RENORMING OF \( C(K) \) SPACES

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Abstract. If \( K \) is a scattered Eberlein compact space, then \( C(K)^* \) admits an equivalent dual norm that is uniformly rotund in every direction. The same is shown for the dual to the Johnson-Lindenstrauss space \( JL_2 \).

1. Introduction

We will find classes of Banach spaces whose duals admit equivalent dual norms that are uniformly rotund in every direction (URED) or pointwise uniformly rotund (p-UR) (definitions are given below). The notion of p-UR covers both the weak and weak\(^*\) uniform rotundity (W\(^*\)UR). It can be shown from the Šmulian theorem (see, e.g., [5, p. 63]), that the existence of a dual p-UR norm on \( X^* \) implies the existence of a “big” set in \( X^{**} \) on which the bidual norm is uniformly Gâteaux differentiable.

In Section 2 we will prove a three-space-like theorem for the following properties of a Banach space \( X \): \( X^* \) admits an equivalent dual URED (p-UR) norm. This result enables us to renorm duals to spaces, such as the Johnson-Lindenstrauss space or \( C(K) \) for \( K \) scattered with \( K^{(\omega)} = \emptyset \), by dual norms that are simultaneously locally uniformly rotund (LUR) and p-UR. On the example of \( C(K) \), where \( K \) is the so-called “two arrow” compact space, it is shown that properties of the duals to be equivalently renormed by dual URED norm (or p-UR norm) are not three space properties.

In Section 3, we will apply previous results, use a result from [1] and a method from [9], [11]. It will be proved that if \( K \) is an Eberlein and scattered compact space, then \( C(K)^* \) admits an equivalent dual LUR and p-UR norm.

Recently it was shown in [6] that if \( X^* \) admits weak\(^*\) uniformly rotund norm, then \( X \) is a subspace of weakly compactly generated space. However, in [12] Th. 1) it is shown that if \( X \) has an unconditional Schauder basis and \( X^* \) admits an equivalent (not necessarily dual) URED norm, then \( X^* \) admits an equivalent dual weak\(^*\) uniformly rotund norm. Hence the space \( JL_2 \) from Section 2 shows that Theorem 1 in [12] does not hold without the assumption of unconditional Schauder...
basis. From the result in Section 3 we can deduce that if \( K \) is scattered Eberlein compact, that is not uniform Eberlein compact, then \( C(K)^* \) is a dual to the weakly compactly generated space and admits an equivalent dual p-UR norm, but no equivalent dual weak* uniformly rotund norm, i.e., there is weakly compactly generated Banach space \( X \), such that its dual \( X^* \) admits an equivalent dual URED and LUR norm and no W*UR norm.

Let \( (X, \| \|) \) be a Banach space. Let \( S_X \) and \( B_X \) denote the unit sphere and the unit ball respectively, i.e., \( S_X = \{ x \in X ; \| x \| = 1 \} \) and \( B_X = \{ x \in X ; \| x \| \leq 1 \} \).

The norm \( \| \| \) on a Banach space \( X \) is said to be uniformly rotund in every direction (URED for short), if \( \lim_{n \to \infty} \| x_n - y_n \| = 0 \) whenever \( x_n, y_n \in S_X \) are such that \( x_n - y_n = \lambda_n z \) for some \( z \in X, \lambda_n \in \mathbb{R} \) and \( \lim_{n \to \infty} \| x_n + y_n \| = 2 \). We will say that the norm \( \| \| \) on \( X \) is pointwise uniformly rotund (p-UR), if there exists a \( w^* \)-dense set \( F \subset X^* \) such that \( \lim_{n \to \infty} f(x_n - y_n) = 0 \) whenever \( x_n, y_n \in S(X, \| \|) \), \( \lim_{n \to \infty} \| x_n + y_n \| = 2 \), and \( f \in F \). More precisely, we say that the norm is \( p \)-UR with \( F \). Clearly, if the norm is \( p \)-UR, then it is URED. In the case of a dual Banach space \( X = Y^* \), we say that the norm is weak* uniformly rotund (W*UR), if it is \( p \)-UR with \( F = Y \subset Y^{**} \). The norm \( \| \| \) is said to be locally uniformly rotund (LUR), if \( \lim_{n \to \infty} \| x - x_n \| = 0 \) whenever \( x, x_n \in S_X \) are such that \( \lim_{n \to \infty} \| x + x_n \| = 2 \).

A compact space \( K \) is an Eberlein compact if \( K \) is homeomorphic to a weakly compact subset of a Banach space in its weak topology. A compact space \( K \) is a uniform Eberlein compact if \( K \) is homeomorphic to a weakly compact subset of a Hilbert space. A compact space is called scattered if every closed subset \( L \subset K \) has an isolated point in \( L \). For scattered compact spaces the Cantor derivative sets are defined as follows: \( K^{(0)} = K, K^{(1)} = K' \) is the set of all limit points of \( K \). If \( \alpha \) is an ordinal and \( K^{(\beta)} \) are defined for all \( \beta < \alpha \), then we put \( K^{(\alpha)} = (K^{(\beta)})' \) for \( \alpha = \beta + 1 \) and \( K^{(\alpha)} = \bigcap_{\beta < \alpha} K^{(\beta)} \) for a limit ordinal.

If we consider spaces such as \( c_0(\Gamma), \ell_1(\Gamma), l_\infty(\Gamma) \), by the symbol \( e_\gamma \) we mean the standard unit vector.

For more information in this area we refer to [3], [5], [7, Ch. 12], [10] and [14].

2. THE THREE SPACE PROBLEM

**Theorem 1.** Let \( X \) be a Banach space such that \( c_0(\Gamma) \subset X \). Let the dual to \( Y = X/c_0(\Gamma) \) admit an equivalent dual \( p \)-UR (URED) norm. Then \( X^* \) admits an equivalent dual \( p \)-UR (URED) norm.

**Proof.** Let \( i : c_0(\Gamma) \to X \) be the inclusion map and \( q : X \to Y \) be the quotient map. Then the dual mappings are \( i^* : X^* \to c_0(\Gamma)^* \), which is a quotient map and a restriction, and \( q^* : Y^* \to X^* \), which is an inclusion. Because of the lifting property of the space \( \ell_1(\Gamma) = c_0(\Gamma)^* \) there is a bounded linear map \( l : \ell_1(\Gamma) \to X^* \) (the so-called lifting; see, e.g., [3]) such that \( i^*(l(e)) = e \) for all \( e \in \ell_1(\Gamma) \). Hence we have an isomorphism \( X^* \cong \ell_1(\Gamma) \oplus Y^* \), where the duality between \( (f, g) \in \ell_1(\Gamma) \oplus Y^* \) and \( x \in X \) is given by the formula

\[
\langle (f, g), x \rangle = \langle l(f), x \rangle + \langle q^*(g), x \rangle.
\]

Let \( \| \|_{Y^*} \) be a dual norm on \( Y^* \) which is \( p \)-UR with \( F \). We will prove that there is an equivalent dual norm \( \| \|_u \) on \( X^* \), that is, \( p \)-UR with \( \mathcal{G} = \{ (e_\gamma, 0) ; \gamma \in \Gamma \} \cup \{ (0, f) ; f \in F \} \), where we identify \( X^{**} \) with \( l_\infty(\Gamma) \oplus Y^{**} \) and where \{ \( e_\gamma \); \( \gamma \in \Gamma \) \} denote the standard unit vectors in \( c_0(\Gamma) \subset l_\infty(\Gamma) \). The proof that the norm \( \| \|_u \) is URED if \( \| \|_{Y^*} \) is URED proceeds in the same way.
Let \( \| \cdot \|_{X^*} \) be a dual norm on \( X^* \) which extends the norm \( \| \cdot \|_{Y^*} \). Let \( \| \cdot \|_{\ell_1(\Gamma)} \) be the standard norm on \( \ell_1(\Gamma) \). We choose \( a > 1 \) such that
\[
a^{-1} \|(f, g)\|_{X^*} \leq \|f\|_{\ell_1(\Gamma)} + \|g\|_{Y^*} \leq a \|(f, g)\|_{X^*}.
\]
Put
\[
\|(f, g)\|_w = (\|f\|_{\ell_1(\Gamma)}^2 + \|f\|_{\ell_2(\Gamma)}^2 + \|g\|_{Y^*}^2)^{\frac{1}{2}}.
\]
This is an equivalent norm on \( X^* \sim \ell_1(\Gamma) \oplus Y^* \). The norm \( \| \cdot \|_w \) need not be a dual norm, but it is p-UR with \( G \). This convexity property will be used at the end of this proof. To have a dual norm, let us define
\[
\|(f, g)\| = \|(f, g)\|_w + \|f\|_{\ell_1(\Gamma)}.
\]

**Observation.** The norm \( \| \cdot \| \) is a dual norm on \( X^* \).

**Proof of the Observation.** We will follow the proof published in [8] and show that the unit ball is closed in the weak* topology. To prove this, let \( \{(f_\alpha, g_\alpha)\}_{\alpha \in A} \) be a net in the unit ball in \((X^*, \| \cdot \|)\), which weak* converges to \((f, g)\). Because \( c_0(\Gamma) \subset X \) and \( \ell_1(\Gamma) \cong c_0(\Gamma)^* \), \( \{f_\alpha\}_{\alpha \in A} \) converges coordinatewise to \( f \). To see this, choose \( x \in c_0(\Gamma) \). We have
\[
\langle (f_\alpha, g_\alpha), i(x) \rangle = \langle l(f_\alpha), i(x) \rangle + \langle g_\alpha^*(x), i(x) \rangle = \langle i^*(l(f_\alpha)), x \rangle + \langle g_\alpha, q(i(x)) \rangle = \langle f_\alpha, x \rangle.
\]
To estimate the norm of \((f, g)\) we will decompose \( f_\alpha \) in a special way. For each \( \alpha \in A \), we can find elements \( f^1_\alpha, f^2_\alpha \in \ell_1(\Gamma) \) such that \( f_\alpha = f^1_\alpha + f^2_\alpha \), the supports of \( f^1_\alpha, f^2_\alpha \) are disjoint and \( \lim_{\alpha \in A} \|f_\alpha - f^1_\alpha\|_{\ell_1(\Gamma)} = 0 \). By passing to a subnet, we can assume that \( \{(f^2_\alpha, 0)\}_{\alpha \in A} \) weak* converges to some \((0, g^1)\) and \( \{(0, g_\alpha)\}_{\alpha \in A} \) weak* converges to \((0, g_2) = (0, g - g_1)\). Then
\[
\|f\|_{\ell_1(\Gamma)} \leq \liminf_{\alpha \in A} \|f^1_\alpha\|_{\ell_1(\Gamma)},
\]
\[
\|g_1\|_{Y^*} = \|\langle (0, g_1) \rangle\|_{X^*} \leq \liminf_{\alpha \in A} \|\langle (f^2_\alpha, 0) \rangle\|_{X^*} \leq a \liminf_{\alpha \in A} \|f^2_\alpha\|_{\ell_1(\Gamma)},
\]
\[
\|g_2\|_{Y^*} = \|\langle (0, g_2) \rangle\|_{X^*} = \liminf_{\alpha \in A} \|\langle (0, g_\alpha) \rangle\|_{X^*} = \liminf_{\alpha \in A} \|g_\alpha\|_{Y^*},
\]
where we used that \( \| \cdot \|_{X^*} \) is the dual norm. It follows from previous estimates that
\[
\|(f, g)\| = \|f\|_{\ell_1(\Gamma)} + \|f^2_\alpha\|_{\ell_1(\Gamma)} + \|g_1 + g_2\|_{Y^*}^{\frac{1}{2}} \leq \|f\|_{\ell_1(\Gamma)} + \|g_1\|_{Y^*} + \|f^2_\alpha\|_{\ell_1(\Gamma)} + \|g_2\|_{Y^*}^{\frac{1}{2}} \leq \liminf_{\alpha \in A} \left( \|f^1_\alpha\|_{\ell_1(\Gamma)} + a \|f^2_\alpha\|_{\ell_1(\Gamma)} + \|f^2_\alpha\|_{\ell_2(\Gamma)} + \|g_\alpha\|_{Y^*}^{\frac{1}{2}} \right) \leq \limsup_{\alpha \in A} \|\langle (f_\alpha, g_\alpha) \rangle\| \leq 1.
\]
Thus dual unit ball is \( w^* \)-closed and the Observation is proved.

We will continue with the proof of Theorem 1. Let us define the norm \( \| \cdot \|_w \) on \( X^* \) by the formula
\[
\|(f, g)\|_w^2 = \|(f, g)\|^2 + \|f\|_{\ell_1(\Gamma)}^2.
\]
It is a dual norm, because it is \( w^* \)-lower semicontinuous as is the seminorm \( \| \cdot \|_{\ell_1(\Gamma)} \) on \( X^* \). We prove that it is p-UR with \( G \). To do this, we use the following fact, which can be found with the proof in [5] Ch. II.
Fact. Let $a_n, b_n$ be bounded elements of a Banach space $(E, \|\cdot\|)$ such that 
$$\lim_{n \to \infty} (2\|a_n\|^2 + 2\|b_n\|^2 - \|a_n + b_n\|^2) = 0.$$ 
Then $\lim_{n \to \infty} (\|a_n\| - \|b_n\|) = 0$ and $\lim_{n \to \infty} (\|a_n\| + \|b_n\| - \|a_n + b_n\|) = 0$.

Now assume that $x_n = (x_n^1, x_n^2), y_n = (y_n^1, y_n^2) \in X^*$ satisfy
\begin{equation}
\lim_{n \to \infty} (2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2) = 0.
\end{equation}
By the previous Fact we have
\begin{equation}
\lim_{n \to \infty} (2\|x_n^1\|^2 + 2\|y_n^1\|^2 - \|x_n^1 + y_n^1\|^2) = 0.
\end{equation}
And again by the Fact
\begin{equation}
\lim_{n \to \infty} (\|x_n^1\| - \|y_n^1\|, \|x_n^1\| + \|y_n^1\|) = 0.
\end{equation}
Hence by the Fact
\begin{equation}
\lim_{n \to \infty} (\|x_n\| + \|y_n\| - \|x_n + y_n\|) = 0.
\end{equation}
By (1) and the Fact it follows that
\begin{equation}
\lim_{n \to \infty} (2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2) = 0.
\end{equation}
Hence by the Fact
\begin{equation}
\lim_{n \to \infty} (\|x_n\| + \|y_n\|) - \lim_{n \to \infty} \|x_n + y_n\| = 0.
\end{equation}
By (2) and (3) we have
\begin{equation}
\lim_{n \to \infty} (\|x_n\|_w + \|y_n\|_w) = \lim_{n \to \infty} \|x_n + y_n\|_w, \\
\lim_{n \to \infty} \|x_n\|_w = \lim_{n \to \infty} \|y_n\|_w.
\end{equation}
Hence
\begin{equation}
\lim_{n \to \infty} \|\frac{x_n}{\|x_n\|_w} + \frac{y_n}{\|y_n\|_w}\|_w = \lim_{n \to \infty} \|\frac{x_n}{\|x_n\|_w} + \frac{y_n}{\|y_n\|_w}\|_w = 2.
\end{equation}
The norm $\|\cdot\|_w$ on $\ell_1(\Gamma) \oplus Y^*$ is p-UR with $G$; thus, for all $G \in G$ we have
\begin{equation}
\lim_{n \to \infty} G\left(\frac{x_n}{\|x_n\|_w} - \frac{y_n}{\|y_n\|_w}\right) = 0,
\end{equation}
which finishes the proof of Theorem 1. \hfill \Box

Remark. Moreover, if the norm $\|\cdot\|_{Y^*}$ on $Y^*$ is LUR, the norm $\|\cdot\|_u$ on $X^*$ is LUR as well because the norm $\|\cdot\|_w$ is.

Corollary 2. Let $K$ be a scattered compact space with $K^{(\omega)} = \emptyset$. Then $C(K)^*$ admits an equivalent dual norm that is simultaneously LUR and p-UR with $\mathbb{F} = \{e_k; k \in K\} \subset c_0(K) \subset C(K)^{**}$.

Proof. By a compactness argument, there is some $n \in \mathbb{N}$ such that $K^{(n)} = \emptyset$. We shall prove the corollary by an induction. If $n = 0, 1$, the claim is trivial. Let $n > 1$.

It is easy to see that the space $Y = \{f \in C(K); f|_{K'} = 0\}$ is isometric to the space $c_0(K \setminus K')$. Moreover, $C(K)/Y = C(K')$, so we can use Theorem 1. \hfill \Box

In fact, the following theorem holds.

Theorem (Deville). Let $K$ be a scattered compact space, such that $K^{(\omega_1)} = \emptyset$. Then $C(K)^*$ admits an equivalent dual norm, that is LUR and p-UR.
Proof. See [5, Theorem 7.4.7]. There is an equivalent dual LUR norm constructed on $C(K)^*$. One can compute that this norm is, moreover, p-UR with $F = \{e_k; k \in K\}$. \hfill \Box

Note that it is shown in [8] (see also [13]) that there is a Banach space $JL_2$ with the following properties:

1. $c_0 \subset JL_2$, $JL_2/c_0 = \ell_2(\Gamma)$, where the cardinality of the set $\Gamma$ is a continuum,
2. $JL_2$ is not a subspace of any WCG space; in particular, $JL_2$ is not isomorphic to the $c_0 \oplus \ell_2(\Gamma)$,
3. there is an equivalent dual LUR norm on $JL_2^*$.

From Theorem 1 we can obtain a stronger result.

**Theorem 3.** There is an equivalent dual norm on the space $JL^*$ that is LUR and p-UR with $F$, where $F$ is the canonical imbedding of $c_0 \oplus \ell_2(\Gamma)$ into $\ell_\infty \oplus \ell_2(\Gamma) \cong JL_2^*$.

It is shown in [5] pp. 299–305, that if $K$ is a so-called “two arrow” compact space, then $C([0,1]) \subset C(K)$, $C(K)/C([0,1]) = c_0([0,1])$ and $C(K)$ has no equivalent Gâteaux smooth norm. It means that there is no dual equivalent strictly convex norm on $C(K)^*$. It means that $C(K)^*$ does not admit an equivalent dual URED norm, although both $C([0,1])^*$ and $c_0([0,1])^*$ do admit an equivalent dual p-UR norms.

3. **Scattered Eberlein Compact Spaces**

**Theorem 4.** Let $K$ be a scattered compact space such that $K = \bigcup_{n=1}^\infty K_n$, and for all $n \in \mathbb{N}$ let $C(K_n)^*$ admit an equivalent dual p-UR norm with $F_n = \{e_k; k \in K_n\}$. Then $C(K)^*$ admits an equivalent dual p-UR norm with $F = \{e_k; k \in K\}$.

Proof. This proof is similar to the proof of Theorem 2.7.16 in [5], which states, that the space $L_1(\Omega)$ admits a norm that is LUR and URED.

As in [9], we can define the operator $T : C(K) \to \sum_{\ell_2} C(K_n)$ by the formula $T(f) = (\frac{1}{n^2} f|_{K_n})$. For $k \in K$ put $N(k) = \{n \in \mathbb{N}; k \in K_n\}$. For $k \in K$, $n \in N(k)$ let $k_n$ denote a copy of $k$ in $K_n$. For $A \subset K$ put $A = \{k_n; k \in A, n \in N(k)\}$. By Rudin’s Theorem (see [7]) $C(K)^*$ is isometric to the space $\ell_1(K)$ and the canonical norm $\|\|_1$ is a dual norm on $C(K)^*$, the same holds for $K_n$’s. Without loss of generality, we can assume, that the p-UR norms are uniformly close to the original norms on $C(K_n)^*$. Hence $(\sum_{\ell_2} C(K_n))^* = \sum_{\ell_2} \ell_1(K_n)$ and $(\sum_{\ell_2} C(K_n))^*$ admits an equivalent dual norm $\|\|_{*}$, which is p-UR with $G = \{e_{k_n}; k \in K, n \in N(k)\}$. The dual operator $T^* : (\sum_{\ell_2} C(K_n))^* \to C(K)^*$ is given by

$$T^*(y^*) = \left( \sum_{n \in N(k)} \frac{1}{n^2} y^*(k_n) \right)_{k \in K}.$$

The range of $T^*$ is a dense set in $C(K)^*$. Now, we shall use the standard LUR renorming method. For $n \in \mathbb{N}$ and $x \in \ell_1(K)$ we define

$$|x|^2_n = \inf \left\{ \|x - T^* y\|_1^2 + \frac{1}{n} \|y\|_2^2; y \in \sum_{\ell_2} C(K_n)^* \right\},$$

$$\|x\|^2 = \|x\|_1^2 + \sum_{n=1}^\infty 2^{-n} |x|^2_n.$$
This is a dual norm and we will prove that it is p-UR. Choose \( x_i, y_i \in l_1(K) \) such that \( \|x_i\|_1 \leq 1, \|y_i\|_1 \leq 1 \) and \[
\lim_{i \to \infty} \left( 2\|x_i\|^2 + 2\|y_i\|^2 - \|x_i + y_i\|^2 \right) = 0.
\]

Then for all \( n \in \mathbb{N} \),

\[
\lim_{i \to \infty} \left( 2|x_i|^2_n + 2|y_i|^2_n - |x_i + y_i|^2_n \right) = 0.
\]

The infimum in the definition of \( |.|_n \) is attained (see [5, p. 44]); e.g., for all \( i, n \in \mathbb{N} \) there are \( u_i^{(n)}, v_i^{(n)} \in \sum_{\ell_2} C(K_n)^* \) such that

\[
|x_i|^2_n = \|x_i - T^*u_i^{(n)}\|^2_n + \frac{1}{n}\|u_i^{(n)}\|^2_\Sigma,
\]

\[
|y_i|^2_n = \|y_i - T^*v_i^{(n)}\|^2_n + \frac{1}{n}\|v_i^{(n)}\|^2_\Sigma.
\]

From (2) we get

\[
\|u_i^{(n)}\|_\Sigma \leq n|x_i|_n \leq n\|x_i\|_1 \leq n,
\]

and by the same manner we have \( \|v_i^{(n)}\|_\Sigma \leq n \); therefore, for all \( k \in K \) and \( l \in N(k) \)

\[
(\bar{u}_i^{(n)} - \bar{v}_i^{(n)})(\tilde{k}_l) \leq \|\bar{u}_i^{(n)} - \bar{v}_i^{(n)}\|_1 \leq c, \|u_i^{(n)} - v_i^{(n)}\|_\Sigma \leq 2cn,
\]

where \( c \) is a constant of the equivalence of norms \( \|\| \) and \( \|\|_\Sigma \). From (1), (2), (3)

we have

\[
\lim_{i \to \infty} \left( 2\|u_i^{(n)}\|^2_\Sigma + 2\|v_i^{(n)}\|^2_\Sigma - \|u_i^{(n)} + v_i^{(n)}\|^2_\Sigma \right) = 0.
\]

The norm \( \|\|_\Sigma \) is p-UR and hence for all \( k \in K, m \in \mathbb{N} \) and \( n \in N(k) \)

\[
\lim_{i \to \infty} \left( u_i^{(m)} - v_i^{(m)}\right)(\tilde{k}_n) = 0.
\]

We can assume (by passing to a subsequence), that \( \lim_{i \to \infty} |x_i|_n = d_n \). For every \( x \in l_1(K), |x|_n \) is a nonincreasing sequence, hence there is \( d = \lim_{n \to \infty} d_n \). By passing to a subsequence again, we can assume, moreover, that \( \lim_{i \to \infty} |y_i|_n = d_n \).

Choose \( \varepsilon > 0 \) and \( k \in K \). Put \( A = K \setminus \{ k \} \). Let \( m \in \mathbb{N} \) be such that \( d_m < d + \varepsilon \).

Then

\[
|(x_i - y_i)(k)| \leq |(x_i - T^*u_i^{(m)}(k))| + |(T^*u_i^{(m)} - T^*v_i^{(m)}(k))| + |(T^*v_i^{(m)} - y_i)(k)|.
\]

Considering the second term, we have

\[
\left| T^*(u_i^{(m)} - v_i^{(m)})(k) \right| = \left| \sum_{n \in N(k)} \frac{1}{n}\left( u_i^{(m)} - v_i^{(m)}\right)(\tilde{k}_n) \right|
\]

\[
\leq \sum_{n \leq n_0, n \in N(k)} \left| \frac{1}{n}\left( u_i^{(m)} - v_i^{(m)}\right)(\tilde{k}_n) \right| + \varepsilon,
\]

where \( n_0 \) depends only on \( \varepsilon \) (because of (4)) and the sum is finite and therefore tends to 0 for \( i \to \infty \) because of (5). It remains to prove, that \( |(x_i - T^*u_i^{(m)}(k))| < \varepsilon \). We can assume that \( k \in K_{n_0} \). Put \( y = s_i + (u_i^{(m)}|_{\tilde{k}_l}) \), where \( s_i(l) = n_0^2x_i(l) \) if \( l = \tilde{k}_{n_0} \),
and \( s_i(l) = 0 \) otherwise. Considering this \( y \) in the definition of \(|x_i|_n \) we get
\[
|x_i|_n^2 \leq \|(x_i - T^* u_i^{(m)})|_A\|^2 + \frac{1}{n} s_i + (u_i^{(m)}|_A)\|^2,
\]
\[
\leq \|(x_i - T^* u_i^{(m)})|_A\|^2 + \frac{1}{n} (s_i + \|u_i^{(m)}|_A\|^2),
\]
\[
\leq \|(x_i - T^* u_i^{(m)})|_A\|^2 + \frac{1}{n} (n_0^2 c + mc^2)^2
\]
because
\[
\|u_i^{(m)}|_A\|^2 \leq c\|u_i^{(m)}|_A\| \leq c\|u_i^{(m)}\|_1 \leq mc^2,
\]
where we used that the canonical norm \( \|\cdot\|_1 \) on \( X^* \) is a lattice norm.
Hence for all \( n \in \mathbb{N} \)
\[
\limsup_{i \to \infty} \|(x_i - T^* u_i^{(m)})|_A\|^2 \geq \lim_{i \to \infty} |x_i|_n^2 - \frac{1}{n} (n_0^2 + mc^2)^2.
\]
Finally,
\[
\limsup_{i \to \infty} \|(x_i - T^* u_i^{(m)})|_A\|^2 \geq d^2.
\]
For all \( i \in \mathbb{N} \) we have
\[
\left| (x_i - T^* u_i^{(m)})(k) \right| = \left| x_i - T^* u_i^{(m)} \right| - \left| (x_i - T^* u_i^{(m)})|_A \right| \leq |x_i|_m - \left| (x_i - T^* u_i^{(m)})|_A \right|.
\]
Hence we get
\[
\liminf_{i \to \infty} \left| (x_i - T^* u_i^{(m)})(k) \right| \leq d_m - d \leq \varepsilon.
\]
The same holds for the third term and this concludes the proof.

**Theorem 5.** Let \( K \) be a scattered Eberlein compact space. Then \( C(K)^* \) admits an equivalent dual norm that is LUR and p-UR with \( \mathcal{F} = \{ e_k; k \in K \} \). In particular, \( C(K)^* \) admits an equivalent dual norm that is LUR and URED.

**Proof.** K. Alster proved in [1] that if \( K \) is a scattered Eberlein compact space, then \( K \) is a strong Eberlein compact space, e.g., \( K \subset \{ 0, 1 \}^\Gamma \) for some \( \Gamma \). Hence
\[
K = \bigcup_{n=1}^\infty K_n, \quad \text{where} \quad K_n = \{ x \in K; \text{card}(\{ \gamma \in \Gamma; x(\gamma) = 1 \}) \leq n \}.
\]
The \( K_n \)'s are uniform Eberlein compact spaces, they are scattered and \( K^{(n+1)} = \emptyset \). Hence by Corollary 2, \( C(K_n)^* \) admits an equivalent dual norm that is both p-UR with \( \mathcal{F} = \{ e_k; k \in K_n \} \) and LUR. Thus we can use the preceding theorem to finish the proof.

4. **Open question**

It is shown in [12] Th. 1 that if \( X \) has an unconditional Schauder basis and \( X^* \) admits an equivalent URED norm, then \( X^* \) admits an equivalent dual weak* uniformly rotund norm. Because there is a scattered Eberlein compact space \( K \) that is not uniform Eberlein compact (see, e.g., [2] Example 1.10), the space \( C(K)^* \) admits an equivalent dual p-UR (and hence URED norm) but does not admit any equivalent dual W*UR norm. But we do not know the answer to the following questions. Is there any reflexive Banach space \( X \) such that \( X \) admits an equivalent URED norm and does not admit any equivalent p-UR (and hence W*UR) norm?
Is there any Banach space that admits an equivalent URED norm and does not admit any p-UR norm?

References


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