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BEHAVIOR OF THE BERGMAN KERNEL AND METRIC NEAR CONVEX BOUNDARY POINTS

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Abstract. The boundary behavior of the Bergman metric near a convex boundary point z_0 of a pseudoconvex domain $D \subset \mathbb{C}^n$ is studied. It turns out that the Bergman metric at points $z \in D$ in the direction of a fixed vector $X_0 \in \mathbb{C}^n$ tends to infinity, when z is approaching z_0, if and only if the boundary of D does not contain any analytic disc through z_0 in the direction of X_0.

For a domain $D \subset \mathbb{C}^n$ we denote by $L_h^2(D)$ the Hilbert space of all holomorphic functions f that are square-integrable and by $||f||_D$ the L_2-norm of f. Let $K_D(z)$ be the restriction on the diagonal of the Bergman kernel function of D. It is well known (cf. [5]) that

$$K_D(z) = \sup\{|f(z)|^2 : f \in L_h^2(D),\ ||f||_D \le 1\}.$$

If $K_D(z) > 0$ for some point $z \in D$, then the Bergman metric $B_D(z;X),\ X \in \mathbb{C}^n$, is well defined and can be given by the equality

$$B_D(z;X) = \frac{M_D(z;X)}{\sqrt{K_D(z)}},$$

where

$$M_D(z;X) = \sup\{|f'(z)X| : f \in L_h^2(D),\ ||f||_D = 1,\ f(z) = 0\}.$$

We say that a boundary point z_0 of a domain $D \subset \mathbb{C}^n$ is *convex* if there is a neighborhood U of this point such that $D \cap U$ is convex.

In [4], Herbort proved the following

Theorem 1. *Let z_0 be a convex boundary point of a bounded pseudoconvex domain $D \subset \mathbb{C}^n$ whose boundary contains no nontrivial germ of an analytic curve near z_0. Then*

$$\lim_{z \to z_0} B_D(z;X) = \infty$$

for any $X \in \mathbb{C}^n \setminus \{0\}$.

Herbort's proof is mainly based on Ohsawa's $\bar{\partial}$-technique. The main purpose of this note is to generalize Theorem 1 using more elementary methods.

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For a convex boundary point z_0 of a domain $D \subset \mathbb{C}^n$ we denote by $L(z_0)$ the set of all $X \in \mathbb{C}^n$ for which there exists a number $\varepsilon_X > 0$ such that $z_0 + \lambda X \in \partial D$ for all complex numbers λ, $|\lambda| \le \varepsilon_X$. Note that $L(z_0)$ is a complex linear space.

Then our result is the following one.

Theorem 2. *Let z_0 be a convex boundary point of a bounded pseudoconvex domain $D \subset \mathbb{C}^n$ and let $X \in \mathbb{C}^n$. Then*

(a) $\liminf\limits_{z\to z_0} K_D(z)\operatorname{dist}^2(z,\partial D) \in (0,\infty]$*;*

(b) $\lim\limits_{z\to z_0} B_D(z;X) = \infty$ *if and only if $X \notin L(z_0)$. Moreover, this limit is locally uniform in $X \notin L(z_0)$;*

(c) *if $L(z_0) = \{0\}$, then* (a) *and* (b) *are still true without the assumption that D is bounded.*

Proof of Theorem 2. To prove (a) and (b) we will use the following localization theorem for the Bergman kernel and metric [2].

Theorem 3. *Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and let $V \subset\subset U$ be open neighborhoods of a point $z_0 \in \partial D$. Then there exists a constant $\tilde{C} \ge 1$ such that*

$$\tilde{C}^{-1} K_{D\cap U}(z) \le K_D(z) \le K_{D\cap U}(z),$$
$$\tilde{C}^{-1} B_{D\cap U}(z;X) \le B_D(z;X) \le \tilde{C} B_{D\cap U}(z;X)$$

for any $z \in D \cap V$ and any $X \in \mathbb{C}^n$. (Here $K_{D\cap U}(z)$ and $B_{D\cap U}(z;\cdot)$ denote the Bergman kernel and metric of the connected component of $D \cap U$ that contains z.)

So, we may assume that D is convex.

To prove part (a) of Theorem 2, for any $z \in D$ we choose a point $\tilde{z} \in \partial D$ such that $||z - \tilde{z}|| = \operatorname{dist}(z,\partial D)$. We denote by l the complex line through z and $\tilde{z}$. By the Oshawa-Takegoshi extension theorem for L^2–holomorphic functions [7], it follows that there exists a constant $C_1 > 0$ only depending on the diameter of D (not on l) such that

$$K_D(z) \ge C_1 K_{D\cap l}(z). \tag{1}$$

Since $D \cap l$ is convex, it is contained in an open half-plane Π of the l-plane with $\tilde{z} \in \partial\Pi$. Then

$$K_{D\cap l}(z) \ge K_\Pi(z) = \frac{1}{4\pi \operatorname{dist}^2(z,\partial\Pi)}. \tag{2}$$

Now, part (a) of Theorem 2 follows from the inequalities (1), (2) and the fact that $\operatorname{dist}(z,\partial\Pi) \le ||z - \tilde{z}|| = \operatorname{dist}(z,\partial D)$.

To prove part (b) of Theorem 2, we denote by $N(z_0)$ the complex affine space through z_0 that is orthogonal to $L(z_0)$. Set $E(z_0) = D \cap N(z_0)$. Note that $E(z_0)$ is a nonempty convex set. So, part (b) of Theorem 2 will be a consequence of the following:

Theorem 4. *Let z_0 be a boundary point of a bounded convex domain $D \subset \mathbb{C}^n$. Then:*

(i) $\lim\limits_{z\to z_0} B_D(z;X) = \infty$ *locally uniformly in $X \notin L(z_0)$;*

(ii) *for any compact set* $K \subset\subset E(z_0)$ *there exists a constant* $C > 0$ *such that*

$$B_D(z;X) \le C||X||, \quad z \in K^0,\ X \in L(z_0),$$

where $K^0 := \{z_0 + tz : \ z \in K,\ 0 < t \le 1\}$ *is the cone generated by* K.

Proof of Theorem 4. To prove (i) we will use the well-known fact that the Carathéodory metric $C_D(z;X)$ of D does not exceed $B_D(z;X)$. On the other hand, we have the following simple geometric inequality [1]:

$$C_D(z;X) \ge \frac{1}{2d(z;X)},$$

where $d(z;X)$ denotes the distance from z to the boundary of D in the X-direction, i.e., $d(z;X) := \sup\{r : z + \lambda X \in D,\ \lambda \in \mathbb{C},\ |\lambda| < r\}$. So, if we assume that (i) does not hold, then we may find a number $a > 0$ and sequences $D \supset (z_j)_j,\ z_j \to z_0,\ \mathbb{C}^n \supset (X_j)_j,\ X_j \to X \notin L(z_0)$, such that $B_D(z_j;X_j) \le \dfrac{1}{2a}$. Hence $d(z_j;X_j) \ge a$ which implies that for $|\lambda| \le a$ the points $z_0 + \lambda X$ belong to $\bar{D}$ and, in view of convexity, they belong to ∂D. This means that $X \in L(z_0)$, a contradiction.

To prove part (ii) of Theorem 4, we may assume that $z_0 = 0$ and $L := L(0) = \{z \in \mathbb{C}^n : z_1 = \ldots = z_k = 0\}$ for some $k < n$. Then $N := N(0) = \{z \in \mathbb{C}^n : z_{k+1} = \ldots = z_n = 0\}$. From now on we will write any point $z \in \mathbb{C}^n$ in the form $z = (z', z'')$, $z' \in \mathbb{C}^k$, $z'' \in \mathbb{C}^{n-k}$. Note that $L \in \partial D$ near 0, i.e., there exists a $c > 0$ such that

$$\{0'\} \times \Delta''_c \in \partial D, \tag{3}$$

where $\Delta''_c \subset \mathbb{C}^{n-k}$ is the polydisc with center at the origin and radius c. Since $K \subset\subset E := E(0)$ and since E is convex, there exists an $\alpha > 1$ such that $K \subset\subset E_\alpha$, where $E_\alpha := \{z : \alpha z \in E\}$. Note that $K^0 \subset E_\alpha$. Using (3), the equality

$$(z', z'') = \frac{1}{\alpha}(\alpha z', 0'') + (1 - \frac{1}{\alpha})(0', (1 - \frac{1}{\alpha})^{-1} z''),$$

and the convexity of D, it follows that

$$F_\alpha \times \Delta''_\varepsilon \subset D, \tag{4}$$

where $\varepsilon := c(1 - \dfrac{1}{\alpha})$ and where F_α is the projection of E_α in $\mathbb{C}^k$ (we can identify E_α with F_α). For $\delta := c(\alpha - 1)$ we obtain in the same way that

$$\tilde{D} := D \cap (\mathbb{C}^k \times \Delta''_\delta) \subset F_{\frac{1}{\alpha}} \times \Delta''_\delta. \tag{5}$$

Now, let $(z, X) \in K^0 \times L$. Note that $z = (z', 0'')$ and $X = (0', X'')$. Then, using (4) and the product properties of the Bergman kernel and metric, we have

$$\begin{aligned} M_D(z;X) &\le M_{F_\alpha \times \Delta''_\varepsilon}(z;X) \\ &= M_{\Delta''_\varepsilon}(0'';X'')\sqrt{K_{F_\alpha}(z')} \le C_1||X||\sqrt{K_{F_\alpha}(z')} \end{aligned} \tag{6}$$

for some constant $C_1 > 0$. On the other hand, since $K^0 \subset\subset \mathbb{C}^k \times \Delta''_\delta$, by virtue of Theorem 3 there exists a constant $\tilde{C} \ge 1$ such that

$$K_D(z) \ge \tilde{C}^{-1} K_{\tilde{D}}(z).$$

Moreover, in view of (5), we have

$$K_{\tilde{D}}(z) \ge K_{F_{\frac{1}{\alpha}}}(z') K_{\Delta''_\delta}(0'')$$

and hence

$$K_D(z) \geq (C_2)^2 K_{F_{\frac{1}{\alpha}}}(z') \tag{7}$$

for some constant $C_2 > 0$. Now, by (6) and (7), it follows that

$$B_D(z;X) = \frac{M_D(z;X)}{\sqrt{K_D(z)}} \leq \frac{C_1}{C_2}||X||\sqrt{\frac{K_{F_\alpha}(z')}{K_{F_{\frac{1}{\alpha}}}(z')}}. \tag{8}$$

Note that $z' \to \alpha^{-2}z'$ is a biholomorphic mapping from $F_{\frac{1}{\alpha}}$ onto F_α and, therefore,

$$K_{F_{\frac{1}{\alpha}}}(z') = \alpha^{-4k} K_{F_\alpha}(\alpha^{-2}z'). \tag{9}$$

In view of (8) and (9), in order to finish (ii) we have to find a constant $C_3 > 0$ such that

$$K_{F_\alpha}(z') \leq C_3 K_{F_\alpha}(\alpha^{-2}z') \tag{10}$$

for any $z' \in H^0$ with $H^0 := \{tz' : \ z' \in H, \ 0 < t \leq 1\}$, where H is the projection of K into $\mathbb{C}^k$ (we can identify K with H).

To do this, note first that $\gamma := \operatorname{dist}(H, \partial F_\alpha) > 0$ since $K \subset\subset E_\alpha$. Fix $\tau \in (0,1]$ and $z' \in H^0$, and denote by $T_{\tau,z'}$ the translation that maps the origin in the point $\tau z'$. It is easy to check that

$$T_{\tau,z'}(\bar{F}_\alpha \cap B_\gamma) \subset F_\alpha, \tag{11}$$

where B_γ is the ball in $\mathbb{C}^k$ with center at the origin and radius γ. To prove (10), we will consider the following two cases:

Case I. $z' \in H^0 \setminus B_{\frac{\gamma}{2}} \subset\subset F_\alpha$: Then

$$K_{F_\alpha}(z') \leq \frac{m_1}{m_2} K_{F_\alpha}(\alpha^{-2}z'), \tag{12}$$

where $m_1 := \sup_{H^0 \setminus B_{\frac{\gamma}{2}}} K_{F_\alpha}$ and $m_2 := \inf K_{F_\alpha}$.

Case II. $z' \in H^0 \cap B_{\frac{\gamma}{2}}$: By Theorem 3 there exists a constant $\tilde{C}_3 \geq 1$ such that $\tilde{C}_3 K_{F_\alpha} \geq K_{F_\alpha \cap B_\gamma}$ on $F_\alpha \cap B_{\frac{\gamma}{2}}$. In particular,

$$\tilde{C}_3 K_{F_\alpha}(\alpha^{-2}z') \geq K_{F_\alpha \cap B_\gamma}(\alpha^{-2}z'). \tag{13}$$

On the other hand, by (11) with data $T := T_{1-\alpha^{-2},z'}$ it follows that

$$K_{F_\alpha \cap B_\gamma}(\alpha^{-2}z') = K_{T(F_\alpha \cap B_\gamma)}(z') \geq K_{F_\alpha}(z'). \tag{14}$$

Now, (12), (13), and (14) imply that (10) holds for $C_3 := \max\{\frac{m_1}{m_2}, \tilde{C}_3\}$. This completes the proofs of Theorem 4 and part (b) of Theorem 2. □

Remark. The approximation (5) of the domain $D \cap (\mathbb{C}^k \times \Delta''_\delta)$ by the domain $E_{\frac{1}{\alpha}} \times \Delta''_\delta$ can be replaced by using the Oshawa-Takegoshi theorem [7] with the data D and N.

Finally, part (c) of Theorem 2 will be a consequence of the following two theorems.

Theorem 5 ([6])**.** *Let* $D \subset \mathbb{C}^n$ *be a pseudoconvex domain and let* U *be an open neighborhood of a local (holomorphic) peak point* $z_0 \in \partial D$*. Then*

$$\lim_{z \to z_0} \frac{K_D(z)}{K_{D\cap U}(z)} = 1$$

and

$$\lim_{z \to z_0} \frac{B_D(z;X)}{B_{D\cap U}(z;X)} = 1$$

locally uniformly in $X \in \mathbb{C}^n \setminus \{0\}$*.*

Theorem 6. *Let* z_0 *be a boundary point of a bounded convex domain* $D \subset \mathbb{C}^n$*. Then the following conditions are equivalent:*

1. z_0 *is a (holomorphic) peak point;*
2. z_0 *is the unique analytic curve in* $\bar{D}$ *containing* z_0*;*
3. $L(z_0) = \{0\}$*.*

Note that the only nontrivial implication is (3) $\Longrightarrow$ (1). It is contained in [8]. Now, part (c) of Theorem 2 is a consequence of this implication, Theorem 5, and part (b) of Theorem 2.

Proof of Theorem 6. The implication (2) $\Longrightarrow$ (3) is trivial.

The implication (1) $\Longrightarrow$ (2) easily follows by the maximum principle and the fact that there are a neighborhood U of z_0 and a vector $X \in \mathbb{C}^n$ such that $(\bar{D} \cap U) + (0,1]X \subset D$ (cf. (11)).

Denote by $A^0(D)$ the algebra of holomorphic functions on D which are continuous on $\bar{D}$. Now, following [8] we shall prove the implication (3) $\Longrightarrow$ (1); namely, (3) implies that z_0 is a peak point with respect to $A^0(D)$. This is equivalent to the fact (cf. [3]) that the point mass at z_0 is the unique element of the set $A(z_0)$ of all representing measures for z_0 with respect to $A^0(D)$, i.e. supp $\mu = \{z_0\}$ for any $\mu \in A(z_0)$.

Let $\mu \in A(z_0)$. Since D is convex, we may assume that $z_0 = 0$ and $D \subset \{z \in \mathbb{C}^n : \operatorname{Re}(z_1) < 0\}$. Note that if a is a positive number such that $a \inf_{z \in D} \operatorname{Re}(z_1) > -1$ (D is bounded), then the function $f_1(z) = \exp(z_1 + a z_1^2)$ belongs to $A^0(D)$ and $|f_1(z)| < 1$ for $z \in \bar{D} \setminus \{z_1 = 0\}$. This easily implies (cf. [3]) that supp $\mu \subset D_1 := \partial D \cap \{z_1 = 0\}$. Since $L(0) = 0$, the origin is a boundary point of the compact convex set D_1. As above, we may assume that $D_1 \subset \{z \in \mathbb{C}^n : \operatorname{Re}(z_2) \le 0\}$ (z_2 is independent of z_1) and then construct a function $f_2 \in A^0(D)$ such that $|f_2(z)| < 1$ for $z \in D_1 \setminus \{z_2 = 0\}$. This implies that supp $\mu \subset D_1 \cap \{z_2 = 0\}$. Repeating this argument we conclude that supp $\mu = \{0\}$, which completes the proofs of Theorems 6 and 2. ☐

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