LOCAL RADIAL PHRAGMÉN-LINDELÖF ESTIMATES FOR PLURISUBHARMONIC FUNCTIONS ON ANALYTIC VARIETIES

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Abstract. We give a sufficient condition for a local radial Phragmén-Lindelöf principle on analytic varieties. This condition is expressed in terms of existence of hyperbolic directions.

Introduction 1. In a basic paper Hörmander [12] characterized when a given linear partial differential operator \( P(D) \) with constant coefficients is surjective on the space \( A(\Omega) \) of all real-analytic functions on an open convex subset \( \Omega \) of \( \mathbb{R}^n \). His characterization was given in terms of global and also of local conditions of Phragmén-Lindelöf type for plurisubharmonic functions on the zero variety of the symbol \( P \). Since then, it was shown in a number of papers that similar Phragmén-Lindelöf conditions on algebraic varieties can be used to characterize other properties of (systems of) such operators (see, e.g., Andreotti and Nacinovich [1], Boiti and Nacinovich [3], Braun, Meise, and Vogt [9], Franken and Meise [11], Kaneko [13], Meise, Taylor, and Vogt [15], Momm [18], Palamodov [20], Zampieri [24]).

This work motivates the challenging complex analysis problem of characterizing geometrically the varieties for which such Phragmén-Lindelöf estimates are valid. The present authors, along with D. Vogt, have studied this question in [4], [5], [6], [8], [9], [14], [15], [16], [17]. The main result of this paper, Theorem 10, gives a local geometric condition on an analytic variety near a real point \( \xi \) which guarantees that any plurisubharmonic function \( u \) on the variety that vanishes on its real points can grow only linearly, \( u(z) = O(|z - \xi|) \), near \( \xi \). The geometric condition, which is described in Definition 9, is expressed in terms of hyperbolicity and is the local analog of a global version given in [5]. Unfortunately, the condition is not necessary, as we will show in Example 14. As in the global case, the proof of the theorem is based on a result of Sibony-Wong type for homogeneous algebraic varieties. We were led to Theorem 10 because it is a key result from pluripotential theory that is needed in our recent characterization in [8] of those surfaces in \( \mathbb{C}^3 \) that satisfy the local Phragmén-Lindelöf condition. This characterization is applied in [8] to extend Hörmander’s characterization of the surjective \( P(D) \) on \( A(\mathbb{R}^n) \) from \( n = 3 \) to \( n = 4 \).
To formulate the results clearly, we need some preparation:

**Notation 2.** Throughout the paper, $|·|$ denotes the Euclidean norm on $\mathbb{C}^n$. For $\xi \in \mathbb{C}^n$ and $r > 0$ we let

$$B(\xi, r) := \{ z \in \mathbb{C}^n : |z - \xi| < r \},$$

and for $\xi \in \mathbb{R}^n$, $|\xi| = 1$, $\epsilon > 0$, and a zero neighborhood $D \subset B(0, 1)$ we define the truncated cone $\Gamma(\xi, D, \epsilon)$ with profile $D$ by

$$\Gamma(\xi, D, \epsilon) := \bigcup_{0 < t < \epsilon} t(\xi + D).$$

**Definition 3.** (a) An analytic variety $V$ in an open set $G$ in $\mathbb{C}^n$ is defined to be a closed analytic subset of points of $V$. It is no restriction to assume $V$ a subvariety of $G$ and let $\Omega$ be an open subset of $V$. A function $u : \Omega \to [-\infty, \infty]$ is called plurisubharmonic if it is locally bounded above, plurisubharmonic in the usual sense on $\Omega_{reg}$, the set of all regular points of $V$ in $\Omega$, and satisfies

$$u(z) = \limsup_{\zeta \in \Omega_{reg}, \zeta \to z} u(\zeta)$$

at the singular points of $V$ in $\Omega$. By $\text{PSH}(\Omega)$ we denote the set of all plurisubharmonic functions on $\Omega$.

It is easy to check that the following definition is equivalent to the one given in Meise, Taylor, and Vogt [10], 2.3 (see Lemma 7 below).

**Definition 4.** Let $V$ be an analytic variety in some ball $B(\xi, r)$ for $\xi \in V \cap \mathbb{R}^n$ and $r > 0$. We say that $V$ satisfies the condition $\text{RPL}_{\text{loc}}(\xi)$ if the following holds:

There exist $A > 0$ and $0 < r_2 \leq r_1 \leq r$ such that each plurisubharmonic function $u$ on $V \cap B(\xi, r_1)$ which satisfies

(a) $u(z) \leq 1, \quad z \in V \cap B(\xi, r_1)$, and

(b) $u(z) \leq 0, \quad z \in V \cap B(\xi, r_2) \cap \mathbb{R}^n$,

already satisfies

(c) $u(z) \leq A|z - \xi|, \quad z \in V \cap B(\xi, r_1)$.

**Definition 5.** Let $V \subset \mathbb{C}^n$ be an analytic variety in some ball $B(p, r)$, $p \in \mathbb{C}^n$, $r > 0$. Let $T_pV$ denote the tangent cone to $V$ at $p$ in the sense of Whitney [23], 7.1G. To describe $T_pV$ in an equivalent way, let $f$ be analytic in some neighborhood of a point $p$. Then the localization $f_p$ of $f$ at the point $p$ is defined as the lowest degree homogeneous polynomial in the Taylor series expansion of $f$ at $p$ which does not vanish. With this notation we have

$$T_pV = \{ z \in \mathbb{C}^n : f_p(z) = 0 \text{ for all } f, \text{ analytic near } p \text{ and vanishing on } V \},$$

by Whitney [23], 7.4D.

**Proposition 6.** Let $V$ be an analytic variety in some open set in $\mathbb{C}^n$. If $V$ satisfies $\text{RPL}_{\text{loc}}(\xi)$ for some $\xi \in V \cap \mathbb{R}^n$, then $T_0V$ satisfies $\text{RPL}_{\text{loc}}(0)$.

**Proof.** It is no restriction to assume $\xi = 0$. We may also assume that $V$ is a subvariety of $B(0, 1)$ and that $\text{RPL}_{\text{loc}}(0)$ holds for the parameters $A > 0, \ 0 < r \leq 1 = r_1$. Then define the following varieties $V_j$ in $B(0, 1)$ for $j \in \mathbb{N}$:

$$V_0 := T_0V \cap B(0, 1), \quad V_j := \{ z \in B(0, 1) : z/j \in V \} = j(V \cap B(0, 1/j)), \quad j \in \mathbb{N}.$$
Furthermore, define, for $j \in \mathbb{N}_0$, the extremal functions $U_j : V_j \to \mathbb{R}$ as in Meise, Taylor, and Vogt [16], 4.1:

$$U_j(z) := \sup \{ u(z) : u \in \text{PSH}(V_j), \; u(z) \leq 1 \text{ for } z \in V_j, \; u(z) \leq 0 \text{ for } z \in V_j \cap \mathbb{R}^n \cap B(0, 1/2) \}.$$ 

Then $\lim_{j \to \infty} V_j = T_0 V$, either as currents on $\mathbb{C}^n$ (see Chirka [10], 16.1, Proposition 2) or in the sense of Meise, Taylor, and Vogt [16], 4.3. By 4.4 of [16], if $z_j \in V_j$ and $\lim_{j \to \infty} z_j =: z \in V_0$, then

$$u_0(z) \leq \liminf_{j \to \infty} U_j(z_j).$$

Consequently, the proposition is proved once we show

(\ast) There exists $B > 0$ such that $U_j(z) \leq B |z|$ for each $j \in \mathbb{N}$ and each $z \in V_j$.

To prove (\ast) we define (as in [16], 2.9)

$$H : \mathbb{C}^n \to \mathbb{R}, \quad H(z) = \frac{1}{2}(|\text{Im } z|^2 - |\text{Re } z|^2),$$

and note that $H$ is pluriharmonic on $\mathbb{C}^n$. Next fix $j \in \mathbb{N}$ and $u \in \text{PSH}(V_j)$ satisfying

$$u(z) \leq 1 \text{ for } z \in V_j \quad \text{and} \quad u(z) \leq 0 \text{ for } z \in V_j \cap \mathbb{R}^n \cap B(0, 1/2)$$

and define $w : V \to \mathbb{R}$ by

$$w(z) := \begin{cases} \max(\frac{1}{j} u(jz) + \frac{3}{j} H(jz), 3|\text{Im } z|), & \text{if } |z| < \frac{1}{2j}, \\ 3|\text{Im } z|, & \text{if } |z| \geq \frac{1}{2j}. \end{cases}$$

For $z \in V$ with $|z| = \frac{1}{2j}$ we have

$$\frac{1}{j} u(jz) + \frac{3}{j} H(jz) \leq \frac{1}{j} + \frac{3}{j} (|\text{Im } jz| - \frac{1}{2}) \leq 3|\text{Im } z| - \frac{1}{2j},$$

This implies $w \in \text{PSH}(V)$. Moreover, the maximum principle gives $w(z) \leq 3$ for $z \in V$. The hypotheses on $u$, the definition of $w$ and $H$, $\|w\| \leq 0$ imply $w(z) \leq 0$ for $z \in V \cap \mathbb{R}^n$. Since $V$ satisfies RPL$_{loc}(0)$ for the parameters $A > 0$ and $0 < r \leq 1$, we obtain

$$w(z) \leq 3A|z|, \quad z \in V.$$

By the definition of $w$ and the definition of $H$ we now get for $z \in V$, $|z| < \frac{1}{2j}$:

$$3A|z| \geq w(z) \geq \frac{1}{j} u(jz) + \frac{3}{j} H(jz) \geq \frac{1}{j} u(jz) - \frac{3}{2j} |jz|^2.$$

This implies

$$u(jz) \leq (3A + \frac{3}{2} |jz|)|jz| \leq (3A + \frac{3}{4})|jz|$$

and hence

$$(1) \quad u(\zeta) \leq 3(A + 1)|\zeta|, \quad \zeta \in V_j \cap B(0, 1/2).$$

Moreover, for $\zeta \in V_j$ satisfying $1/2 \leq |\zeta| < 1$ the hypotheses on $u$ imply $u(\zeta) \leq 1 \leq 2|\zeta|$. Hence (1) holds on $V_j$. Condition (\ast) follows because $U_j(z)$ is the upper envelope of all such $u(z)$. This completes the proof. \qed
Let $B$ be a real cone at $p$. Then $V$ satisfies $RPL_{\text{loc}}(\xi)$ if and only if the following condition is satisfied:

For all choices $0 < r_2 \leq r_1 \leq r$ there exists $A > 0$ such that each $u \in \text{PSH}(V \cap B(\xi, r_1))$ which satisfies

$(\alpha)$ $u(z) \leq 1$ for $z \in V \cap B(\xi, r_1),$
$(\beta)$ $u(z) \leq 0$ for $z \in V \cap \mathbb{R}^n \cap B(\xi, r_2),$
also satisfies

$(\gamma)$ $u(z) \leq A|z - \xi|$ for $z \in V \cap B(\xi, r_1).$

The converse implication in Proposition 6 does not hold, as the following example shows.

Example 8. The variety $V := \{(x, y) \in \mathbb{C}^2 : x + iy^2 = 0\}$ does not satisfy $RPL_{\text{loc}}(0)$, although $T_0 V$ satisfies $RPL_{\text{loc}}(0)$. The last statement follows easily from a classical result (see Nevanlinna [19], 38) since $T_0 V = \{(x, y) \in \mathbb{C}^2 : x = 0\}$.

To see that $V$ does not satisfy $RPL_{\text{loc}}(0)$, note that by Meise, Taylor, and Vogt [10], 2.8, a necessary condition for $V$ to satisfy $RPL_{\text{loc}}(0)$ is that $V$ satisfies the dimension condition at 0. By [10], 2.6, this means that $V \cap \mathbb{R}^2$ has to have real dimension 1. Since $V \cap \mathbb{R}^2 = \{0\}$, this is not the case.

Because of Proposition 6 and Example 8 it appears interesting to know conditions which imply that a given variety $V$ inherits $RPL_{\text{loc}}$ at $\xi \in V \cap \mathbb{R}^n$ from its tangent cone $T_{\xi}V$. As we indicated in the Introduction, we also need to know such a condition in our paper [8]. To present a sufficient condition which covers the intended applications, we have to introduce some more notation.

Definition 9. (a) Let $V$ be an analytic variety in $\mathbb{C}^n$ of pure dimension $k \geq 1$, $p \in V$ and $\pi : \mathbb{C}^n \to \mathbb{C}^n$ a projection map. We say that $\pi$ is a noncharacteristic projection for $V$ at $p$ if $\text{im} \pi$ and $\ker \pi$ are spanned by real vectors, rank $\pi = k$, and $T_p V \cap \ker \pi = \{0\}$.

(b) Let $V$ be an analytic variety in some open set in $\mathbb{C}^n$ and $p \in V \cap \mathbb{R}^n$. $V$ is said to be 1-hyperbolic at $p$ with respect to $\xi \in T_p V \cap \mathbb{R}^n$, $\xi \neq 0$, if there exist a cone $\Gamma = \Gamma(\xi, D, \epsilon)$ and a noncharacteristic projection for $V$ at $p$ such that $\pi : (V - p) \cap \Gamma \to \pi((V - p) \cap \Gamma)$ is proper and $z \in (V - p) \cap \Gamma$ is real whenever $\pi(z)$ is real.

The expression “1-hyperbolicity” stems from our paper [8], where the more general concept of “d-hyperbolicity” is used.

Now we can formulate the main result of the present paper:

Theorem 10. Let $V$ be an analytic variety of pure dimension $k \geq 1$ in some ball $B(\xi, r)$ in $\mathbb{C}^n$, where $\xi \in V \cap \mathbb{R}^n$. Assume that for each irreducible component $W$ of $T_{\xi}V$ there is $\eta \in W \cap \mathbb{R}^n$ such that $V$ is 1-hyperbolic at $\xi$ with respect to $\eta$ and to $-\eta$. Then $V$ satisfies $RPL_{\text{loc}}(\xi)$.

The proof of Theorem 10 is based on a result of Sibony-Wong type (compare Sibony and Wong [21] and Siciak [22] for the original result) which we recall from [5], 3.1, for the convenience of the reader.
Theorem 11. Let $W$ be a homogeneous algebraic variety of pure dimension $k$ in $\mathbb{C}^n$ and let $\pi: W \to \mathbb{C}^k$, $\pi(z', z'') = z'$, be proper. Let $\Gamma \subset \mathbb{C}^k$ be an open complex cone, let $\tilde{\Gamma} := \pi^{-1}(\Gamma)$, and let $R \subset \Gamma$ be a complex cone which is nonpluripolar in $\Gamma$. Then there exists a constant $A \geq 1$ such that for each $u \in \text{PSH}(\tilde{\Gamma})$ which is positively homogeneous and satisfies

\begin{equation}
\begin{aligned}
& \text{(a)} \quad u(z) \leq |z| \text{ for } z \in R, \\
& \text{(b)} \quad v(z') := \max\{u(z', z'') : (z', z'') \in \Gamma\} \text{ for } z' \in \Gamma', \text{ extends to a plurisubharmonic function } \tilde{v} \text{ on } \mathbb{C}^k, \\
& \text{each extension } \tilde{v} \text{ satisfies} \\
& \text{(c)} \quad \tilde{v}(z') \leq A|z'|, \quad z' \in \mathbb{C}^k.
\end{aligned}
\end{equation}

In particular, $u$ satisfies $u(z) \leq A|z|$ for all $z \in W$.

The following geometric preparation for the proof of Theorem 12 might also be of some interest by itself. While in Definition 9 it appears that 1-hyperbolicity depends on a special projection direction, it in fact does not—there are always nearby points at which $V$ is 1-hyperbolic with respect to every real noncharacteristic projection.

Proposition 12. Let $V$ be a purely $k$-dimensional analytic variety in some open neighborhood of the origin in $\mathbb{C}^n$, and let $\xi_0 \in T_0V \cap \mathbb{R}^n$ with $|\xi_0| = 1$ be given. If $V$ is 1-hyperbolic at the origin with respect to $\xi_0$ and to $-\xi_0$ and $U$ is an arbitrary neighborhood of $\xi_0$, then there is $\xi \in T_0V \cap U \cap \mathbb{R}^n$ which is a regular point of $T_0V$ and has the following property:

For each projection $\pi$ which is noncharacteristic for $V$ at 0 and which satisfies $T_0(\xi_0) \cap \ker \pi = \{0\}$ there are $\epsilon > 0$ and a profile $C$ such that for $\sigma = \pm 1$ the cone $\Gamma := \Gamma(\sigma \xi, C, \epsilon)$ is such that $\pi: V \cap \Gamma \to \pi(V \cap \Gamma)$ is proper and $z \in V \cap \Gamma$ is real whenever $\pi(z)$ is real.

Proof. Since $V$ is 1-hyperbolic at 0 with respect to $\xi_0$ and to $-\xi_0$, there are a noncharacteristic projection $\pi_0$, a constant $\epsilon_0 > 0$, and a profile $C_0$ such that, for the cones $\Gamma_0^\pm := \Gamma(\pm \xi_0, C_0, \epsilon_0)$, the projection $\pi_0: V \cap \Gamma_0^\pm \to \pi_0(V \cap \Gamma_0^\pm)$ is proper and $z \in V \cap \Gamma_0^\pm$ is real whenever $\pi_0(z)$ is real. Since $\pi_0$ is noncharacteristic, $\xi_0 \notin \ker \pi_0$. We may assume $\xi_0 = (0, \ldots, 0, 1)$.

Since $\pi_0$ is noncharacteristic, it induces a description of $T_0V$ as analytic cover above a neighborhood of 0 in $\pi_0$. Since branching occurs only over a proper analytic subset, there is $\xi \in T_0\mathbb{V} \cap U \cap \mathbb{R}^n$ such that $\pi_0: T_0V \to \im \pi_0$ is unbranched in a suitable neighborhood of $\xi$, in particular, $\xi$ is a regular point of $T_0V$. Then there are a cone $\Gamma_1 = \Gamma(\xi, C_1, \epsilon_1) \subset \Gamma_0^+$ and a holomorphic map $g: \pi_0(\Gamma_1) \to \ker \pi_0$ such that

\begin{equation}
T_0V \cap \Gamma_1 = \{v + g(v) : v \in \pi_0(\Gamma_1)\}.
\end{equation}

Let $C_2 \subset C_1$ be relatively compact, and set $\Gamma_2 := \Gamma(\xi, C_2, \epsilon_1)$. We claim that $u \in T_0V \cap \Gamma_2$ is real whenever $\pi_0(u)$ is real. To see this, fix $w \in T_0V \cap \Gamma_2$ with $\pi_0(w)$ real. Near the origin, the projection $\pi_0$ induces a description of $V$ as an analytic cover above some neighborhood of 0 in $\pi_0$. Since $T_0V$ is the tangent cone, at least one sheet of $V$ must eventually come close to $T_0V \cap \Gamma_2$. Hence, for sufficiently large $j \in \mathbb{N}$, there is $z_j \in V \cap \Gamma_2$ with $\pi_0(z_j) = \pi_0(w)/j$. Then $z_j \in \mathbb{R}^n$ because $\Gamma_2 \subset \Gamma_0^+$. Note that there is $\eta > 0$ such that

\begin{equation}
\eta|\pi_0(z)| \leq |z| \leq \frac{1}{\eta}|\pi_0(z)| \quad \text{for all } z \in \Gamma_0^+.
\end{equation}
Hence the points \( jz \) are all in \( \Gamma_2 \) and satisfy a common upper as well as a common lower bound, where the latter is strictly positive. By compactness, this means that the sequence \((jz_j)_{j \in \mathbb{N}}\) has a subsequence which converges to some \( w_1 \in T_0 V \setminus \{0\} \subset \Gamma_1 \). By definition, we have \( w_1 \in T_0 V \) and \( \pi_0(w_1) = \pi_0(w) \). Since \( T_0 V \cap \Gamma_1 \) is a graph by \( (2) \), this shows \( w = w_1 \in \mathbb{R}^n \).

Fix now \( \pi \) as in the hypothesis, then \( T_\xi(T_0 V) \) is a tangent hyperplane which has trivial intersection with \( \ker \pi \). This implies that near \( \xi \), \( T_0 V \) is a graph over \( \pi \), i.e., there are a cone \( \Gamma_3 = \Gamma(\xi, C_3, \epsilon_3) \) with \( 0 \in C_3 \subset C_2 \) and a holomorphic map \( h: \pi(\Gamma_3) \to \ker \pi \) such that

\[
T_0 V \cap \Gamma_3 = \{ v + h(v) : v \in \pi(\Gamma_3) \}.
\]

It is easy to see that, since \( g \) is real valued for real arguments, so is \( h \). Fix a cone \( \Gamma_4 = \Gamma(\pi_0(\xi), C_4, \epsilon_4) \subset \pi_0(\Gamma_2) \) such that \( z \in \Gamma_3 \) whenever \( z \in V \cap \Gamma_2 \) satisfies \( \pi_0(z) \in \Gamma_4 \). Let \( C_5 \) be a neighborhood of 0, relatively compact in \( C_4 \), and set \( \Gamma_5 = \Gamma(\pi_0(\xi), C_5, \epsilon_4/2) \). It follows from classical estimates of potential theory (e.g., Nevanlinna [19], 38) that there is a constant \( A > 0 \) such that the following holds:

Whenever \( u: \Gamma_4 \to [-\infty, \infty] \) is plurisubharmonic and satisfies \( u(\zeta) \leq |\zeta| \) for each \( \zeta \in \Gamma_4 \), and \( u(\zeta) \leq 0 \) for each \( \zeta \in \Gamma_4 \cap \mathbb{R}^n \), then \( u(\zeta) \leq A|\Im \zeta| \) for each \( \zeta \in \Gamma_5 \).

Next note that we can find \( \delta > 0 \) such that

\[
|\Im \pi_0(z)| = |\pi_0(\Im z)| \leq \frac{1}{\delta}|\Im z|
\]

for all \( z \in \mathbb{C}^n \) and define \( u: \Gamma_4 \to [-\infty, \infty] \) by

\[
u(\zeta) := (A/\delta + 1) \max\{|\Im(z - \pi(z) - h(\pi(z)))| : z \in V \cap \Gamma_3, \pi_0(z) = \zeta\}.
\]

Note that \(|z - \pi(z) - h(\pi(z))| = o(|z|)\) for \( z \in V \cap \Gamma_3 \), hence \( u(\zeta) \leq |\zeta| \) if we assume that \( \epsilon_4 \) is sufficiently small. If \( \zeta \) is real, then so is each \( z \in V \cap \Gamma_3 \) with \( \pi_0(z) = \zeta \), hence \( u(\zeta) \leq 0 \) in that case. By the above, we find \( u(\zeta) \leq A|\Im \zeta| \) for \( \zeta \in \Gamma_5 \). Moreover, if \( z \in V \cap \Gamma_3 \), \( \pi(z) \) is real, and \( \pi_0(z) \in \Gamma_5 \), then

\[
A|\Im \pi_0(z)| \geq u(\pi_0(z)) \geq (A/\delta + 1)|\Im z| \geq \delta(A/\delta + 1)|\Im \pi_0(z)| = (A + \delta)|\Im \pi_0(z)|.
\]

We have shown \((A + \delta)|\Im \pi_0(z)| \leq A|\Im \pi_0(z)|\). This implies that \( \pi_0(z) \) is real, hence also \( z \) is real.

The same arguments apply to the opposite cone \( \Gamma_0^- \).

\begin{proof}
(This proof follows very closely the one of [5], Theorem 5.1.) Contrary to the convention in the remainder of the paper, we will work here with the norm \(|z| := \max_{j=1,...,n}|z_j|\). We may assume \( \zeta = 0 \). Let \( W_1, \ldots, W_q \) denote the irreducible components of \( T_0 V \). By hypothesis, there exist \( \xi_j \in W_j \cap (\mathbb{R}^n \setminus \{0\}) \) so that \( V \) is 1-hyperbolic with respect to \( \xi_j \) and to \(-\xi_j \), \( j = 1, \ldots, q \). It follows from Proposition [12] that, if we perturb the \( \xi_j \) a little if necessary, then there is a common noncharacteristic projection \( \pi \) that works for all \( \pm \xi_j \), \( j = 1, \ldots, q \). We assume it to be \( \pi: (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_k) \) where \( k := \dim V \).

Let \( D \in \mathbb{C}[z_1, \ldots, z_k] \), homogeneous of some degree \( d \), be chosen so that \( T_0 V \) is unbranched over \( \{ w \in \mathbb{C}^k \mid D(w) \neq 0 \} \). (See, e.g., Whitney [23], I.8E, or Chirka

\end{proof}
For $0 < \eta < 1$ let

$$S(\eta) := \{ z \in \mathbb{C}^n \mid |D(z_1, \ldots, z_k)| \leq \eta |z_1, \ldots, z_k|^d \}.$$ 

Since it is possible to perturb the $\xi_j$ a little and still keep the same projection $\pi$, we may assume $D(\xi_j) \neq 0$ and hence $\xi_j \notin S(\eta)$ if $\eta$ is chosen sufficiently small. Pick $\delta > 0$ so small that

$$S(\eta) \cap \bigcup_{j=1}^q \Gamma(\sigma \xi_j, B(0, \delta), \delta) = \emptyset$$

and

(3) if $z \in V \cap \Gamma(\pm \xi_j, B(0, \delta), \delta)$ and $\pi(z)$ is real, then $z$ is real.

Since $T_0 V$ is unbranched over $\pi(S(\eta))$, it is not difficult to see that for sufficiently small positive $\delta_1 < \delta$ the projection $\pi$ induces an analytic map

$$\hat{\pi} : V \cap B(0, \delta_1) \setminus S(\eta) \to T_0 V$$

such that for $z \in V$ the point $\hat{\pi}(z)$ is the one in $T_0 V$ satisfying $\pi(z) = \pi \circ \hat{\pi}(z)$ which is nearest to $z$. Note that if $\eta > 0$ is small enough, then the sets $W_j \setminus S(\eta)$ are connected manifolds.

Fix such a number $\eta$ and set

(4) $R(j) := \mathbb{C} \cdot (\Gamma(\xi_j, B(0, \delta), \delta) \cap W_j \cap \mathbb{R}^n)$ and $\mathcal{R} := \bigcup_{j=1}^q R(j)$.

Since each $W_j \setminus S(\eta)$ is connected, it is not difficult to see that $\mathcal{R}$ is nonpluripolar in $T_0 V \cap B(0, r) \setminus S(\eta)$. Now let $u \in \text{PSH}(V)$ satisfy

$$u(z) \leq 1, \quad z \in V, \quad \text{and} \quad u(z) \leq 0, \quad z \in V \cap \mathbb{R}^n.$$ 

In order to show that for some constant $A_0 > 0$, which does not depend on $u$, the estimate

$$u(z) \leq A_0 |z|, \quad z \in V,$$

holds, we will apply Theorem 11 with $W := T_0 V$, $\Gamma := T_0 V \setminus S(\eta)$, and $\Gamma' := \mathbb{C}^k \setminus \pi(S(\eta))$ to suitable functions, derived from the given function $u$. To define them, fix a small number $\alpha > 0$ and define the function $u_0$ on $T_0 V \cap B(0, \delta_1) \setminus S(\eta)$ by

$$u_0(w) := \max\{u(z) \mid z \in V, \hat{\pi}(z) = w\}$$

and a function $u_1$ on $T_0 V \setminus S(\eta)$ by

$$u_1(w) := \begin{cases} \max(0, u_0(w) - \alpha, 3 + 2(\log|w| - \log \delta_1)), & \text{if } |w| < \delta_1, \\ 3 + 2(\log|w| - \log \delta_1), & \text{otherwise}. \end{cases}$$

It follows from Hörmander [12, 4.4], that the singularities at the points where the maximum moves from one branch to another are removable, i.e., that $u_0$ is plurisubharmonic. This argument together with the fact that $u_0(w) \leq 1 < 3 + 2(\log|w| - \log \delta_1)$ whenever $\delta_1/e < |w| < \delta_1$ also shows that $u_1$ is plurisubharmonic.
We define \( v_0 \) on \( B(0, \delta_1) \subset \mathbb{C}^k \) and \( v_1 \) on \( \mathbb{C}^k \) by
\[
 v_0(w) := \max(u(z) \mid z \in V, \pi(z) = w), \quad |w| < \delta_1,
\]
\[
 v_1(w) := \begin{cases} 
 \max(0, v_0(w) - a, 3 + 2(\log|w| - \log \delta_1)), & \text{if } |w| < \delta_1, \\
 3 + 2(\log|w| - \log \delta_1)), & \text{otherwise.}
\end{cases}
\]

The functions \( v_0 \) and \( v_1 \) are plurisubharmonic.

If \( x \in W_j \cap \Gamma(\zeta, B(0, \delta), \delta_1) \cap \mathbb{R}^n \) for some \( j \) and \( |x| > \delta_1/2 \), then \( u_0(\zeta x) \leq 1 \) for all \( \zeta \in \mathbb{C} \) with \( |\zeta| \leq 1 \) and \( u_0(\zeta x) \leq 0 \) for \( -1 \leq \zeta \leq 1 \). By a standard potential theory argument (see, e.g., Nevanlinna [19, 38], there is a constant \( C > 0 \), not depending on anything, such that \( u_0(\zeta x) \leq C|\zeta| \) provided \( |\zeta| \leq 1/2 \). We have shown
\[
 u_0(w) \leq \frac{2C}{\delta_1}|w|, \quad w \in \mathbb{R} \cap B(0, \delta_1/2).
\]

Next define \( \phi : T_0V \setminus S(\eta) \to [-\infty, \infty] \) by
\[
 \phi(z) := \sup_{\zeta \in \mathbb{C}} \frac{u_1(\zeta z)}{|\zeta|}, \quad z \in T_0V \setminus S(\eta).
\]

Then \( \phi \) is positive homogeneous. Since \( u_1 \) is plurisubharmonic, the upper regularization \( \phi^* \) is plurisubharmonic and positive homogeneous. Let \( z \in \mathbb{R} \). If \( z \in T_0V \cap B(0, \delta_1) \) satisfies \( |z| \geq \delta_1/2 \), then \( u_0(z) \leq 1 \leq (2/\delta_1)|z| \). Since \( 3 + 2(\log|z| - \log \delta_1) \leq 2\sqrt{|z|} \), these considerations and (8) show
\[
 \phi(z) \leq M|z| \quad \text{for } z \in \mathbb{R} \quad \text{where } M := \max(2C/\delta_1, 2\sqrt{e}/\delta_1).
\]

Since each \( z \in \mathbb{R} \) with \( z \neq 0 \) is a regular point of \( T_0V \), the same estimate holds for \( \phi^* \). Now define similarly to \( \phi \) the function \( \psi : \mathbb{C}^k \to [-\infty, \infty] \) by
\[
 \psi(w) := \sup_{\zeta \in \mathbb{C}} \frac{v_1(\zeta w)}{|\zeta|}, \quad w \in \mathbb{C}^k.
\]

\( \psi \) is an extension of \( w \mapsto \max\{\phi(z) \mid z \in T_0V \setminus S(\eta), \pi(z) = w\} \) and consequently \( \psi^* \) extends \( w \mapsto \max\{\phi^*(z) \mid z \in T_0V \setminus S(\eta), \pi(z) = w\} \). Therefore we can apply Theorem [11] with \( W = T_0V \), \( \Gamma = T_0V \setminus S(\eta) \), \( \Gamma' = \mathbb{C}^k \setminus \pi(S(\eta)) \), \( u = \phi^* \), and \( v = \psi^* \). It gives the existence of \( A \geq 1 \) such that
\[
 \psi^*(w) \leq A|w| \quad \text{for all } w \in \mathbb{C}^k.
\]

If \( \psi^* \) is expressed in terms of \( u \), this estimate implies \( \RPL_{\text{loc}}(0) \) if \( a \) tends to 0. \( \square \)

**Corollary 13.** Let \( f \) be a holomorphic function, defined on the ball \( B(0, r) \subset \mathbb{C}^n \) for some \( r > 0 \) which satisfies the following conditions:

(a) \( f \) is real on \( B(0, r) \cap \mathbb{R}^n \).

(b) the localization \( f_0 \) of \( f \) at zero is squarefree and has positive degree.

(c) for each irreducible factor \( q \) of \( f_0 \), the real zero set of \( q \) has dimension \( n-1 \) at zero.

Then \( V = \{ z \in B(0, r) : f(z) = 0 \} \) satisfies \( \RPL_{\text{loc}}(0) \).

**Proof.** The hypotheses imply that \( f \) can be decomposed as \( f = f_0 + g \), where \( f_0 \) is a homogeneous polynomial of degree \( m > 0 \) and where \( g \) is holomorphic on \( B(0, r) \) and satisfies for some constant \( C > 0 \) the estimate
\[
 |g(z)| \leq C|z|^{m+1}, \quad z \in B(0, r/2).
\]
Note that $f_0$ has real coefficients. Then the hypotheses imply $f_0 = \prod_{j=1}^{s} P_j$, where $P_1, \ldots, P_s$ are irreducible homogeneous polynomials with real coefficients which are pairwise not proportional. By Whitney [23], 7.4D (see Definition 5), we have

$$T_0 V = \{ z \in \mathbb{C}^n : f_0(z) = 0 \} = \bigcup_{j=1}^{s} W_j,$$

where the varieties

$$W_j := \{ z \in \mathbb{C}^n : P_j(z) = 0 \}, \quad 1 \leq j \leq s,$$

are the irreducible components of $T_0 V$.

Next note that the hypotheses on $f_0$ imply that for $1 \leq j \leq s$ we can choose $\xi_j \in W \cap B(0, r/2) \cap \mathbb{R}^n$ such that $\text{grad } P_j(\xi_j) \neq 0$ and $\xi_j \notin W_k$ for $k \neq j$. Then we have

$$\text{grad } f_0(\xi_j) = \left( \prod_{k \neq j}^{s} P_k(\xi_j) \right) \text{grad } P_j(\xi_j) \neq 0, \quad 1 \leq j \leq s.$$ 

For fixed $j$, we may assume that $\frac{\partial f_0}{\partial z_n}(\xi_j) \neq 0$ and write $z = (z', z_n)$ for $z \in \mathbb{C}^n$. Then an application of the real and the complex implicit function theorem implies that for suitable numbers $\delta_1, \delta_2 > 0$ the variety $W_j$ in a neighborhood of $\xi_j$ is the graph of a holomorphic function $h_j : B(\xi_j', \delta_1) \rightarrow B(\xi_j, \delta_2)$ which is real over $B(\xi_j', \delta_1) \cap \mathbb{R}^{n-1}$. Of course, we may choose $\delta_1$ and $\delta_2$ so small such that $\frac{\partial h_j}{\partial z_n}(z) \neq 0$ for $z \in B(\xi_j, \delta) \times B(\xi_j, n, \delta)$, where $\delta := \min(\delta_1, \delta_2)$. Since $\xi_j$ is a simple zero of $\lambda \mapsto f_0(\xi_j', \lambda)$, we can choose $0 < \rho < \delta/4$ so that $|f_0(\xi_j', \lambda)| > 0$ for $0 < |\lambda - \xi_j, n| \leq \rho$. Thus

$$a := \inf \{ |f_0(\xi_j', z_n)| : |z_n - \xi_j, n| = \rho \} > 0.$$ 

Hence we can choose $0 < \delta_3 < \delta/2$ such that

$$\inf \{ |f_0(\xi', \lambda)| : |\xi' - \xi_j| \leq \delta_3, |z_n - \xi_j, n| = \rho \} \geq \frac{a}{2}.$$ 

Now note that for $0 < t \leq 1$, $|\xi' - \xi_j| \leq \delta_3$, and $|z_n - \xi_j, n| = \rho$, the estimate implies

$$|g(t\xi', tz_n)| \leq C t^{m+1} |(\xi', z_n)|^{m+1} \leq C (|\xi'_j| + |\xi_j, n| + \delta)^{m+1} t^{m+1},$$

while

$$|f_0(t\xi', tz_n)| = t^m |f_0(\xi', z_n)| \geq \frac{a}{2} t^m.$$ 

These estimates show the existence of $0 < t_0 \leq 1$ such that for each $0 < t \leq t_0$ and each $\xi' \in \mathbb{C}^{n-1}$ with $|\xi' - \xi_j| < \delta_3$ we can apply Rouche’s Theorem to see that the equation $f(t\xi', z_n) = 0$ has exactly one solution $\zeta_n \in \mathbb{C}$ satisfying $|\zeta_n - t\xi_j| < t\rho$. Since $f$ is real for real arguments and the neighborhood in which the root is unique is symmetric with respect to complex conjugation, it follows that $V$ is 1-hyperbolic with respect to $\xi_j$. Since the same arguments apply to $-\xi_j$, the proof is complete. \qed
Example 14. The sufficient condition in Theorem [10] is not necessary. In fact, for
\[ P(s, w_1, w_2) := (s^2 - w_1^2)^2 - w_2(w_1^2 - w_2^2)(w_1^2 + w_2^2) \]
the variety \( V(P) \) satisfies \( \text{RPL}_{\text{loc}}(0) \), although the hypotheses of Theorem [10] are violated. This example is an adaptation of Example 4.10 of Bainbridge [2]. For the proof we refer to [7].

References

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