (residually absolutely irreducible) distinct algebraic representations, all of weight  $\leq t$  for some fixed integer  $t$ , converging to  $\rho : G_L \rightarrow GL_m(K)$ , and  $K_i$  the field of definition of  $\rho_i$ , does  $[K_i : \mathbf{Q}] \rightarrow \infty$  as  $i \rightarrow \infty$ ?

*Remark.*

- It is observed in [R] (this is a remark of Fred Diamond) that in the situation of Question 2 only finitely many of the  $\rho_i$ 's can arise from elliptic curves; this is a consequence of the Mordell conjecture which gives that suitable twists of the classical modular curves  $X(p^n)$  for  $n \gg 0$  have finitely many  $L$ -valued points for a given number field  $L$ .
- If Question 2 has a negative answer, using Proposition 1, we deduce that for a set of primes  $\{r\}$  of density one, the characteristic polynomials of  $\rho_i(\text{Frob}_r)$  are eventually constant. Hence, we deduce that the characteristic polynomials of  $\rho(\text{Frob}_r)$  are defined and integral over a fixed number field  $F$ , i.e.,  $\rho$  is algebraic (in the case when  $\rho$  is infinitely ramified this is linked to the questions below).

**Question 3.** Let  $\rho : G_L \rightarrow GL_m(K)$  be a continuous, residually absolutely irreducible representation that is potentially semistable at places above  $p$ . Then are the following equivalent:

- (1)  $\rho$  is motivic,
- (2)  $\rho$  is finitely ramified,
- (3)  $\rho$  is algebraic?

In the question above, the equivalence of 1 and 2 is the Fontaine-Mazur conjecture recalled above: the possible equivalence of 3 to 1 and 2 is the main thrust of the question. One might even ask the stronger question: If  $\rho : G_L \rightarrow GL_m(K)$ , a continuous, residually absolutely irreducible representation, is algebraic, then is  $\rho$  forced to be both finitely ramified and potentially semistable at places above  $p$ ? All the questions of this section have a positive answer when  $m = 1$ .

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