THE IDEAL PROPERTY IN CROSSED PRODUCTS

CORNEL PASNICU

(Communicated by David R. Larson)

Abstract. We describe the lattice of the ideals generated by projections and prove a characterization of the ideal property for “large” classes of crossed products of commutative $C^*$-algebras by discrete, amenable groups; some applications are also given. We prove that the crossed product of a $C^*$-algebra with the ideal property by a group with the ideal property may fail to have the ideal property; this answers a question of Shuzhou Wang.

1. Introduction

A $C^*$-algebra is said to have the ideal property if its ideals are generated (as ideals) by their projections (in this paper, by an ideal we shall mean a closed, two-sided ideal).

The class of the $C^*$-algebras with the ideal property is important because it includes the $C^*$-algebras of real rank zero ([BPe]) and also the simple, unital $C^*$-algebras. The ideal property has been studied in [Pa1], [Pa2], [PaR], [S]. On the other hand, the crossed products ([Pe]) represent a valuable source of interesting $C^*$-algebras, related to the domain of the dynamical systems. Many important nuclear, simple, unital, separable $C^*$-algebras generated by a subset of elements or many $AH$ algebras are in fact crossed products of commutative, separable $C^*$-algebras by discrete, countable, amenable groups (e.g. the irrational rotation algebras, the Bunce-Deddens algebras and others (see [EH])). It became clear that the nuclear crossed products with the ideal property will play a central role in Elliott’s project of the classification of the nuclear, separable $C^*$-algebras by invariants including $K$-theory ([EH]). Therefore, it is clearly important to know when a crossed product has the ideal property. Our main interest is in the case when the group is discrete and it acts on a commutative $C^*$-algebra. For a discrete, countable, amenable group $G$ and a commutative, separable $C^*$-algebra $A$ let $\alpha$ be an action of $G$ on $A$ which induces a free action of $G$ on the spectrum of $A$. Based mainly on results in [EH], [GL] and [Z], we identify the lattice of the ideals generated by projections of $A\rtimes_\alpha G$ with a lattice of $\alpha$-invariant ideals of $A$ (see Theorem 2.1 below), which allows us to characterize the ideal property for $A\rtimes G$ (see Corollary 2.2 below). Using this result, we deduce that if in addition the spectrum of the $C^*$-algebra $A$ is zero dimensional, or if $A = C(X)$, where $X$ is a compact, Hausdorff topological space and every...
closed, \( \alpha \)-invariant subset of \( X \) is also open, it follows that \( A \rtimes G \) has the ideal property (see Corollary 2.3 and Corollary 2.4 below). In \( \text{PaR} \) we proved, jointly with M. Rørdam, that the minimal tensor product of two \( C^* \)-algebras each having the ideal property does not necessarily have the ideal property. On the other hand, if \( (A, G, \alpha) \) is a \( C^* \)-dynamical system, then one may visualize the crossed product \( A \rtimes G \) as a skew tensor product between \( A \) and \( C^*(G) \), since if \( \alpha \) is trivial, then \( A \rtimes G \cong A \otimes_{\text{max}} C^*(G) \). We answer a question on Shuzhou Wang, proving that there are crossed products of \( C^* \)-algebras with the ideal property by groups \( G \) with the ideal property (i.e., \( C^*(G) \) has the ideal property; see Definition 2.5 below) that do not have the ideal property (see Theorem 2.6 below).

If \( A \) is a \( C^* \)-algebra, the fact that \( I \) is an ideal (closed, two-sided) of \( A \) will be denoted by \( I \triangleleft A \). The projections of a \( C^* \)-algebra \( A \) will be denoted by \( \mathcal{P}(A) \). Let \( G \) be a locally compact group. \( G \) is amenable if there is a left invariant mean on \( L^\infty(G) \) (see \( \text{PaC} \)). Note that if \( G \) is abelian or if it is compact, then \( G \) is amenable (\( \text{PaC} \)). For the definition of \( C^*(G) \), the group \( C^* \)-algebra of a locally compact group \( G \), see \( \text{Pe} \). Let \( (G, X) \) be a topological transformation group. Recall that \( G \) is said to act freely on \( X \) (or we say that the action of \( G \) on \( X \) is a free action) if for every \( x \in X \) and for every \( g \in G, g \) different from the neutral element of \( G \), we have that \( gx \neq x \). Recall also that a \( C^* \)-dynamical system is a triple \( (A, G, \alpha) \) consisting of a \( C^* \)-algebra \( A \), a locally compact group \( G \) and a continuous homomorphism \( \alpha \) of \( G \) into the group \( \text{Aut}(A) \) of automorphisms (i.e., \( * \)-automorphisms) of \( A \) equipped with the topology of pointwise convergence.

Let \( (A, G, \alpha) \) be a \( C^* \)-dynamical system, where \( G \) is a discrete group with neutral element \( e \). Consider an injective \( * \)-representation \( \pi : A \to B(H) \). We shall identify \( A \) with its image by \( \pi \) and \( A \rtimes G \) with the norm closure of \( (\pi \times U)(l^1(G, A)) \) in \( B(l^2(G, H)) \), where

\[
(\tilde{\pi}(a)x)(g) = \pi(\alpha_{g^{-1}}(a))x(g), \\
(U_hx)(g) = x(h^{-1}g)
\]

for each \( a \in A, g, h \in G \) and \( x \in l^2(G, H) \) (see \( \text{Pe} \) Theorem 7.7.5). We will denote by \( E : A \rtimes \alpha \rtimes G \to A \) the canonical conditional expectation given by

\[
E(aU_g) = \begin{cases} 
  a, & \text{if } g = e, \\
  0, & \text{if } g \neq e
\end{cases}
\]

where \( a \in A, \ g \in G \). If \( M \subseteq A \rtimes \alpha \rtimes G \), we shall denote by \( \langle M \rangle \) the (closed and two-sided) ideal of \( A \rtimes \alpha \rtimes G \) generated by \( M \). Suppose now that \( A = C(X) \), where \( X \) is a compact, Hausdorff topological space. A subset \( F \) of \( X \) is called \( \alpha \)-invariant if it is invariant for the action of \( G \) on \( X \) canonically induced by \( \alpha \).

As usual, \( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \) and \( \mathcal{K} \) denotes the \( C^* \)-algebra of compact operators on \( l^2(\mathbb{N}) \).

An \( AF \) algebra is an inductive limit \( C^* \)-algebra of the form \( \lim_{\to} A_n \), where for each \( n \in \mathbb{N}, A_n \) is a finite-dimensional \( C^* \)-algebra (\( \text{BR} \)).
2. Results

**Theorem 2.1.** Let \((A, G, \alpha)\) be a \(C^*\)-dynamical system such that the \(C^*\)-algebra \(A\) is commutative and separable and the group \(G\) is discrete, countable and amenable. Suppose that \(\alpha\) induces a free action of \(G\) on the spectrum of \(A\). Then, the map

\[ \theta : \{ I \triangleleft A \mid I \text{ is } \alpha\text{-invariant} \} \to \{ J \mid J \triangleleft A \rtimes \hat{\alpha} \} \]

given by

\[ \theta(I) := I \rtimes \hat{\alpha} \]

is a lattice isomorphism and it induces canonically a lattice isomorphism

\[ \{ I \triangleleft A \mid I \text{ is } \alpha\text{-invariant and } I \subseteq \langle P(I \rtimes \alpha) \rangle \} \cong \{ J \triangleleft A \rtimes \alpha \mid J \text{ is generated by projections} \}. \]

**Proof.** For every \(x \in A \rtimes G\) and every \(g \in G\), define \(x_g := E(xU_g^*)\). For an arbitrary \(M \subseteq A\), denote \(\mathcal{I}(M) = \{ x \in A \rtimes G \mid x_g \in M \text{ for every } g \in G \}\) (see [Z, 4.15]). Note that since \(G\) is amenable, \(A \rtimes \alpha = A \rtimes G\) ([Z, Théorème 5.1]). Since \(A\) is separable and \(G\) is a countable, amenable group acting freely on the spectrum of \(A\), then it follows by [EH, Corollary 5.16] (as in [Z, Théorème 5.15]) that

\[ \Phi : \{ I \triangleleft A \mid I \text{ is } \alpha\text{-invariant} \} \to \{ J \mid J \triangleleft A \rtimes \alpha \}, \]

defined by

\[ \Phi(I) := \mathcal{I}(I) \]

for each \(\alpha\)-invariant ideal \(I\) of \(A\), is an order isomorphism. Since \(G\) is amenable, it follows by [Z, 5.10] that for each \(\alpha\)-invariant ideal \(I\) of \(A\), we have that

\[ \mathcal{I}(I) = \langle I \rangle. \]

But, by [GL, Proposition 3.11 (ii)] it follows that \(\langle I \rangle = I \rtimes G\) for every \(\alpha\)-invariant ideal \(I\) of \(A\). Again using the fact that \(G\) is amenable, by a result of E.C. Gootman and A.J. Lazar ([GL, Theorem 3.4]) it follows that an ideal \(J\) of \(A \rtimes G\) is \(\hat{\alpha}\)-invariant if and only if \(J = I \rtimes \hat{\alpha}\), for some unique \(\alpha\)-invariant ideal \(\hat{\alpha}\) of \(\hat{A}\), where \(\hat{\alpha}\) is the dual action of \(\hat{G}\) on \(A \rtimes \alpha\) ([T]). Hence, there is an order isomorphism

\[ \Psi : \{ I \triangleleft A \mid I \text{ is } \alpha\text{-invariant} \} \to \{ J \triangleleft A \rtimes \hat{\alpha}G \mid J \text{ is } \hat{\alpha}\text{-invariant} \} \]

given by

\[ \Psi(I) := I \rtimes \hat{\alpha} \]

for each \(\alpha\)-invariant ideal \(I\) of \(A\).

Now, observe that by the above facts, it follows that the order isomorphisms \(\Phi\) and \(\Psi\), which have the same domains of definition, agree on their domain of definition and hence the range of \(\Phi\) is equal to the range of \(\Psi\). But this together with the above discussion imply that \(\theta\) is an order isomorphism. The fact that \(\{ I \triangleleft A \mid I \text{ is } \alpha\text{-invariant} \}\) and \(\{ J \mid J \triangleleft A \rtimes \hat{\alpha} \}\) are lattices and \(\theta\) is a lattice isomorphism follows from [Pa4, Remark 4.3].
Now, suppose that $I \triangleleft A$, $I$ is $\alpha$-invariant and $I \subseteq \langle \mathcal{P} \rangle$. This implies that $\theta(I) = \mathcal{P} \triangleleft \alpha \subseteq \alpha \langle \langle \mathcal{P} \rangle \rangle$, from which we conclude that $\mathcal{P}$ is generated by its projections. On the other hand, if $I \triangleleft A$, $I$ is $\alpha$-invariant and $\theta(I) = \mathcal{P} \triangleleft \alpha \subseteq \alpha \langle \langle \mathcal{P} \rangle \rangle$, then, clearly $I \subseteq \alpha \mathcal{P} = \langle \mathcal{P} \rangle$. These facts, together with [Pa4, Remark 4.3], imply that $\theta$ induces canonically a lattice isomorphism

$$\{ I \triangleleft A \mid I \text{ is } \alpha\text{-invariant and } I \subseteq \langle \alpha \mathcal{P} \rangle \} \cong \{ J \triangleleft \alpha \mathcal{P} \mid J \text{ is generated by projections} \}.$$

**Corollary 2.2.** Let $A$, $G$ and $\alpha$ be as in the hypothesis of Theorem 2.1. Then, the following are equivalent:

(a) $A \times G$ has the ideal property.

(b) For every $\alpha$-invariant ideal $I$ of $A$, we have that $I \subseteq \alpha \langle \mathcal{P} \rangle$.

**Corollary 2.3.** Let $A$, $G$ and $\alpha$ be as in the hypothesis of Theorem 2.1. Moreover, suppose that the spectrum of $A$ has dimension zero. Then, $A \times G$ has the ideal property.

**Proof.** Since $A$ is a commutative, separable $C^*$-algebra with zero-dimensional spectrum, then $A$ is an AF algebra. Hence, $A$ has the ideal property. Now, the result follows by also using Corollary 2.2.

**Corollary 2.4.** Let $A$, $G$ and $\alpha$ be as in the hypothesis of Theorem 2.1. Suppose moreover that $A = C(X)$, where $X$ is a compact, Hausdorff topological space, and that every closed $\alpha$-invariant subset of $X$ is also open. Then, $A \times G$ has the ideal property.

**Proof.** Let $I$ be an $\alpha$-invariant ideal of $A$. Then, there is a closed, $\alpha$-invariant subset $F$ of $X$ such that $I = \{ f \in C(X) \mid f|_F = 0 \}$. Since $X \setminus F$ is closed (because, by hypothesis, $F$ is open) it follows that the map $A \ni f \mapsto f|_F + f|_{X \setminus F} \in C(F) \oplus C(X \setminus F)$ is an isomorphism, which implies that $I \cong C(X \setminus F)$ and hence, $I$ being unital, is generated by projections. Now, the result follows by also using Corollary 2.2.

**Definition 2.5.** Let $G$ be a locally compact group. We say that $G$ has the ideal property if its group $C^*$-algebra $C^*(G)$ has the ideal property.

**Question** (Shuzhou Wang). Let $(A, G, \alpha)$ be a $C^*$-dynamical system such that $A$ and $G$ have the ideal property. Does the crossed product $A \times G$ (always) have the ideal property?

The answer to the above question is “no”, even in the “separable” case, as it follows from the following:

**Theorem 2.6.** There are $C^*$-dynamical systems $(A, G, \alpha)$ with $A$ a separable, simple, unital $C^*$-algebra (and hence, with the ideal property) and with $G$ a separable, commutative topological group with the ideal property such that $A \times G$ does not have the ideal property.
The proof of the above theorem will use the following proposition:

**Proposition 2.7.** Let $G$ be a compact, abelian group. Then, $G$ has the ideal property.

**Proof.** The hypothesis implies that $\hat{G}$ (the dual group of $G$) is a discrete, abelian group. We have that $C^*(G) \cong C_0(\hat{G})$ (see [P, Proposition 7.1.6]). On the other hand, since $\hat{G}$ is discrete, then $\dim (\hat{G}) = 0$ and hence $C_0(\hat{G})$ is an approximately finite-dimensional $C^*$-algebra, i.e., it is the inductive limit of a net of finite-dimensional $C^*$-algebras. Since a finite-dimensional $C^*$-algebra has the ideal property, this implies that $C_0(\hat{G})$ has the ideal property (as an inductive limit of $C^*$-algebras with the ideal property ([Pa4, Proposition 2.3])) and hence, by the above discussion, it follows that $C^*(G)$ has the ideal property, which ends the proof.

**Proof of Theorem 2.6.** Let $(B, \mathbb{Z}, \alpha)$ be a $C^*$-dynamical system with $B = C(X)$, where $X$ is an infinite, compact, connected, metrizable topological space and such that $A := B \rtimes \mathbb{Z}$ is a simple $C^*$-algebra. (Clearly, $A$ is unital and separable and hence with the ideal property, since, obviously, every simple, unital $C^*$-algebra has the ideal property.) Note that $A$ is simple if (and only if) the homeomorphism of $X$ induced by $\alpha(1)$ is minimal (see e.g. [D, Theorem VIII. 3.9]). (Remark that the irrational rotation algebras are examples of such kind of $C^*$-algebras $A$ ([Bl]).)

Now, by Takai duality ([T]), we have that

$$A \rtimes \widehat{\alpha} \cong B \otimes K$$

where $\alpha : T = \hat{\mathbb{Z}} \to \text{Aut}(A)$ is the dual action ([T]). Our claim is that $B \otimes K$ does not have the ideal property and this fact together with the above isomorphism will imply that $A \rtimes T$ does not have the ideal property either. Indeed, let $\phi \neq F = \mathcal{F} \subseteq X$ and let $I := \{f \in B \otimes K = C(X, K) \mid f |_F = 0\}$. Then, clearly $I \alpha B \otimes K$ and $P(I) = \{0\}$, since $X$ is connected, but $I$ is not generated by its projections (because, obviously, $I \neq \{0\}$). Hence, $B \otimes K$ does not have the ideal property. The proof of the theorem is over if we observe that since $T$ is a compact, abelian group, the separable group $G = T$ has the ideal property by Proposition 2.7.

**Acknowledgments**

This material is based upon work supported by, or in part by, the U.S. Army Research Office under grant number DAAD19-00-1-0152. This research was also partially supported by NSF grant DMS-0101060.

**References**


License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use


Department of Mathematics and Computer Science, University of Puerto Rico, Box 23355, San Juan, Puerto Rico 00931-3355

E-mail address: cpasnic@upracd.upr.clu.edu