

RENORMING OF $C(K)$ SPACES

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ABSTRACT. If K is a scattered Eberlein compact space, then $C(K)^*$ admits an equivalent dual norm that is uniformly rotund in every direction. The same is shown for the dual to the Johnson-Lindenstrauss space JL_2 .

1. INTRODUCTION

We will find classes of Banach spaces whose duals admit equivalent dual norms that are uniformly rotund in every direction (URED) or pointwise uniformly rotund (p-UR) (definitions are given below). The notion of p-UR covers both the weak and weak* uniform rotundity (W*UR). It can be shown from the Šmulyan theorem (see, e.g., [5, p. 63]), that the existence of a dual p-UR norm on X^* implies the existence of a “big” set in X^{**} on which the bidual norm is uniformly Gâteaux differentiable.

In Section 2 we will prove a three-space-like theorem for the following properties of a Banach space X : X^* admits an equivalent dual URED (p-UR) norm. This result enables us to renorm duals to spaces, such as the Johnson-Lindenstrauss space or $C(K)$ for K scattered with $K^{(\omega)} = \emptyset$, by dual norms that are simultaneously locally uniformly rotund (LUR) and p-UR. On the example of $C(K)$, where K is the so-called “two arrow” compact space, it is shown that properties of the duals to be equivalently renormed by dual URED norm (or p-UR norm) are not three space properties.

In Section 3, we will apply previous results, use a result from [1] and a method from [9], [11]. It will be proved that if K is an Eberlein and scattered compact space, then $C(K)^*$ admits an equivalent dual LUR and p-UR norm.

Recently it was shown in [6] that if X^* admits weak* uniformly rotund norm, then X is a subspace of weakly compactly generated space. However, in [12, Th. 1] it is shown that if X has an unconditional Schauder basis and X^* admits an equivalent (not necessarily dual) URED norm, then X^* admits an equivalent dual weak* uniformly rotund norm. Hence the space JL_2 from Section 2 shows that Theorem 1 in [12] does not hold without the assumption of unconditional Schauder

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basis. From the result in Section 3 we can deduce that if K is scattered Eberlein compact, that is not uniform Eberlein compact, then $C(K)^*$ is a dual to the weakly compactly generated space and admits an equivalent dual p-UR norm, but no equivalent dual weak* uniformly rotund norm, i.e., there is weakly compactly generated Banach space X , such that its dual X^* admits an equivalent dual URED and LUR norm and no W^* UR norm.

Let $(X, \|\cdot\|)$ be a Banach space. Let S_X and B_X denote the unit sphere and the unit ball respectively, i.e., $S_X = \{x \in X; \|x\| = 1\}$ and $B_X = \{x \in X; \|x\| \leq 1\}$. The norm $\|\cdot\|$ on a Banach space X is said to be *uniformly rotund in every direction* (URED for short), if $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ whenever $x_n, y_n \in S_X$ are such that $x_n - y_n = \lambda_n z$ for some $z \in X, \lambda_n \in \mathbb{R}$ and $\lim_{n \rightarrow \infty} \|x_n + y_n\| = 2$. We will say that the norm $\|\cdot\|$ on X is *pointwise uniformly rotund* (p -UR), if there exists a w^* -dense set $\mathbb{F} \subset X^*$ such that $\lim_{n \rightarrow \infty} f(x_n - y_n) = 0$ whenever $x_n, y_n \in S_{(X, \|\cdot\|)}$, $\lim_{n \rightarrow \infty} \|x_n + y_n\| = 2$, and $f \in \mathbb{F}$. More precisely, we say that the norm is p -UR with \mathbb{F} . Clearly, if the norm is p-UR, then it is URED. In the case of a dual Banach space $X = Y^*$ we say that the norm is *weak* uniformly rotund* (W^* UR), if it is p-UR with $\mathbb{F} = Y \subset Y^{**}$. The norm $\|\cdot\|$ is said to be *locally uniformly rotund* (LUR), if $\lim_{n \rightarrow \infty} \|x - x_n\| = 0$ whenever $x, x_n \in S_X$ are such that $\lim_{n \rightarrow \infty} \|x + x_n\| = 2$.

A compact space K is an *Eberlein compact* if K is homeomorphic to a weakly compact subset of a Banach space in its weak topology. A compact space K is a *uniform Eberlein compact* if K is homeomorphic to a weakly compact subset of a Hilbert space. A compact space is called *scattered* if every closed subset $L \subset K$ has an isolated point in L . For scattered compact spaces the Cantor derivative sets are defined as follows: $K^{(0)} = K, K^{(1)} = K'$ is the set of all limit points of K . If α is an ordinal and $K^{(\beta)}$ are defined for all $\beta < \alpha$, then we put $K^{(\alpha)} = (K^{(\beta)})'$ for $\alpha = \beta + 1$ and $K^{(\alpha)} = \bigcap_{\beta < \alpha} K^{(\beta)}$ for α a limit ordinal.

If we consider spaces such as $c_0(\Gamma), \ell_1(\Gamma), l_\infty(\Gamma)$, by the symbol e_γ we mean the standard unit vector.

For more information in this area we refer to [3], [5], [7, Ch. 12], [10] and [14].

2. THE THREE SPACE PROBLEM

Theorem 1. *Let X be a Banach space such that $c_0(\Gamma) \subset X$. Let the dual to $Y = X/c_0(\Gamma)$ admit an equivalent dual p-UR (URED) norm. Then X^* admits an equivalent dual p-UR (URED) norm.*

Proof. Let $i : c_0(\Gamma) \rightarrow X$ be the inclusion map and $q : X \rightarrow Y$ be the quotient map. Then the dual mappings are $i^* : X^* \rightarrow c_0(\Gamma)^*$, which is a quotient map and a restriction, and $q^* : Y^* \rightarrow X^*$, which is an inclusion. Because of the lifting property of the space $\ell_1(\Gamma) \cong c_0(\Gamma)^*$ there is a bounded linear map $l : \ell_1(\Gamma) \rightarrow X^*$ (the so-called lifting; see, e.g., [4]) such that $i^*(l(e)) = e$ for all $e \in \ell_1(\Gamma)$. Hence we have an isomorphism $X^* \cong \ell_1(\Gamma) \oplus Y^*$, where the duality between $(f, g) \in \ell_1(\Gamma) \oplus Y^*$ and $x \in X$ is given by the formula

$$\langle (f, g), x \rangle = \langle l(f), x \rangle + \langle q^*(g), x \rangle.$$

Let $\|\cdot\|_{Y^*}$ be a dual norm on Y^* which is p-UR with \mathbb{F} . We will prove that there is an equivalent dual norm $\|\cdot\|_u$ on X^* , that is, p-UR with $\mathbb{G} = \{(e_\gamma, 0); \gamma \in \Gamma\} \cup \{(0, f); f \in \mathbb{F}\}$, where we identify X^{**} with $l_\infty(\Gamma) \oplus Y^{**}$ and where $\{e_\gamma; \gamma \in \Gamma\}$ denote the standard unit vectors in $c_0(\Gamma) \subset l_\infty(\Gamma)$. The proof that the norm $\|\cdot\|_u$ is URED if $\|\cdot\|_{Y^*}$ is URED proceeds in the same way.

Let $\|\cdot\|_{X^*}$ be a dual norm on X^* which extends the norm $\|\cdot\|_{Y^*}$. Let $\|\cdot\|_{\ell_1(\Gamma)}$ be the standard norm on $\ell_1(\Gamma)$. We choose $a > 1$ such that

$$a^{-1}\|(f, g)\|_{X^*} \leq \|f\|_{\ell_1(\Gamma)} + \|g\|_{Y^*} \leq a\|(f, g)\|_{X^*}.$$

Put

$$\|(f, g)\|_w = (\|f\|_{\ell_1(\Gamma)}^2 + \|f\|_{\ell_2(\Gamma)}^2 + \|g\|_{Y^*}^2)^{\frac{1}{2}}.$$

This is an equivalent norm on $X^* \cong \ell_1(\Gamma) \oplus Y^*$. The norm $\|\cdot\|_w$ need not be a dual norm, but it is p-UR with \mathbb{G} . This convexity property will be used at the end of this proof. To have a dual norm, let us define

$$\|(f, g)\| = \|(f, g)\|_w + a\|f\|_{\ell_1(\Gamma)}.$$

Observation. The norm $\|\cdot\|$ is a dual norm on X^* .

Proof of the Observation. We will follow the proof published in [8] and show that the unit ball is closed in the weak* topology. To prove this, let $\{(f_\alpha, g_\alpha)\}_{\alpha \in A}$ be a net in the unit ball in $(X^*, \|\cdot\|)$, which weak* converges to (f, g) . Because $c_0(\Gamma) \subset X$ and $\ell_1(\Gamma) \cong c_0(\Gamma)^*$, $\{f_\alpha\}_{\alpha \in A}$ converges coordinatewise to f . To see this, choose $x \in c_0(\Gamma)$. We have

$$\begin{aligned} \langle (f_\alpha, g_\alpha), i(x) \rangle &= \langle l(f_\alpha), i(x) \rangle + \langle q^*(g_\alpha), i(x) \rangle \\ &= \langle i^*(l(f_\alpha)), x \rangle + \langle g_\alpha, q(i(x)) \rangle = \langle f_\alpha, x \rangle. \end{aligned}$$

To estimate the norm of (f, g) we will decompose f_α in a special way. For each $\alpha \in A$, we can find elements $f_\alpha^1, f_\alpha^2 \in \ell_1(\Gamma)$ such that $f_\alpha = f_\alpha^1 + f_\alpha^2$, the supports of f_α^1, f_α^2 are disjoint and $\lim_{\alpha \in A} \|f_\alpha - f_\alpha^1\|_{\ell_1(\Gamma)} = 0$. By passing to a subnet, we can assume that $\{(f_\alpha^2, 0)\}_{\alpha \in A}$ weak* converges to some $(0, g^1)$ and $\{(0, g_\alpha)\}_{\alpha \in A}$ weak* converges to $(0, g_2) = (0, g - g_1)$. Then

$$\begin{aligned} \|f\|_{\ell_1(\Gamma)} &\leq \liminf_{\alpha \in A} \|f_\alpha^1\|_{\ell_1(\Gamma)}, \\ \|g_1\|_{Y^*} &= \|(0, g_1)\|_{X^*} \leq \liminf_{\alpha \in A} \|(f_\alpha^2, 0)\|_{X^*} \leq a \liminf_{\alpha \in A} \|f_\alpha^2\|_{\ell_1(\Gamma)}, \\ \|g_2\|_{Y^*} &= \|(0, g_2)\|_{X^*} \leq \liminf_{\alpha \in A} \|(0, g_\alpha)\|_{X^*} = \liminf_{\alpha \in A} \|g_\alpha\|_{Y^*}, \end{aligned}$$

where we used that $\|\cdot\|_{X^*}$ is the dual norm. It follows from previous estimates that

$$\begin{aligned} \|(f, g)\| &= a\|f\|_{\ell_1(\Gamma)} + (\|f\|_{\ell_1(\Gamma)}^2 + \|f\|_{\ell_2(\Gamma)}^2 + \|g_1 + g_2\|_{Y^*}^2)^{\frac{1}{2}} \\ &\leq a\|f\|_{\ell_1(\Gamma)} + \|g_1\|_{Y^*} + (\|f\|_{\ell_1(\Gamma)}^2 + \|f\|_{\ell_2(\Gamma)}^2 + \|g_2\|_{Y^*}^2)^{\frac{1}{2}} \\ &\leq \liminf_{\alpha \in A} \left(a\|f_\alpha^1\|_{\ell_1(\Gamma)} + a\|f_\alpha^2\|_{\ell_1(\Gamma)} + (\|f_\alpha^1\|_{\ell_1(\Gamma)}^2 + \|f_\alpha^2\|_{\ell_2(\Gamma)}^2 + \|g_\alpha\|_{Y^*}^2)^{\frac{1}{2}} \right) \\ &\leq \limsup_{\alpha \in A} \|(f_\alpha, g_\alpha)\| \leq 1. \end{aligned}$$

Thus dual unit ball is w^* -closed and the Observation is proved. \square

We will continue with the proof of Theorem 1. Let us define the norm $\|\cdot\|_u$ on X^* by the formula

$$\|(f, g)\|_u^2 = \|(f, g)\|^2 + \|f\|_{\ell_1(\Gamma)}^2.$$

It is a dual norm, because it is w^* -lower semicontinuous as is the seminorm $\|\cdot\|_{\ell_1(\Gamma)}$ on X^* . We prove that it is p-UR with \mathbb{G} . To do this, we use the following fact, which can be found with the proof in [5, Ch. II].

Fact. Let a_n, b_n be bounded elements of a Banach space $(E, \|\cdot\|)$ such that

$$\lim_{n \rightarrow \infty} (2\|a_n\|^2 + 2\|b_n\|^2 - \|a_n + b_n\|^2) = 0.$$

Then $\lim_{n \rightarrow \infty} (\|a_n\| - \|b_n\|) = 0$ and $\lim_{n \rightarrow \infty} (\|a_n\| + \|b_n\| - \|a_n + b_n\|) = 0$.

Now assume that $x_n = (x_n^1, x_n^2), y_n = (y_n^1, y_n^2) \in X^*$ satisfy

$$(1) \quad \lim_{n \rightarrow \infty} (2\|x_n\|_u^2 + 2\|y_n\|_u^2 - \|x_n + y_n\|_u^2) = 0.$$

By the previous Fact we have

$$\lim_{n \rightarrow \infty} (2\|x_n^1\|_{\ell_1(\Gamma)}^2 + 2\|y_n^1\|_{\ell_1(\Gamma)}^2 - \|x_n^1 + y_n^1\|_{\ell_1(\Gamma)}^2) = 0.$$

And again by the Fact

$$(2) \quad \begin{aligned} & \lim_{n \rightarrow \infty} (\|x_n^1\|_{\ell_1(\Gamma)} - \|y_n^1\|_{\ell_1(\Gamma)}) = 0, \\ & \lim_{n \rightarrow \infty} (\|x_n^1\|_{\ell_1(\Gamma)} + \|y_n^1\|_{\ell_1(\Gamma)}) - \lim_{n \rightarrow \infty} \|x_n^1 + y_n^1\|_{\ell_1(\Gamma)} = 0. \end{aligned}$$

By (1) and by the Fact it follows that

$$\lim_{n \rightarrow \infty} (2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2) = 0.$$

Hence by the Fact

$$(3) \quad \lim_{n \rightarrow \infty} (\|x_n\| + \|y_n\|) - \lim_{n \rightarrow \infty} \|x_n + y_n\| = 0.$$

By (2) and (3) we have

$$\begin{aligned} \lim_{n \rightarrow \infty} (\|x_n\|_w + \|y_n\|_w) &= \lim_{n \rightarrow \infty} \|x_n + y_n\|_w, \\ \lim_{n \rightarrow \infty} \|x_n\|_w &= \lim_{n \rightarrow \infty} \|y_n\|_w. \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} \left\| \frac{x_n}{\|x_n\|_w} + \frac{y_n}{\|y_n\|_w} \right\|_w = \lim_{n \rightarrow \infty} \left\| \frac{x_n}{\|x_n\|_w} + \frac{y_n}{\|x_n\|_w} \right\|_w = 2.$$

The norm $\|\cdot\|_w$ on $\ell_1(\Gamma) \oplus Y^*$ is p-UR with \mathbb{G} ; thus, for all $G \in \mathbb{G}$ we have

$$\lim_{n \rightarrow \infty} G\left(\frac{x_n}{\|x_n\|_w} - \frac{y_n}{\|x_n\|_w}\right) = 0,$$

which finishes the proof of Theorem 1. \square

Remark. Moreover, if the norm $\|\cdot\|_{Y^*}$ on Y^* is LUR, the norm $\|\cdot\|_u$ on X^* is LUR as well because the norm $\|\cdot\|_w$ is.

Corollary 2. *Let K be a scattered compact space with $K^{(\omega)} = \emptyset$. Then $C(K)^*$ admits an equivalent dual norm that is simultaneously LUR and p-UR with $\mathbb{F} = \{e_k; k \in K\} \subset c_0(K) \subset C(K)^{**}$.*

Proof. By a compactness argument, there is some $n \in \mathbb{N}$ such that $K^{(n)} = \emptyset$. We shall prove the corollary by an induction. If $n = 0, 1$, the claim is trivial. Let $n > 1$. It is easy to see that the space $Y = \{f \in C(K); f|_{K'} = 0\}$ is isometric to the space $c_0(K \setminus K')$. Moreover, $C(K)/Y = C(K')$, so we can use Theorem 1. \square

In fact, the following theorem holds.

Theorem (Deville). *Let K be a scattered compact space, such that $K^{(\omega_1)} = \emptyset$. Then $C(K)^*$ admits an equivalent dual norm, that is LUR and p-UR.*

Proof. See [5, Theorem 7.4.7]. There is an equivalent dual LUR norm constructed on $C(K)^*$. One can compute that this norm is, moreover, p-UR with $\mathbb{F} = \{e_k; k \in K\}$. \square

Note that it is shown in [8] (see also [13]) that there is a Banach space \mathbb{JL}_2 with the following properties:

- (1) $c_0 \subset \mathbb{JL}_2$, $\mathbb{JL}_2/c_0 = \ell_2(\Gamma)$, where the cardinality of the set Γ is a continuum,
- (2) \mathbb{JL}_2 is not a subspace of any WCG space; in particular, \mathbb{JL}_2 is not isomorphic to the $c_0 \oplus \ell_2(\Gamma)$,
- (3) there is an equivalent dual LUR norm on \mathbb{JL}_2^* .

From Theorem 1 we can obtain a stronger result.

Theorem 3. *There is an equivalent dual norm on the space \mathbb{JL}^* that is LUR and p-UR with \mathbb{F} , where \mathbb{F} is the canonical imbedding of $c_0 \oplus \ell_2(\Gamma)$ into $\ell_\infty \oplus \ell_2(\Gamma) \cong \mathbb{JL}_2^{**}$.*

It is shown in [5, pp. 299–305], that if K is a so-called “two arrow” compact space, then $C([0, 1]) \subset C(K)$, $C(K)/C([0, 1]) = c_0([0, 1])$ and $C(K)$ has no equivalent Gâteaux smooth norm. It means that there is no dual equivalent strictly convex norm on $C(K)^*$. It means that $C(K)^*$ does not admit an equivalent dual URED norm, although both $C([0, 1])^*$ and $c_0([0, 1])^*$ do admit an equivalent dual p-UR norms.

3. SCATTERED EBERLEIN COMPACT SPACES

Theorem 4. *Let K be a scattered compact space such that $K = \bigcup_{n=1}^\infty K_n$, and for all $n \in \mathbb{N}$ let $C(K_n)^*$ admit an equivalent dual p-UR norm with $\mathbb{F}_n = \{e_k; k \in K_n\}$. Then $C(K)^*$ admits an equivalent dual p-UR norm with $\mathbb{F} = \{e_k; k \in K\}$.*

Proof. This proof is similar to the proof of Theorem 2.7.16 in [5], which states, that the space $L_1(\Omega)$ admits a norm that is LUR and URED.

As in [9], we can define the operator $T : C(K) \rightarrow \sum_{\ell_2} C(K_n)$ by the formula $T(f) = (\frac{1}{n^2}f|_{K_n})$. For $k \in K$ put $N(k) = \{n \in \mathbb{N}; k \in K_n\}$. For $k \in K$, $n \in N(k)$ let \tilde{k}_n denote a copy of k in K_n . For $A \subset K$ put $\tilde{A} = \{\tilde{k}_n; k \in A, n \in N(k)\}$. By Rudin’s Theorem (see [7]) $C(K)^*$ is isometric to the space $\ell_1(K)$ and the canonical norm $\|\cdot\|_1$ is a dual norm on $C(K)^*$, the same holds for K_n ’s. Without loss of generality, we can assume, that the p-UR norms are uniformly close to the original norms on $C(K_n)^*$. Hence $(\sum_{\ell_2} C(K_n))^* \cong \sum_{\ell_2} \ell_1(K_n)$ and $(\sum_{\ell_2} C(K_n))^*$ admits an equivalent dual norm $\|\cdot\|_\Sigma$, which is p-UR with $\mathbb{G} = \{e_{\tilde{k}_n}; k \in K, n \in N(k)\}$. The dual operator $T^* : (\sum_{\ell_2} C(K_n))^* \rightarrow C(K)^*$ is given by

$$T^*(y^*) = \left(\sum_{n \in N(k)} \frac{1}{n^2} y^*(\tilde{k}_n) \right)_{k \in K}.$$

The range of T^* is a dense set in $C(K)^*$. Now, we shall use the standard LUR renorming method. For $n \in \mathbb{N}$ and $x \in \ell_1(K)$ we define

$$|x|_n^2 = \inf \left\{ \|x - T^*y\|_1^2 + \frac{1}{n} \|y\|_\Sigma^2; y \in \sum_{\ell_2} C(K_n)^* \right\},$$

$$\|x\|^2 = \|x\|_1^2 + \sum_{n=1}^\infty 2^{-n} |x|_n^2.$$

This is a dual norm and we will prove that it is p-UR. Choose $x_i, y_i \in l_1(K)$ such that $\|x_i\|_1 \leq 1, \|y_i\|_1 \leq 1$ and

$$\lim_{i \rightarrow \infty} (2\|x_i\|^2 + 2\|y_i\|^2 - \|x_i + y_i\|^2) = 0.$$

Then for all $n \in \mathbb{N}$,

$$(1) \quad \lim_{i \rightarrow \infty} (2|x_i|_n^2 + 2|y_i|_n^2 - |x_i + y_i|_n^2) = 0.$$

The infimum in the definition of $|\cdot|_n$ is attained (see [5, p. 44]); e.g., for all $i, n \in \mathbb{N}$ there are $u_i^{(n)}, v_i^{(n)} \in \sum_{\ell_2} C(K_n)^*$ such that

$$(2) \quad \begin{aligned} |x_i|_n^2 &= \|x_i - T^*u_i^{(n)}\|_1^2 + \frac{1}{n}\|u_i^{(n)}\|_\Sigma^2, \\ |y_i|_n^2 &= \|y_i - T^*v_i^{(n)}\|_1^2 + \frac{1}{n}\|v_i^{(n)}\|_\Sigma^2. \end{aligned}$$

From (2) we get

$$(3) \quad \|u_i^{(n)}\|_\Sigma \leq n|x_i|_n \leq n\|x_i\|_1 \leq n,$$

and by the same manner we have $\|v_i^{(n)}\|_\Sigma \leq n$; therefore, for all $k \in K$ and $l \in N(k)$

$$(4) \quad (u_i^{(n)} - v_i^{(n)})(\tilde{k}_l) \leq \|u_i^{(n)} - v_i^{(n)}\|_1 \leq c\|u_i^{(n)} - v_i^{(n)}\|_\Sigma \leq 2cn,$$

where c is a constant of the equivalence of norms $\|\cdot\|_1$ and $\|\cdot\|_\Sigma$. From (1), (2), (3) we have

$$\lim_{i \rightarrow \infty} (2\|u_i^{(n)}\|_\Sigma^2 + 2\|v_i^{(n)}\|_\Sigma^2 - \|u_i^{(n)} + v_i^{(n)}\|_\Sigma^2) = 0.$$

The norm $\|\cdot\|_\Sigma$ is p-UR and hence for all $k \in K, m \in \mathbb{N}$ and $n \in N(k)$

$$(5) \quad \lim_{i \rightarrow \infty} (u_i^{(m)} - v_i^{(m)})(\tilde{k}_n) = 0.$$

We can assume (by passing to a subsequence), that $\lim_{i \rightarrow \infty} |x_i|_n = d_n$. For every $x \in l_1(K)$, $|x|_n$ is a nonincreasing sequence, hence there is $d = \lim_{n \rightarrow \infty} d_n$. By passing to a subsequence again, we can assume, moreover, that $\lim_{i \rightarrow \infty} |y_i|_n = d_n$. Choose $\varepsilon > 0$ and $k \in K$. Put $A = K \setminus \{k\}$. Let $m \in \mathbb{N}$ be such that $d_m < d + \varepsilon$. Then

$$|(x_i - y_i)(k)| \leq |(x_i - T^*u_i^{(m)})(k)| + |(T^*u_i^{(m)} - T^*v_i^{(m)})(k)| + |(T^*v_i^{(m)} - y_i)(k)|.$$

Considering the second term, we have

$$\begin{aligned} |T^*(u_i^{(m)} - v_i^{(m)})(k)| &= \left| \sum_{n \in N(k)} \frac{1}{n^2} (u_i^{(m)} - v_i^{(m)})(\tilde{k}_n) \right| \\ &\leq \sum_{n \leq n_0, n \in N(k)} \left| \frac{1}{n^2} (u_i^{(m)} - v_i^{(m)})(\tilde{k}_n) \right| + \varepsilon, \end{aligned}$$

where n_0 depends only on ε (because of (4)) and the sum is finite and therefore tends to 0 for $i \rightarrow \infty$ because of (5). It remains to prove, that $|(x_i - T^*u_i^{(m)})(k)| < \varepsilon$. We can assume that $k \in K_{n_0}$. Put $y = s_i + (u_i^{(m)})|_{\tilde{A}}$, where $s_i(l) = n_0^2 x_i(k)$ if $l = \tilde{k}_{n_0}$,

and $s_i(l) = 0$ otherwise. Considering this y in the definition of $|x_i|_n$ we get

$$\begin{aligned} |x_i|_n^2 &\leq \|(x_i - T^*u_i^{(m)})|_A\|_1^2 + \frac{1}{n} \|s_i + (u_i^{(m)})|_{\tilde{A}}\|_{\Sigma}^2 \\ &\leq \|(x_i - T^*u_i^{(m)})|_A\|_1^2 + \frac{1}{n} (\|s_i\|_{\Sigma} + \|u_i^{(m)}|_{\tilde{A}}\|_{\Sigma})^2, \\ &\leq \|(x_i - T^*u_i^{(m)})|_A\|_1^2 + \frac{1}{n} (n_0^2c + mc^2)^2 \end{aligned}$$

because

$$\|u_i^{(m)}|_{\tilde{A}}\|_{\Sigma} \leq c \|u_i^{(m)}|_{\tilde{A}}\|_1 \leq c \|u_i^{(m)}\|_1 \leq mc^2,$$

where we used that the canonical norm $\|\cdot\|_1$ on X^* is a lattice norm.

Hence for all $n \in \mathbb{N}$

$$\limsup_{i \rightarrow \infty} \left\| (x_i - T^*u_i^{(m)})|_A \right\|_1^2 \geq \lim_{i \rightarrow \infty} |x_i|_n^2 - \frac{1}{n} (cn_0^2 + mc^2)^2.$$

Finally,

$$\limsup_{i \rightarrow \infty} \left\| (x_i - T^*u_i^{(m)})|_A \right\|_1^2 \geq d^2.$$

For all $i \in \mathbb{N}$ we have

$$\begin{aligned} \left| (x_i - T^*u_i^{(m)})(k) \right| &= \left\| x_i - T^*u_i^{(m)} \right\|_1 - \left\| (x_i - T^*u_i^{(m)})|_A \right\|_1 \\ &\leq |x_i|_m - \left\| (x_i - T^*u_i^{(m)})|_A \right\|_1. \end{aligned}$$

Hence we get

$$\liminf_{i \rightarrow \infty} \left| (x_i - T^*u_i^{(m)})(k) \right| \leq d_m - d \leq \varepsilon.$$

The same holds for the third term and this concludes the proof. \square

Theorem 5. *Let K be a scattered Eberlein compact space. Then $C(K)^*$ admits an equivalent dual norm that is LUR and p -UR with $\mathbb{F} = \{e_k; k \in K\}$. In particular, $C(K)^*$ admits an equivalent dual norm that is LUR and URED.*

Proof. K. Alster proved in [1] that if K is a scattered Eberlein compact space, then K is a strong Eberlein compact space, e.g., $K \subset \{0, 1\}^{\Gamma}$ for some Γ . Hence

$$K = \bigcup_{n=1}^{\infty} K_n, \text{ where } K_n = \left\{ x \in K; \text{card}(\{\gamma \in \Gamma; x(\gamma) = 1\}) \leq n \right\}.$$

The K_n 's are uniform Eberlein compact spaces, they are scattered and $K^{(n+1)} = \emptyset$. Hence by Corollary 2, $C(K_n)^*$ admits an equivalent dual norm that is both p -UR with $\mathbb{F} = \{e_k; k \in K_n\}$ and LUR. Thus we can use the preceding theorem to finish the proof. \square

4. OPEN QUESTION

It is shown in [12, Th. 1] that if X has an unconditional Schauder basis and X^* admits an equivalent URED norm, then X^* admits an equivalent dual weak* uniformly rotund norm. Because there is a scattered Eberlein compact space K that is not uniform Eberlein compact (see, e.g., [2, Example 1.10]), the space $C(K)^*$ admits an equivalent dual p -UR (and hence URED norm) but does not admit any equivalent dual W^* UR norm. But we do not know the answer to the following questions. Is there any reflexive Banach space X such that X admits an equivalent URED norm and does not admit any equivalent p -UR (and hence W^* UR) norm?

Is there any Banach space that admits an equivalent URED norm and does not admit any p-UR norm?

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