

A ZERO TOPOLOGICAL ENTROPY MAP WITH RECURRENT POINTS NOT F_σ

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Dedicated to Professor Jaroslav Smítal on the occasion of his 60th birthday

ABSTRACT. We show that there is a continuous map χ of the unit interval into itself of type 2^∞ which has a trajectory disjoint from the set $\text{Rec}(\chi)$ of recurrent points of χ , but contained in the closure of $\text{Rec}(\chi)$. In particular, $\text{Rec}(\chi)$ is not closed. A function ψ of type 2^∞ , with nonclosed set of recurrent points, was found by H. Chu and J. Xiong [Proc. Amer. Math. Soc. **97** (1986), 361–366]. However, there is no trajectory contained in $\overline{\text{Rec}(\psi)} \setminus \text{Rec}(\psi)$, since any point in $\overline{\text{Rec}(\psi)}$ is eventually mapped into $\text{Rec}(\psi)$. Moreover, our construction is simpler.

We use χ to show that there is a continuous map of the interval of type 2^∞ for which the set of recurrent points is not an F_σ set. This example disproves a conjecture of A. N. Sharkovsky et al., from 1989. We also provide another application of χ .

1. INTRODUCTION AND PRELIMINARIES

The main result of our paper is in Section 3. It states that, for any f in the class $C(I, I)$ of continuous maps of the unit interval, the condition “the set $\text{Rec}(f)$ of recurrent points is an F_σ set” is strictly stronger than “ f is of type less than or equal to 2^∞ ” (cf. Theorems 3.3 and 3.4). Recall that a point $x \in I$ is *recurrent* if it belongs to its ω -limit set $\omega_f(x)$. The map f is of *type 2^n* or 2^∞ if the periods of periodic points of f are the numbers $1, 2, \dots, 2^n$, or all powers of 2, respectively. Recall that $f \in C(I, I)$ has zero topological entropy if and only if it is of type $\leq 2^\infty$ (Misiurewicz [10]; cf. also [3]).

Section 2 contains an auxiliary result. Theorem 2.2 exhibits a map of type 2^∞ with nonclosed set $\text{Rec}(f)$, and with some other properties. We make use of this result in the proof of Theorem 3.3, but it is interesting in itself. Section 4 contains some remarks, an open problem, and another application of Theorem 3.3.

In the sequel, we use the standard terminology and notation as, e.g., in [3]. So we only recall a few items. By f^n we denote the n th iterate of f . The *orbit* (resp. *trajectory*) of a set A , written $\text{Orb}_f(A)$ (resp. $\text{Traj}_f(A)$), is the smallest

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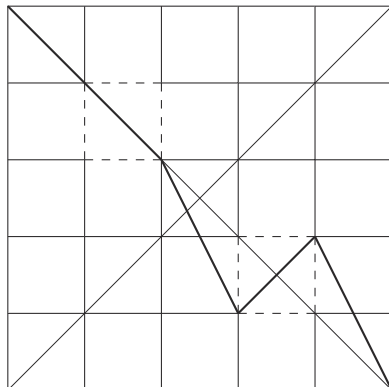
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set containing A that is closed under both images and preimages with respect to f (resp. the smallest set containing A closed under f). A periodic point p of period n is *repelling* if the one-sided derivatives of f^n at p have absolute values greater than 1. By $\text{Per}(f)$ we denote the set of periodic points of f , by $\|\cdot\|$ the uniform norm, and by \overline{A} the closure of a set A .

2. NON-CLOSED SET OF RECURRENT POINTS

We exhibit an auxiliary map $\chi \in C(I, I)$ of type 2^∞ for which the set of recurrent points is not closed. A map with similar property is given in [5]. However, our construction is simpler and the map has some additional properties which are summarized in Theorem 2.2 below. We use this example in the proof of our main result. But there are other applications; cf. Remark 4.2.

We start with a continuous map $\mu : I \rightarrow I$, which is given by $\mu(0) = 1$, $\mu(\frac{2}{5}) = \frac{3}{5}$, $\mu(\frac{3}{5}) = \frac{1}{5}$, $\mu(\frac{4}{5}) = \frac{2}{5}$, and $\mu(1) = 0$, and which is linear on the intermediate intervals. Its graph is displayed in the next figure.



For simplicity, put $I_1 = [\frac{1}{5}, \frac{2}{5}]$ and $J_1 = [\frac{3}{5}, \frac{4}{5}]$. It is easy to see that $\text{Per}(\mu) = \{0, 1, \frac{7}{15}\} \cup I_1 \cup J_1$. Specifically, $\frac{7}{15}$ is a repelling fixed point, $\{0, 1\}$ a repelling periodic trajectory of period 2, and μ^2 is the identity on $I_1 \cup J_1$. Hence, μ is of type 2^2 and

(2.1) the isolated points of $\text{Per}(\mu)$ are repelling.

Before stating the next proposition recall that a *Cantor set* $Q \subset I$ is any nonempty, compact set without isolated points containing no intervals. A point $x \in Q$ is *extremal* if it is isolated in Q from the left or from the right. In other words, extremal points of Q are the end-points of intervals contiguous to Q , except for 0 and 1.

Proposition 2.1. *There is a continuous map χ_∞ in $C(I, I)$ such that*

- (i) χ_∞ is of type 2^∞ with a unique infinite ω -limit set, a Cantor set Q ;
- (ii) χ_∞ is increasing in a neighborhood of any point x in Q different from $a = \min Q$, and χ_∞ is decreasing at a ;
- (iii) χ_∞ is a bijection on the set A of extremal points in Q ;
- (iv) the points in $\text{Per}(\chi_\infty)$ are isolated and repelling;
- (v) any point in the orbit of a in Q is in the closure of $\text{Per}(\chi_\infty)$ from each side;
- (vi) the χ_∞ -preimage of any point $x \in I$ is finite.

Proof. Let α, β be the increasing affine mappings such that $\alpha(I) = I_1$ and $\beta(I) = J_1$ and, for a positive integer n , let $I_n = \alpha^n(I)$ and $J_n = \beta^n(I)$. Let $\{\chi_k\}_{k=1}^\infty$ be the sequence of maps in $C(I, I)$ such that $\chi_1 = \mu$, and χ_2 is obtained by pasting into the square $I_1 \times J_1$ a five times reduced copy of the graph of μ , i.e.,

$$(2.2) \quad \chi_2(x) = \begin{cases} \mu(x) & \text{if } x \notin I_1, \\ \frac{1}{5}\mu(5x - 1) + \frac{3}{5} & \text{if } x \in I_1. \end{cases}$$

Note that I_1 is a maximal periodic interval both of χ_1 and χ_2 , of period 2, with $\chi_1(I_1) = \chi_2(I_1) = J_1$. Also, I_2 is a periodic interval of χ_1 and χ_2 , of period 4, which is maximal for χ_2 but not for χ_1 . Since $\chi_1(I_2) = \chi_2(I_2) = J_2$, and since the length of I_2 and J_2 is $1/25$, we can paste into the square $I_2 \times J_2$ the 25-times reduced copy of the graph of μ , to obtain χ_3 . We proceed inductively. χ_{n+1} is obtained from χ_n by pasting into the square $I_n \times J_n$ the 5^n -times reduced copy of the graph of μ .

Consequently,

$$(2.3) \quad \text{Per}(\chi_{k+1}) = (\text{Per}(\chi_k) \setminus \text{Traj}_{\chi_k} I_k) \cup \text{Traj}_{\chi_k} \alpha^k(\text{Per}(\mu)).$$

Since $\|\chi_{k+1} - \chi_k\| = \frac{1}{5}\|\chi_k - \chi_{k-1}\|$ for $k \geq 1$, the sequence $\{\chi_k\}_{k=1}^\infty$ has a uniform limit χ_∞ .

Now we can prove the properties (i)–(vi).

(i) Since I_k is a maximal periodic interval of χ_k of period 2^k , and the periods of points in $\text{Per}(\mu)$ are at most 4, χ_{k+1} is of type 2^{k+2} . Block’s Stability Theorem [2] implies that χ_∞ is of type 2^∞ . Since, for any $s \geq k$, I_k is a maximal periodic interval of χ_s of period 2^k , the same is also true for χ_∞ . Moreover, $\text{Traj}_{\chi_\infty}(I_k)$ is the unique trajectory of maximal periodic intervals of χ_∞ of period 2^k , and since its Lebesgue measure is $(\frac{2}{5})^k$, $Q = \bigcap_{k=1}^\infty \text{Traj}_{\chi_\infty}(I_k)$ is a Cantor set (of measure zero). It is well known that then Q is the unique infinite ω -limit set for χ_∞ (cf., e.g., [4]).

(ii) Clearly, χ_k is monotone on any interval from the trajectory of I_k , and since $\chi_\infty(x) = \chi_k(x)$ for $x \notin I_k$, χ_∞ is monotone on any interval from the trajectory of I_k , except for I_k itself. To finish the proof note that

$$(2.4) \quad \{a\} = \bigcap_{k=1}^\infty I_k.$$

(iii) By (ii), $\chi_\infty(x) \in A$ if $x \in A$. Hence, $\chi_\infty(A) \subset A$. Since $\chi_\infty(Q) = Q$, any point $y \in A$ has a preimage $x \in Q$. By (ii), x must be in A .

(iv) By (2.3), $\text{Per}(\chi_k) \setminus \text{Orb}_{\chi_k}(I_k)$ is a finite set. Thus, the points of $\text{Per}(\chi_\infty)$ are isolated. The rest follows by (2.1).

(v) The end-points $a_k < b_k$ of I_k are in $\text{Per}(\chi_k) \subset \text{Per}(\chi_\infty)$. By (2.4), point a satisfies the condition. By (ii), the same is true for the points $\chi_\infty(a)$ and $\chi_\infty^{-1}(a) \cap Q$. The result now follows by induction.

(vi) This follows from (2.4) since χ_∞ is piecewise linear on $I \setminus I_k$, and is constant on no subinterval of I . □

Theorem 2.2. *There is a map $\chi \in C(I, I)$ of type 2^∞ such that*

- (i) χ has a unique infinite maximal ω -limit set $\tilde{\omega} = R \cup P$, where R is a Cantor set, and $P = \{v_n\}_{n=-\infty}^\infty$ is an infinite set of points isolated in $\tilde{\omega}$ such that $\chi(v_n) = v_{n+1}$, for any n ;

- (ii) $P \subset \overline{\text{Per}(\chi)} \setminus \text{Per}(\chi)$;
- (iii) the set $\text{Rec}(\chi)$ is not closed;
- (iv) any point in $\text{Per}(\chi)$ is isolated in $\omega(\chi)$, and repelling;
- (v) χ is monotone in a neighborhood of any $p \in P$.

Proof. We obtain χ from the map χ_∞ described in Proposition 2.1 by Denjoy's method of blowing up the orbit of $a = \min Q$ (see [6]; for applications of this method just to maps of type 2^∞ see [4]). Thus, we replace any point in $\text{Orb}_{\chi_\infty}(a)$ by a suitable compact interval, and compress the rest to fit the interval I . This is possible by Proposition 2.1 (vi). We obtain a nondecreasing continuous map φ from I onto I such that $\chi_\infty \circ \varphi = \varphi \circ \chi$. By Proposition 2.1 (iii), $\text{Orb}_{\chi_\infty}(a) \cap A$ can be rearranged as $\{a_n\}_{n=-\infty}^\infty$, where $a_0 = a$ such that $\chi_\infty(a_n) = a_{n+1}$, for any n .

Thus, any a_n is replaced by a closed wandering interval A_n so that $\varphi(A_n) = \chi_\infty^n(a) = a_n$. Let R be the Cantor set such that $\varphi(R) = Q$. Since $\omega_{\chi_\infty}(a) = Q$, we have $\omega_\chi(x) = R$, for any $x \in A_n$. Since the blowing up an orbit preserves the ordering, Proposition 2.1 (v) implies that for any n the end-points u_n and v_n of A_n belong to $\overline{\text{Per}(\chi)}$ and hence, to the set $\omega(\chi)$ of ω -limit points of χ . Just one of the end-points, say point v_n , is not in R . Consequently, v_n is an isolated point of a maximal ω -limit set $\tilde{\omega}$ of χ , which is of the form $\tilde{\omega} = R \cup \{v_n\}_{n=-\infty}^\infty$ (for well-known facts concerning maximal ω -limit sets of maps of type 2^∞ see, e.g., [3] or [4]). This proves (i) and (ii).

Since $\tilde{\omega}$ is χ -invariant and any point v_n is isolated in $\tilde{\omega}$, $v_n \notin \text{Rec}(\chi)$. But $\overline{\text{Per}(\chi)} = \text{Rec}(\chi)$ (cf., e.g., [3]), hence (ii) implies (iii).

The properties (iv) and (v) follow by Proposition 2.1 (ii) and (v). \square

3. THE SET OF RECURRENT POINTS NOT OF TYPE F_σ

Sharkovsky and mathematicians from his school claimed that, for any map $f \in C(I, I)$, the property of “zero topological entropy” is equivalent to “ $\text{Rec}(f)$ is an F_σ set” (cf. [11], [12], [8], [7], [14], [13], for example). Unfortunately, this is not the case: we give an example of a map $f \in C(I, I)$ with zero topological entropy with $\text{Rec}(f)$ not an F_σ set. We also show that the other implication is true.

Proposition 3.1. *There is a sequence $\{f_n\}_{n=0}^\infty$ of maps in $C(I, I)$ such that, for any n ,*

- (i) f_n is of type 2^∞ ;
- (ii) any point in $\text{Per}(f_n)$ is isolated in $\omega(f_n)$, and repelling;
- (iii) for any $u \in \text{Per}(f_n)$ there is a $v \in \overline{\text{Per}(f_{n+1})} \setminus \text{Rec}(f_{n+1})$ such that $|u - v| < 1/2^n$;
- (iv) $\omega(f_{n+1}) \supset \omega(f_n)$ and $\text{Per}(f_{n+1}) \supset \text{Per}(f_n)$;
- (v) $f_n \upharpoonright \omega(f_n) = f_{n+1} \upharpoonright \omega(f_n)$;
- (vi) $\|f_n - f_{n+1}\| < 1/2^n$.

Proof. Let $f_0 = \chi$ be as in Theorem 2.2. Assume by induction that f_n is given. Let α be the unique fixed point of f_0 in I . The idea is to let $\text{Per}(f_{n+1})$ be a disjoint union of small affine copies of $\text{Per}(f_0)$. Any such copy K intersects $\text{Per}(f_n)$ exactly at one point p , whose relative position in K is the same as the position of α in $\text{Per}(f_0)$.

By (ii), $\text{Per}(f_n)$ is countable so there is a sequence $\{p_i\}_{i=1}^\infty$ in $\text{Per}(f_n)$ that contains exactly one point from any periodic trajectory of a point. By (i), the

period of p_i is $2^{k(i)}$. Again by (ii), for any i there is a compact neighbourhood U_i of p_i such that $U_i \cap \text{Per}(f_n) = \{p_i\}$. Moreover, taking U_i sufficiently small, we may assume that $f_n^l(U_i) \cap f_n^m(U_j) \neq \emptyset$, $0 \leq l, m < 2^{k(i)}$ imply $i = j$ and $l = m$. Consequently, for any $i > 0$ and any l with $0 \leq l < 2^{k(i)}$ there is a compact interval V_i^l contained in the interior of $f_n^l(U_i)$ and containing $f_n^l(p_i)$ with the following two properties: All the intervals V_i^l , for a fixed i , have the same length, and the point $f_n^l(p_i)$ divides V_i^l in the ratio $\alpha/(1 - \alpha)$.

For compact intervals K, L , let $\xi_{K,L}$ denote the linear increasing map from K onto L , and define the map $f_{n+1} : I \rightarrow I$ as follows:

$$(3.1) \quad f_{n+1}(x) = f_n(x) \text{ if } x \notin f_n^l(U_i),$$

$$(3.2) \quad f_{n+1} \mid V_i^0 = \xi_{I, V_i^1} \circ f_0 \circ \xi_{V_i^0, I},$$

$$(3.3) \quad f_{n+1} \mid V_i^l = \xi_{V_i^l, V_i^{l+1}},$$

for any i and any $l = 1, \dots, 2^{k(i)} - 1$, letting $V_i^{2^{k(i)}} = V_i^0$. Then extend f_{n+1} linearly onto the whole of I .

To finish the proof it suffices to verify that f_{n+1} satisfies conditions (i)–(vi), with n replaced by $n + 1$. Conditions (iv) and (v) are obvious, and (ii) follows by Theorem 2.2 (iv). For a continuous map φ of the interval,

$$(3.4) \quad \text{Per}(\varphi) \subset \text{Rec}(\varphi) \subset \overline{\text{Per}(\varphi)} = \overline{\text{Rec}(\varphi)} \subset \omega(\varphi);$$

cf., e.g., [3]. Consequently, by Theorem 2.2 (ii), $D = \overline{\text{Per}(\chi)} \setminus \text{Rec}(\chi) \neq \emptyset$. To prove (iii) note that, locally at u , the set $\overline{\text{Per}(f_{n+1})} \setminus \text{Rec}(f_{n+1})$ is an affine copy of D . Hence to meet condition (iii) it suffices to take the intervals U_i sufficiently small, of diameter less than 2^{-n} . This will also imply (vi).

Thus, it remains to verify condition (i). We have to show that f_{n+1} has no periodic point of period $\neq 2^m$. Assume the contrary: let p be a periodic point of f_{n+1} of period $k \neq 2^m$. The trajectory of p must be disjoint from $V = \bigcup_{i=0}^\infty \bigcup_{l=0}^{2^{k(i)}-1} f_n^l(V_i^0)$ since V is an invariant set and, by (3.2) and (3.3), $f_{n+1} \mid V$ has zero topological entropy. On the other hand, by (3.1), the trajectories of p cannot be contained in $I \setminus \bigcup_{i=0}^\infty \bigcup_{l=0}^{2^{k(i)}-1} f_n^l(U_i)$. Hence, we may assume that $p \in U_i \setminus V_i$. Since by (ii) the periodic point p_i of f_n is repelling, by taking V_i sufficiently small we get the same for p_i as a periodic point of f_{n+1} . This means that the trajectory of p is eventually in the complement of U_i , and this is a contradiction. Thus, f_{n+1} satisfies condition (i). □

The following result is known but we are not able to give a reference. Recall, that X is nonwandering if, for any open set $G \neq \emptyset$ in X there is an integer $n > 1$ such that $\varphi^n(G) \cap G \neq \emptyset$.

Lemma 3.2. *Let (X, d) be a compact metric space, and $\varphi \in C(X, X)$. If X is nonwandering, then $\text{Rec}(\varphi)$ is a G_δ set.*

Proof. Our argument is a modification of the proof of Theorem 1.27 in [9]. Let d denote the metric on X , and define

$$F(x) = \inf_{n \geq 1} d(x, \varphi^n x).$$

Because of the well-known fact that the set of points of continuity of any function F with domain and range in metric spaces is a G_δ set, it suffices to show that $\text{Rec}(\varphi)$ is the set of points of continuity of F .

First suppose that x_0 is a point of continuity of F . If $F(x_0) = 0$, then x_0 is a recurrent point. If $F(x_0) > 0$, then, for some $a > 0$ and some neighbourhood V of x_0 , $F(x) > a$ throughout this neighbourhood. We may assume that $\text{diam}(V) < a$. Since V is not a wandering set, for some n , $\varphi^n V \cap V \neq \emptyset$ or, for some $x \in V$, $\varphi^n x \in V$. But then $F(x) < a$, which is a contradiction.

To finish the proof, let $x_0 \in \text{Rec}(\varphi)$. Then for any $\varepsilon > 0$ there is an integer $n_0 \geq 1$ satisfying $d(x_0, \varphi^{n_0} x_0) < \varepsilon$. Since φ^{n_0} is continuous, there is a $\delta > 0$ such that $d(\varphi^{n_0} x_0, \varphi^{n_0} x) < \varepsilon$ whenever $d(x, x_0) < \delta$. Hence, for any x with $d(x, x_0) < \min\{\delta, \varepsilon\}$,

$$d(x, \varphi^{n_0} x) \leq d(x, x_0) + d(x_0, \varphi^{n_0} x_0) + d(\varphi^{n_0} x_0, \varphi^{n_0} x) < 3\varepsilon.$$

Consequently,

$$\begin{aligned} F(x) &= \inf_{n \geq 1} d(x, \varphi^n x) \leq d(x, \varphi^{n_0} x) < 3\varepsilon, \\ F(x_0) &= \inf_{n \geq 1} d(x_0, \varphi^n x_0) \leq d(x_0, \varphi^{n_0} x_0) < \varepsilon, \end{aligned}$$

and finally, $|F(x) - F(x_0)| < 3\varepsilon$, i.e., F is continuous in x_0 . \square

Theorem 3.3. *There is a map $f \in C(I, I)$ with zero topological entropy such that $\text{Rec}(f)$ is not an F_σ set.*

Proof. Let f_n be as in Proposition 3.1. By (vi), put $f = \lim_{n \rightarrow \infty} f_n$. Then by (i) and (vi), f is of type $\geq 2^\infty$, and by the Block's Stability Theorem [2] f is of type $\leq 2^\infty$. Thus, f is of type 2^∞ . Moreover,

$$(3.5) \quad \text{Rec}(f) \text{ and } \overline{\text{Per}(f)} \setminus \text{Rec}(f) \text{ are dense in } \overline{\text{Per}(f)}.$$

Indeed, $\text{Rec}(f)$ is dense in $\overline{\text{Per}(f)}$ by (3.4). The other set is dense since by (iii), (iv) and (v), $\bigcup_{n=1}^\infty \overline{\text{Per}(f_n)} \setminus \text{Rec}(f_n)$ is dense in $\overline{\text{Per}(f)}$ and, by (3.4) and (v), $\overline{\text{Per}(f_n)} \setminus \text{Rec}(f_n) = \overline{\text{Per}(f_n)} \setminus \text{Rec}(f)$. By Lemma 3.2, $\overline{\text{Per}(f)} \setminus \text{Rec}(f)$ is an F_σ set. But then $\text{Rec}(f)$ cannot be F_σ since otherwise, by (3.5), $\overline{\text{Per}(f)}$ would be a countable union of nowhere dense closed sets and hence, of the first category, contrary to the Baire theorem. \square

Theorem 3.4. *Let $f \in C(I, I)$. If $\text{Rec}(f)$ is an F_σ set, then f has zero topological entropy or, equivalently, is of type $\leq 2^\infty$.*

Proof. Let Σ be the set of infinite sequences $\alpha = (a_1, a_2, \dots)$ with $a_k = 0$ or 1 , and let $\sigma : \Sigma \rightarrow \Sigma$ be the shift operator $(a_1, a_2, \dots) \mapsto (a_2, a_3, \dots)$. Then $\text{Rec}(\sigma)$ is not an F_σ set. Indeed, by Lemma 3.2, $\text{Rec}(\sigma)$ is a G_δ set, hence its complement $Y = \Sigma \setminus \text{Rec}(\sigma)$ is an F_σ set. It is easy to see that both $\text{Rec}(\sigma)$ and Y are dense in X . Hence, if $\text{Rec}(\sigma)$ is F_σ , then X would be the union of a countable family of nowhere dense closed sets and hence, of the first category, contrary to the Baire theorem.

Now assume that $f \in C(I, I)$ has positive topological entropy. Then, by [1] (cf. also [3]), there is an integer $n \geq 1$, and a compact set $X \subset I$ such that $f^n(X) = X$, and $f^n|_X$ is semiconjugate to σ , i.e., there is a continuous surjective map $h : X \rightarrow \Sigma$ which is one-to-one with countably many exceptions where the map is two-to-one, and

$$(3.6) \quad h \circ f^n(x) = \sigma \circ h(x), \quad x \in X.$$

Moreover, for any $y \in \Sigma$,

$$(3.7) \quad h^{-1}(y) = \{z_0, z_1\} \text{ and } z_0 < z_1 \text{ imply } X \cap (z_0, z_1) = \emptyset.$$

Denote $g = f^n|X$. By Lemma 2.1 in [1],

$$(3.8) \quad \omega_\sigma(h(x)) = h(\omega_g(x)) \text{ for any } x \in X.$$

We prove that

$$(3.9) \quad h(\text{Rec}(g)) = \text{Rec}(\sigma).$$

If $x \in \text{Rec}(g)$, i.e., if $x \in \omega_g(x)$, then $h(x) \in h(\omega_g(x))$; hence, by (3.8), $h(x) \in \omega_\sigma(h(x))$, i.e., $h(x) \in \text{Rec}(\sigma)$. This shows that $h(\text{Rec}(g)) \subset \text{Rec}(\sigma)$. To prove the converse inclusion, assume that $y \in \text{Rec}(\sigma)$, and let $h^{-1}(y) = \{z_0, z_1\}$, possibly with $z_0 = z_1$. If $z_0 = z_1$, then $z_0 \notin \omega_g(z_0)$ would imply $h(z_0) \notin h(\omega_g(z_0))$; hence, by (3.8), $y \notin \text{Rec}(\sigma)$. This contradiction shows that $y \in h(\text{Rec}(g))$. If $z_0 < z_1$ and $z_i \notin \omega_g(z_j)$, for any $i, j \in \{0, 1\}$, then, by (3.7), $(\omega_g(z_0) \cup \omega_g(z_1)) \cap \{z_0, z_1\} = \emptyset$ and again, since h is at most two-to-one, $y \notin \text{Rec}(\sigma)$ which is impossible. Thus, we have $z_i \in \omega_g(z_j)$, for some i, j . But then, by (3.7), $z_i \in \omega_g(z_i)$, for some i , and again $y \in h(\text{Rec}(g))$. This proves (3.9).

Now to finish the proof of Theorem 3.4, assume that $\text{Rec}(f)$ is an F_σ set. Since, for any continuous map φ of a compact metric space, and any positive integer m , $\text{Rec}(\varphi) = \text{Rec}(\varphi^m)$ (cf., e.g., [3], Lemma 4.25), we conclude that $\text{Rec}(f^n)$, and hence, $\text{Rec}(g) = \text{Rec}(f^n)|X$, is an F_σ set. But then, by (3.9), $\text{Rec}(\sigma)$ would be an F_σ set, a contradiction. \square

4. CONCLUDING REMARKS

Remark 4.1. In fact, Sharkovsky et al., among others, stated that the next three conditions are equivalent: (P_1) f has zero topological entropy; (P_2) the set of periodic points of f is a G_δ set; (P_3) the set of recurrent points of f is an F_σ set. Unfortunately, this is not true: no two of these conditions are equivalent.

Recently [15] we proved that (P_2) implies (P_1) and we exhibited a map satisfying (P_1) , but not (P_2) . This map, however, has the property (P_3) . By the present paper, Theorems 3.3 and 3.4, (P_3) implies but is not implied by (P_1) . There is still an open problem if (P_2) implies (P_3) . Our conjecture is that the answer is positive. Under this hypothesis, we get the following ordering: (P_2) is stronger than (P_3) , and (P_3) is stronger than (P_1) .

Remark 4.2. Let G be a (continuous) triangular map of the square I^2 defined by $(x, y) \mapsto (\chi(x), g_x(y))$ where χ with ω -limit set $\tilde{\omega} = R \cup P$ is the map described in Theorem 2.2, and g_x is continuous and nondecreasing for any x . Let g_x be the identity for any $x \in R$. It is possible to specify the maps g_x so that $\text{Per}(\chi) \times I = \text{Per}(G)$ and, for some $a \in P$, the points $u = (a, 0)$ and $v = (a, 1)$ satisfy conditions

$$\liminf_{n \rightarrow \infty} |G^n(u) - G^n(v)| = 0, \quad \limsup_{n \rightarrow \infty} |G^n(u) - G^n(v)| = 1.$$

Thus, G restricted to $P \times I$ is chaotic in the sense of Li and Yorke. On the other hand, the points u, v are not recurrent (since no point in P is), but both belong to the closure of $\text{Per}(G)$, hence, to the center $C(G) := \overline{\text{Rec}(G)} = \omega(\chi) \times I$ of G . Consequently, G is chaotic on the center of G . Now G can be extended to the whole of I^2 so that $\{a\} \times I$ intersects $\omega(G)$ exactly at one point. Consequently, G is not

chaotic on $\omega(G)$. Such a map G cannot be obtained when χ is replaced by the map from [5] since there the trajectory of any point a in P is eventually in R .

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