

## LOCAL RADIAL PHRAGMÉN-LINDELÖF ESTIMATES FOR PLURISUBHARMONIC FUNCTIONS ON ANALYTIC VARIETIES

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**ABSTRACT.** We give a sufficient condition for a local radial Phragmén-Lindelöf principle on analytic varieties. This condition is expressed in terms of existence of hyperbolic directions.

**Introduction 1.** In a basic paper Hörmander [12] characterized when a given linear partial differential operator  $P(D)$  with constant coefficients is surjective on the space  $\mathcal{A}(\Omega)$  of all real-analytic functions on an open convex subset  $\Omega$  of  $\mathbb{R}^n$ . His characterization was given in terms of global and also of local conditions of Phragmén-Lindelöf type for plurisubharmonic functions on the zero variety of the symbol  $P$ . Since then, it was shown in a number of papers that similar Phragmén-Lindelöf conditions on algebraic varieties can be used to characterize other properties of (systems of) such operators (see, e.g., Andreotti and Nacinovich [1], Boiti and Nacinovich [3], Braun, Meise, and Vogt [9], Franken and Meise [11], Kaneko [13], Meise, Taylor, and Vogt [15], Momm [18], Palamodov [20], Zampieri [24]).

This work motivates the challenging complex analysis problem of characterizing geometrically the varieties for which such Phragmén-Lindelöf estimates are valid. The present authors, along with D. Vogt, have studied this question in [4], [5], [6], [8], [9], [14], [15], [16], [17]. The main result of this paper, Theorem 10, gives a local geometric condition on an analytic variety near a real point  $\xi$  which guarantees that any plurisubharmonic function  $u$  on the variety that vanishes on its real points can grow only linearly,  $u(z) = O(|z - \xi|)$ , near  $\xi$ . The geometric condition, which is described in Definition 9, is expressed in terms of hyperbolicity and is the local analog of a global version given in [5]. Unfortunately, the condition is not necessary, as we will show in Example 14. As in the global case, the proof of the theorem is based on a result of Sibony-Wong type for homogeneous algebraic varieties. We were led to Theorem 10 because it is a key result from pluripotential theory that is needed in our recent characterization in [8] of those surfaces in  $\mathbb{C}^3$  that satisfy the local Phragmén-Lindelöf condition. This characterization is applied in [8] to extend Hörmander's characterization of the surjective  $P(D)$  on  $\mathcal{A}(\mathbb{R}^n)$  from  $n = 3$  to  $n = 4$ .

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To formulate the results clearly, we need some preparation:

**Notation 2.** Throughout the paper,  $|\cdot|$  denotes the Euclidean norm on  $\mathbb{C}^n$ . For  $\xi \in \mathbb{C}^n$  and  $r > 0$  we let

$$B(\xi, r) := \{z \in \mathbb{C}^n : |z - \xi| < r\},$$

and for  $\xi \in \mathbb{R}^n$ ,  $|\xi| = 1$ ,  $\epsilon > 0$ , and a zero neighborhood  $D \subset B(0, 1)$  we define the truncated cone  $\Gamma(\xi, D, \epsilon)$  with profile  $D$  by

$$\Gamma(\xi, D, \epsilon) := \bigcup_{0 < t < \epsilon} t(\xi + D).$$

**Definition 3.** (a) An analytic variety  $V$  in an open set  $G$  in  $\mathbb{C}^n$  is defined to be a closed analytic subset of  $G$  (see Chirka [10], 2.1).

(b) Let  $V$  be an analytic variety in some open set in  $\mathbb{C}^n$  and let  $\Omega$  be an open subset of  $V$ . A function  $u: \Omega \rightarrow [-\infty, \infty]$  is called plurisubharmonic if it is locally bounded above, plurisubharmonic in the usual sense on  $\Omega_{\text{reg}}$ , the set of all regular points of  $V$  in  $\Omega$ , and satisfies

$$u(z) = \limsup_{\zeta \in \Omega_{\text{reg}}, \zeta \rightarrow z} u(\zeta)$$

at the singular points of  $V$  in  $\Omega$ . By  $\text{PSH}(\Omega)$  we denote the set of all plurisubharmonic functions on  $\Omega$ .

It is easy to check that the following definition is equivalent to the one given in Meise, Taylor, and Vogt [16], 2.3 (see Lemma 7 below).

**Definition 4.** Let  $V$  be an analytic variety in some ball  $B(\xi, r)$  for  $\xi \in V \cap \mathbb{R}^n$  and  $r > 0$ . We say that  $V$  satisfies the condition  $\text{RPL}_{\text{loc}}(\xi)$  if the following holds:

There exist  $A > 0$  and  $0 < r_2 \leq r_1 \leq r$  such that each plurisubharmonic function  $u$  on  $V \cap B(\xi, r_1)$  which satisfies

- ( $\alpha$ )  $u(z) \leq 1$ ,  $z \in V \cap B(\xi, r_1)$ , and
- ( $\beta$ )  $u(z) \leq 0$ ,  $z \in V \cap B(\xi, r_2) \cap \mathbb{R}^n$ ,

already satisfies

- ( $\gamma$ )  $u(z) \leq A|z - \xi|$ ,  $z \in V \cap B(\xi, r_1)$ .

**Definition 5.** Let  $V \subset \mathbb{C}^n$  be an analytic variety in some ball  $B(p, r)$ ,  $p \in \mathbb{C}^n$ ,  $r > 0$ . Let  $T_p V$  denote the tangent cone to  $V$  at  $p$  in the sense of Whitney [23], 7.1G. To describe  $T_p V$  in an equivalent way, let  $f$  be analytic in some neighborhood of a point  $p$ . Then the localization  $f_p$  of  $f$  at the point  $p$  is defined as the lowest degree homogeneous polynomial in the Taylor series expansion of  $f$  at  $p$  which does not vanish. With this notation we have

$$T_p V = \{z \in \mathbb{C}^n : f_p(z) = 0 \text{ for all } f, \text{ analytic near } p \text{ and vanishing on } V\},$$

by Whitney [23], 7.4D.

**Proposition 6.** Let  $V$  be an analytic variety in some open set in  $\mathbb{C}^n$ . If  $V$  satisfies  $\text{RPL}_{\text{loc}}(\xi)$  for some  $\xi \in V \cap \mathbb{R}^n$ , then  $T_\xi V$  satisfies  $\text{RPL}_{\text{loc}}(0)$ .

*Proof.* It is no restriction to assume  $\xi = 0$ . We may also assume that  $V$  is a subvariety of  $B(0, 1)$  and that  $\text{RPL}_{\text{loc}}(0)$  holds for the parameters  $A > 0$ ,  $0 < r \leq 1 = r_1$ . Then define the following varieties  $V_j$  in  $B(0, 1)$  for  $j \in \mathbb{N}_0$ :

$$V_0 := T_0 V \cap B(0, 1), \quad V_j := \{z \in B(0, 1) : z/j \in V\} = j(V \cap B(0, 1/j)), \quad j \in \mathbb{N}.$$

Furthermore, define, for  $j \in \mathbb{N}_0$ , the extremal functions  $U_j: V_j \rightarrow \mathbb{R}$  as in Meise, Taylor, and Vogt [16], 4.1:

$$U_j(z) := \sup\{u(z) : u \in \text{PSH}(V_j), u(z) \leq 1 \text{ for } z \in V_j, \\ u(z) \leq 0 \text{ for } z \in V_j \cap \mathbb{R}^n \cap \overline{B(0, 1/2)}\}.$$

Then  $\lim_{j \rightarrow \infty} V_j = T_0V$ , either as currents on  $\mathbb{C}^n$  (see Chirka [10], 16.1, Proposition 2) or in the sense of Meise, Taylor, and Vogt [16], 4.3. By 4.4 of [16], if  $z_j \in V_j$  and  $\lim_{j \rightarrow \infty} z_j =: z \in V_0$ , then

$$U_0(z) \leq \liminf_{j \rightarrow \infty} U_j(z_j).$$

Consequently, the proposition is proved once we show

(\*) There exists  $B > 0$  such that  $U_j(z) \leq B|z|$  for each  $j \in \mathbb{N}$  and each  $z \in V_j$ .

To prove (\*) we define (as in [16], 2.9)

$$H: \mathbb{C}^n \rightarrow \mathbb{R}, \quad H(z) = \frac{1}{2}(|\text{Im } z|^2 - |\text{Re } z|^2),$$

and note that  $H$  is pluriharmonic on  $\mathbb{C}^n$ . Next fix  $j \in \mathbb{N}$  and  $u \in \text{PSH}(V_j)$  satisfying

$$u(z) \leq 1 \text{ for } z \in V_j \quad \text{and} \quad u(z) \leq 0 \text{ for } z \in V_j \cap \mathbb{R}^n \cap B(0, 1/2)$$

and define  $w: V \rightarrow \mathbb{R}$  by

$$w(z) := \begin{cases} \max(\frac{1}{j}u(jz) + \frac{3}{j}H(jz), 3|\text{Im } z|), & \text{if } |z| < \frac{1}{2j}, \\ 3|\text{Im } z|, & \text{if } |z| \geq \frac{1}{2j}. \end{cases}$$

For  $z \in V$  with  $|z| = \frac{1}{2j}$  we have

$$\frac{1}{j}u(jz) + \frac{3}{j}H(jz) \leq \frac{1}{j} + \frac{3}{j}(|\text{Im } jz| - \frac{1}{2}) \leq 3|\text{Im } z| - \frac{1}{2j}.$$

This implies  $w \in \text{PSH}(V)$ . Moreover, the maximum principle gives  $w(z) \leq 3$  for  $z \in V$ . The hypotheses on  $u$ , the definition of  $w$  and  $H|_{\mathbb{R}^n} \leq 0$  imply  $w(z) \leq 0$  for  $z \in V \cap \mathbb{R}^n$ . Since  $V$  satisfies  $\text{RPL}_{\text{loc}}(0)$  for the parameters  $A > 0$  and  $0 < r \leq 1$ , we obtain

$$w(z) \leq 3A|z|, \quad z \in V.$$

By the definition of  $w$  and the definition of  $H$  we now get for  $z \in V$ ,  $|z| < \frac{1}{2j}$ :

$$3A|z| \geq w(z) \geq \frac{1}{j}u(jz) + \frac{3}{j}H(jz) \geq \frac{1}{j}u(jz) - \frac{3}{2j}|jz|^2.$$

This implies

$$u(jz) \leq (3A + \frac{3}{2}|jz|)|jz| \leq (3A + \frac{3}{4})|jz|$$

and hence

$$(1) \quad u(\zeta) \leq 3(A + 1)|\zeta|, \quad \zeta \in V_j \cap B(0, 1/2).$$

Moreover, for  $\zeta \in V_j$  satisfying  $1/2 \leq |\zeta| < 1$  the hypotheses on  $u$  imply  $u(\zeta) \leq 1 \leq 2|\zeta|$ . Hence (1) holds on  $V_j$ . Condition (\*) follows because  $U_j(z)$  is the upper envelope of all such  $u(z)$ . This completes the proof.  $\square$

Using localization arguments as in the proof of Proposition 6, one can show that the particular choice of  $r_1$  and  $r_2$  in the definition of  $\text{RPL}_{\text{loc}}$  does not matter.

**Lemma 7.** *Let  $V$  be an analytic variety in  $B(\xi, r)$  for some  $r > 0$  and  $\xi \in V \cap \mathbb{R}^n$ . Then  $V$  satisfies  $\text{RPL}_{\text{loc}}(\xi)$  if and only if the following condition is satisfied:*

*For all choices  $0 < r_2 \leq r_1 \leq r$  there exists  $A > 0$  such that each  $u \in \text{PSH}(V \cap B(\xi, r_1))$  which satisfies*

- ( $\alpha$ )  $u(z) \leq 1$  for  $z \in V \cap B(\xi, r_1)$ ,
- ( $\beta$ )  $u(z) \leq 0$  for  $z \in V \cap \mathbb{R}^n \cap B(\xi, r_2)$ ,

*also satisfies*

- ( $\gamma$ )  $u(z) \leq A|z - \xi|$  for  $z \in V \cap B(\xi, r_1)$ .

The converse implication in Proposition 6 does not hold, as the following example shows.

**Example 8.** The variety  $V := \{(x, y) \in \mathbb{C}^2 : x + iy^2 = 0\}$  does not satisfy  $\text{RPL}_{\text{loc}}(0)$ , although  $T_0V$  satisfies  $\text{RPL}_{\text{loc}}(0)$ . The last statement follows easily from a classical result (see Nevanlinna [19], 38) since  $T_0V = \{(x, y) \in \mathbb{C}^2 : x = 0\}$ .

To see that  $V$  does not satisfy  $\text{RPL}_{\text{loc}}(0)$ , note that by Meise, Taylor, and Vogt [16], 2.8, a necessary condition for  $V$  to satisfy  $\text{RPL}_{\text{loc}}(0)$  is that  $V$  satisfies the dimension condition at 0. By [16], 2.6, this means that  $V \cap \mathbb{R}^2$  has to have real dimension 1. Since  $V \cap \mathbb{R}^2 = \{0\}$ , this is not the case.

Because of Proposition 6 and Example 8 it appears interesting to know conditions which imply that a given variety  $V$  inherits  $\text{RPL}_{\text{loc}}$  at  $\xi \in V \cap \mathbb{R}^n$  from its tangent cone  $T_\xi V$ . As we indicated in the Introduction, we also need to know such a condition in our paper [8]. To present a sufficient condition which covers the intended applications, we have to introduce some more notation.

**Definition 9.** (a) Let  $V$  be an analytic variety in  $\mathbb{C}^n$  of pure dimension  $k \geq 1$ ,  $p \in V$  and  $\pi : \mathbb{C}^n \rightarrow \mathbb{C}^n$  a projection map. We say that  $\pi$  is a noncharacteristic projection for  $V$  at  $p$  if  $\text{im } \pi$  and  $\ker \pi$  are spanned by real vectors,  $\text{rank } \pi = k$ , and  $T_p V \cap \ker \pi = \{0\}$ .

(b) Let  $V$  be an analytic variety in some open set in  $\mathbb{C}^n$  and  $p \in V \cap \mathbb{R}^n$ .  $V$  is said to be 1-hyperbolic at  $p$  with respect to  $\xi \in T_p V \cap \mathbb{R}^n$ ,  $\xi \neq 0$ , if there exist a cone  $\Gamma = \Gamma(\xi, D, \epsilon)$  and a noncharacteristic projection for  $V$  at  $p$  such that  $\pi : (V - p) \cap \Gamma \rightarrow \pi((V - p) \cap \Gamma)$  is proper and  $z \in (V - p) \cap \Gamma$  is real whenever  $\pi(z)$  is real.

The expression “1-hyperbolicity” stems from our paper [8], where the more general concept of “ $d$ -hyperbolicity” is used.

Now we can formulate the main result of the present paper:

**Theorem 10.** *Let  $V$  be an analytic variety of pure dimension  $k \geq 1$  in some ball  $B(\xi, r)$  in  $\mathbb{C}^n$ , where  $\xi \in V \cap \mathbb{R}^n$ . Assume that for each irreducible component  $W$  of  $T_\xi V$  there is  $\eta \in W \cap \mathbb{R}^n$  such that  $V$  is 1-hyperbolic at  $\xi$  with respect to  $\eta$  and to  $-\eta$ . Then  $V$  satisfies  $\text{RPL}_{\text{loc}}(\xi)$ .*

The proof of Theorem 10 is based on a result of Sibony-Wong type (compare Sibony and Wong [21] and Siciak [22] for the original result) which we recall from [5], 3.1, for the convenience of the reader.

**Theorem 11.** *Let  $W$  be a homogeneous algebraic variety of pure dimension  $k$  in  $\mathbb{C}^n$  and let  $\pi : W \rightarrow \mathbb{C}^k$ ,  $\pi(z', z'') = z'$ , be proper. Let  $\Gamma' \subset \mathbb{C}^k$  be an open complex cone, let  $\Gamma := \pi^{-1}(\Gamma')$ , and let  $R \subset \Gamma$  be a complex cone which is nonpluripolar in  $\Gamma$ . Then there exists a constant  $A \geq 1$  such that for each  $u \in \text{PSH}(\Gamma)$  which is positively homogeneous and satisfies*

- (a)  $u(z) \leq |z|$  for  $z \in R$ ,
- (b)  $v(z') := \max\{u(z', z'') : (z', z'') \in \Gamma\}$  for  $z' \in \Gamma'$ , extends to a plurisubharmonic function  $\tilde{v}$  on  $\mathbb{C}^k$ ,

each extension  $\tilde{v}$  satisfies

- (c)  $\tilde{v}(z') \leq A|z'|$ ,  $z' \in \mathbb{C}^k$ .

In particular,  $u$  satisfies  $u(z) \leq A|z|$  for all  $z \in W$ .

The following geometric preparation for the proof of Theorem 10 might also be of some interest by itself. While in Definition 9 it appears that 1-hyperbolicity depends on a special projection direction, it in fact does not—there are always nearby points at which  $V$  is 1-hyperbolic with respect to every real noncharacteristic projection.

**Proposition 12.** *Let  $V$  be a purely  $k$ -dimensional analytic variety in some open neighborhood of the origin in  $\mathbb{C}^n$ , and let  $\xi_0 \in T_0V \cap \mathbb{R}^n$  with  $|\xi_0| = 1$  be given. If  $V$  is 1-hyperbolic at the origin with respect to  $\xi_0$  and to  $-\xi_0$  and  $U$  is an arbitrary neighborhood of  $\xi_0$ , then there is  $\xi \in T_0V \cap U \cap \mathbb{R}^n$  which is a regular point of  $T_0V$  and has the following property:*

*For each projection  $\pi$  which is noncharacteristic for  $V$  at 0 and which satisfies  $T_\xi(T_0V) \cap \ker \pi = \{0\}$  there are  $\epsilon > 0$  and a profile  $C$  such that for  $\sigma = \pm 1$  the cone  $\Gamma := \Gamma(\sigma\xi, C, \epsilon)$  is such that  $\pi : V \cap \Gamma \rightarrow \pi(V \cap \Gamma)$  is proper and  $z \in V \cap \Gamma$  is real whenever  $\pi(z)$  is real.*

*Proof.* Since  $V$  is 1-hyperbolic at 0 with respect to  $\xi_0$  and to  $-\xi_0$ , there are a noncharacteristic projection  $\pi_0$ , a constant  $\epsilon_0 > 0$ , and a profile  $C_0$  such that, for the cones  $\Gamma_0^\pm := \Gamma(\pm\xi_0, C_0, \epsilon_0)$ , the projection  $\pi_0 : V \cap \Gamma_0^\pm \rightarrow \pi_0(V \cap \Gamma_0^\pm)$  is proper and  $z \in V \cap \Gamma_0^\pm$  is real whenever  $\pi_0(z)$  is real. Since  $\pi_0$  is noncharacteristic,  $\xi_0 \notin \ker \pi_0$ . We may assume  $\xi_0 = (0, \dots, 0, 1)$ .

Since  $\pi_0$  is noncharacteristic, it induces a description of  $T_0V$  as analytic cover above a neighborhood of 0 in  $\text{im } \pi_0$ . Since branching occurs only over a proper analytic subset, there is  $\xi \in T_0V \cap U \cap \mathbb{R}^n$  such that  $\pi_0 : T_0V \rightarrow \text{im } \pi_0$  is unbranched in a suitable neighborhood of  $\xi$ , in particular,  $\xi$  is a regular point of  $T_0V$ . Then there are a cone  $\Gamma_1 = \Gamma(\xi, C_1, \epsilon_1) \subset \Gamma_0^+$  and a holomorphic map  $g : \pi_0(\Gamma_1) \rightarrow \ker \pi_0$  such that

$$(2) \quad T_0V \cap \Gamma_1 = \{v + g(v) : v \in \pi_0(\Gamma_1)\}.$$

Let  $C_2 \subset C_1$  be relatively compact, and set  $\Gamma_2 := \Gamma(\xi, C_2, \epsilon_1)$ . We claim that  $w \in T_0V \cap \Gamma_2$  is real whenever  $\pi_0(w)$  is real. To see this, fix  $w \in T_0V \cap \Gamma_2$  with  $\pi_0(w)$  real. Near the origin, the projection  $\pi_0$  induces a description of  $V$  as an analytic cover above some neighborhood of 0 in  $\text{im } \pi_0$ . Since  $T_0V$  is the tangent cone, at least one sheet of  $V$  must eventually come close to  $T_0V \cap \Gamma_2$ . Hence, for sufficiently large  $j \in \mathbb{N}$ , there is  $z_j \in V \cap \Gamma_2$  with  $\pi_0(z_j) = \pi_0(w)/j$ . Then  $z_j \in \mathbb{R}^n$  because  $\Gamma_2 \subset \Gamma_0^+$ . Note that there is  $\eta > 0$  such that

$$\eta|\pi_0(z)| \leq |z| \leq \frac{1}{\eta}|\pi_0(z)| \quad \text{for all } z \in \Gamma_0^+.$$

Hence the points  $z_j$  are all in  $\Gamma_2$  and satisfy a common upper as well as a common lower bound, where the latter is strictly positive. By compactness, this means that the sequence  $(z_j)_{j \in \mathbb{N}}$  has a subsequence which converges to some  $w_1 \in \overline{\Gamma_2} \setminus \{0\} \subset \Gamma_1$ . By definition, we have  $w_1 \in T_0V$  and  $\pi_0(w_1) = \pi_0(w)$ . Since  $T_0V \cap \Gamma_1$  is a graph by (2), this shows  $w = w_1 \in \mathbb{R}^n$ .

Fix now  $\pi$  as in the hypothesis, then  $T_\xi(T_0V)$  is a tangent hyperplane which has trivial intersection with  $\ker \pi$ . This implies that near  $\xi$ ,  $T_0V$  is a graph over  $\text{im } \pi$ , i.e., there are a cone  $\Gamma_3 = \Gamma(\xi, C_3, \epsilon_3)$  with  $0 \in \overline{C_3} \subset C_2$  and a holomorphic map  $h: \pi(\Gamma_3) \rightarrow \ker \pi$  such that

$$T_0V \cap \Gamma_3 = \{v + h(v) : v \in \pi(\Gamma_3)\}.$$

It is easy to see that, since  $g$  is real valued for real arguments, so is  $h$ . Fix a cone  $\Gamma_4 = \Gamma(\pi_0(\xi), C_4, \epsilon_4) \subset \pi_0(\Gamma_2)$  such that  $z \in \Gamma_3$  whenever  $z \in V \cap \Gamma_2$  satisfies  $\pi_0(z) \in \Gamma_4$ . Let  $C_5$  be a neighborhood of 0, relatively compact in  $C_4$ , and set  $\Gamma_5 = \Gamma(\pi_0(\xi), C_5, \epsilon_4/2)$ . It follows from classical estimates of potential theory (e.g., Nevanlinna [19], 38) that there is a constant  $A > 0$  such that the following holds:

Whenever  $u: \Gamma_4 \rightarrow [-\infty, \infty[$  is plurisubharmonic and satisfies  $u(\zeta) \leq |\zeta|$  for each  $\zeta \in \Gamma_4$  and  $u(\zeta) \leq 0$  for each  $\zeta \in \Gamma_4 \cap \mathbb{R}^n$ , then  $u(\zeta) \leq A|\text{Im } \zeta|$  for each  $\zeta \in \Gamma_5$ .

Next note that we can find  $\delta > 0$  such that

$$|\text{Im } \pi_0(z)| = |\pi_0(\text{Im } z)| \leq \frac{1}{\delta} |\text{Im } z| \quad \text{for all } z \in \mathbb{C}^n$$

and define  $u: \Gamma_4 \rightarrow [-\infty, \infty[$  by

$$u(\zeta) := (A/\delta + 1) \max\{|\text{Im}(z - \pi(z) - h(\pi(z)))| : z \in V \cap \Gamma_3, \pi_0(z) = \zeta\}.$$

Note that  $|z - \pi(z) - h(\pi(z))| = o(|z|)$  for  $z \in V \cap \Gamma_3$ , hence  $u(\zeta) \leq |\zeta|$  if we assume that  $\epsilon_3$  is sufficiently small. If  $\zeta$  is real, then so is each  $z \in V \cap \Gamma_3$  with  $\pi_0(z) = \zeta$ , hence  $u(\zeta) \leq 0$  in that case. By the above, we find  $u(\zeta) \leq A|\text{Im } \zeta|$  for  $\zeta \in \Gamma_5$ . Moreover, if  $z \in V \cap \Gamma_3$ ,  $\pi(z)$  is real, and  $\pi_0(z) \in \Gamma_5$ , then

$$\begin{aligned} A|\text{Im } \pi_0(z)| &\geq u(\pi_0(z)) \geq (A/\delta + 1)|\text{Im } z| \\ &\geq \delta(A/\delta + 1)|\text{Im } \pi_0(z)| = (A + \delta)|\text{Im } \pi_0(z)|. \end{aligned}$$

We have shown  $(A + \delta)|\text{Im } \pi_0(z)| \leq A|\text{Im } \pi_0(z)|$ . This implies that  $\pi_0(z)$  is real, hence also  $z$  is real.

The same arguments apply to the opposite cone  $\Gamma_0^-$ . □

*Proof of Theorem 10.* (This proof follows very closely the one of [5], Theorem 5.1.) Contrary to the convention in the remainder of the paper, we will work here with the norm  $|z| := \max_{j=1, \dots, n} |z_j|$ . We may assume  $\xi = 0$ . Let  $W_1, \dots, W_q$  denote the irreducible components of  $T_0V$ . By hypothesis, there exist  $\xi_j \in W_j \cap (\mathbb{R}^n \setminus \{0\})$  so that  $V$  is 1-hyperbolic with respect to  $\xi_j$  and to  $-\xi_j$ ,  $j = 1, \dots, q$ . It follows from Proposition 12 that, if we perturb the  $\xi_j$  a little if necessary, then there is a common noncharacteristic projection  $\pi$  that works for all  $\pm \xi_j$ ,  $j = 1, \dots, q$ . We assume it to be  $\pi: (z_1, \dots, z_n) \mapsto (z_1, \dots, z_k)$  where  $k := \dim V$ .

Let  $D \in \mathbb{C}[z_1, \dots, z_k]$ , homogeneous of some degree  $d$ , be chosen so that  $T_0V$  is unbranched over  $\{w \in \mathbb{C}^k \mid D(w) \neq 0\}$ . (See, e.g., Whitney [23], I.8E, or Chirka

[10], 1.3.) For  $0 < \eta < 1$  let

$$S(\eta) := \{z \in \mathbb{C}^n \mid |D(z_1, \dots, z_k)| \leq \eta |z_1, \dots, z_k|^d\}.$$

Since it is possible to perturb the  $\xi_j$  a little and still keep the same projection  $\pi$ , we may assume  $D(\xi_j) \neq 0$  and hence  $\xi_j \notin S(\eta)$  if  $\eta$  is chosen sufficiently small. Pick  $\delta > 0$  so small that

$$S(\eta) \cap \bigcup_{\substack{j=1 \\ \sigma=\pm 1}}^q \Gamma(\sigma \xi_j, B(0, \delta), \delta) = \emptyset$$

and

(3) if  $z \in V \cap \Gamma(\pm \xi_j, B(0, \delta), \delta)$  and  $\pi(z)$  is real, then  $z$  is real.

Since  $T_0V$  is unbranched over  $\pi(S(\eta))$ , it is not difficult to see that for sufficiently small positive  $\delta_1 < \delta$  the projection  $\pi$  induces an analytic map

$$\tilde{\pi}: V \cap B(0, \delta_1) \setminus S(\eta) \rightarrow T_0V$$

such that for  $z \in V$  the point  $\tilde{\pi}(z)$  is the one in  $T_0V$  satisfying  $\pi(z) = \pi \circ \tilde{\pi}(z)$  which is nearest to  $z$ . Note that if  $\eta > 0$  is small enough, then the sets  $W_j \setminus S(\eta)$  are connected manifolds.

Fix such a number  $\eta$  and set

$$(4) \quad R(j) := \mathbb{C} \cdot (\Gamma(\xi_j, B(0, \delta), \delta) \cap W_j \cap \mathbb{R}^n) \quad \text{and} \quad \mathcal{R} := \bigcup_{j=1}^q R(j).$$

Since each  $W_j \setminus S(\eta)$  is connected, it is not difficult to see that  $\mathcal{R}$  is nonpluripolar in  $T_0V \cap B(0, r) \setminus S(\eta)$ . Now let  $u \in \text{PSH}(V)$  satisfy

$$u(z) \leq 1, \quad z \in V, \quad \text{and} \quad u(z) \leq 0, \quad z \in V \cap \mathbb{R}^n.$$

In order to show that for some constant  $A_0 > 0$ , which does not depend on  $u$ , the estimate

$$u(z) \leq A_0 |z|, \quad z \in V,$$

holds, we will apply Theorem 11 with  $W := T_0V$ ,  $\Gamma := T_0V \setminus S(\eta)$ , and  $\Gamma' := \mathbb{C}^k \setminus \pi(S(\eta))$  to suitable functions, derived from the given function  $u$ . To define them, fix a small number  $a > 0$  and define the function  $u_0$  on  $T_0V \cap B(0, \delta_1) \setminus S(\eta)$  by

$$u_0(w) := \max\{u(z) \mid z \in V, \tilde{\pi}(z) = w\}$$

and a function  $u_1$  on  $T_0V \setminus S(\eta)$  by

$$u_1(w) := \begin{cases} \max(0, u_0(w) - a, 3 + 2(\log|w| - \log \delta_1)), & \text{if } |w| < \delta_1, \\ 3 + 2(\log|w| - \log \delta_1), & \text{otherwise.} \end{cases}$$

It follows from Hörmander [12], 4.4, that the singularities at the points where the maximum moves from one branch to another are removable, i.e., that  $u_0$  is plurisubharmonic. This argument together with the fact that  $u_0(w) \leq 1 < 3 + 2(\log|w| - \log \delta_1)$  whenever  $\delta_1/e < |w| < \delta_1$  also shows that  $u_1$  is plurisubharmonic.

We define  $v_0$  on  $B(0, \delta_1) \subset \mathbb{C}^k$  and  $v_1$  on  $\mathbb{C}^k$  by

$$v_0(w) := \max\{u(z) \mid z \in V, \pi(z) = w\}, \quad |w| < \delta_1,$$

$$v_1(w) := \begin{cases} \max(0, v_0(w) - a, 3 + 2(\log|w| - \log \delta_1)), & \text{if } |w| < \delta_1, \\ 3 + 2(\log|w| - \log \delta_1), & \text{otherwise.} \end{cases}$$

The functions  $v_0$  and  $v_1$  are plurisubharmonic.

If  $x \in W_j \cap \Gamma(\xi_j, B(0, \delta), \delta_1) \cap \mathbb{R}^n$  for some  $j$  and  $|x| > \delta_1/2$ , then  $u_0(\zeta x) \leq 1$  for all  $\zeta \in \mathbb{C}$  with  $|\zeta| \leq 1$  and  $u_0(\xi x) \leq 0$  for  $-1 \leq \xi \leq 1$ . By a standard potential theory argument (see, e.g., Nevanlinna [19], 38), there is a constant  $C > 0$ , not depending on anything, such that  $u_0(\zeta x) \leq C|\zeta|$  provided  $|\zeta| \leq 1/2$ . We have shown

$$(5) \quad u_0(w) \leq \frac{2C}{\delta_1}|w|, \quad w \in \mathcal{R} \cap B(0, \delta_1/2).$$

Next define  $\phi: T_0V \setminus S(\eta) \rightarrow [-\infty, \infty[$  by

$$\phi(z) := \sup_{\zeta \in \mathbb{C}} \frac{u_1(\zeta z)}{|\zeta|}, \quad z \in T_0V \setminus S(\eta).$$

Then  $\phi$  is positive homogeneous. Since  $u_1$  is plurisubharmonic, the upper regularization  $\phi^*$  is plurisubharmonic and positive homogeneous. Let  $z \in \mathcal{R}$ . If  $z \in T_0V \cap B(0, \delta_1)$  satisfies  $|z| \geq \delta_1/2$ , then  $u_0(z) \leq 1 \leq (2/\delta_1)|z|$ . Since  $3 + 2(\log|z| - \log \delta_1) \leq 2\sqrt{e}|w|$ , these considerations and (5) show

$$\phi(z) \leq M|z| \quad \text{for } z \in \mathcal{R} \quad \text{where } M := \max(2C/\delta_1, 2\sqrt{e}/\delta_1).$$

Since each  $z \in \mathcal{R}$  with  $z \neq 0$  is a regular point of  $T_0V$ , the same estimate holds for  $\phi^*$ . Now define similarly to  $\phi$  the function  $\psi: \mathbb{C}^k \rightarrow [-\infty, \infty[$  by

$$\psi(w) := \sup_{\zeta \in \mathbb{C}} \frac{v_1(\zeta w)}{|\zeta|}, \quad w \in \mathbb{C}^k.$$

$\psi$  is an extension of  $w \mapsto \max\{\phi(z) \mid z \in T_0V \setminus S(\eta), \pi(z) = w\}$  and consequently  $\psi^*$  extends  $w \mapsto \max\{\phi^*(z) \mid z \in T_0V \setminus S(\eta), \pi(z) = w\}$ . Therefore we can apply Theorem 11 with  $W = T_0V$ ,  $\Gamma = T_0V \setminus S(\eta)$ ,  $\Gamma' = \mathbb{C}^k \setminus \pi(S(\eta))$ ,  $u = \phi^*$ , and  $v = \psi^*$ . It gives the existence of  $A \geq 1$  such that

$$\psi^*(w) \leq A|w| \quad \text{for all } w \in \mathbb{C}^k.$$

If  $\psi^*$  is expressed in terms of  $u$ , this estimate implies  $\text{RPL}_{\text{loc}}(0)$  if  $a$  tends to 0.  $\square$

**Corollary 13.** *Let  $f$  be a holomorphic function, defined on the ball  $B(0, r) \subset \mathbb{C}^n$  for some  $r > 0$ , which satisfies the following conditions:*

- (a)  $f$  is real on  $B(0, r) \cap \mathbb{R}^n$ ,
- (b) the localization  $f_0$  of  $f$  at zero is squarefree and has positive degree,
- (c) for each irreducible factor  $q$  of  $f_0$ , the real zero set of  $q$  has dimension  $n - 1$  at zero.

Then  $V = \{z \in B(0, r) : f(z) = 0\}$  satisfies  $\text{RPL}_{\text{loc}}(0)$ .

*Proof.* The hypotheses imply that  $f$  can be decomposed as  $f = f_0 + g$ , where  $f_0$  is a homogeneous polynomial of degree  $m > 0$  and where  $g$  is holomorphic on  $B(0, r)$  and satisfies for some constant  $C > 0$  the estimate

$$(6) \quad |g(z)| \leq C|z|^{m+1}, \quad z \in B(0, r/2).$$

Note that  $f_0$  has real coefficients. Then the hypotheses imply  $f_0 = \prod_{j=1}^s P_j$ , where  $P_1, \dots, P_s$  are irreducible homogeneous polynomials with real coefficients which are pairwise not proportional. By Whitney [23], 7.4D (see Definition 5), we have

$$T_0V = \{z \in \mathbb{C}^n : f_0(z) = 0\} = \bigcup_{j=1}^s W_j,$$

where the varieties

$$W_j := \{z \in \mathbb{C}^n : P_j(z) = 0\}, \quad 1 \leq j \leq s,$$

are the irreducible components of  $T_0V$ .

Next note that the hypotheses on  $f_0$  imply that for  $1 \leq j \leq s$  we can choose  $\xi_j \in W \cap B(0, r/2) \cap \mathbb{R}^n$  such that  $\text{grad } P_j(\xi_j) \neq 0$  and  $\xi_j \notin W_k$  for  $k \neq j$ . Then we have

$$\text{grad } f_0(\xi_j) = \left( \prod_{\substack{k=1 \\ k \neq j}}^s P_k(\xi_j) \right) \text{grad } P_j(\xi_j) \neq 0, \quad 1 \leq j \leq s.$$

For fixed  $j$ , we may assume that  $\frac{\partial f_0}{\partial z_n}(\xi_j) \neq 0$  and write  $z = (z', z_n)$  for  $z \in \mathbb{C}^n$ . Then an application of the real and the complex implicit function theorem implies that for suitable numbers  $\delta_1, \delta_2 > 0$  the variety  $W_j$  in a neighborhood of  $\xi_j$  is the graph of a holomorphic function  $h_j : B(\xi'_j, \delta_1) \rightarrow B(\xi_{j,n}, \delta_2)$  which is real over  $B(\xi'_j, \delta_1) \cap \mathbb{R}^{n-1}$ . Of course, we may choose  $\delta_1$  and  $\delta_2$  so small such that  $\frac{\partial f_0}{\partial z_n}(z) \neq 0$  for  $z \in B(\xi'_j, \delta) \times B(\xi_{j,n}, \delta)$ , where  $\delta := \min(\delta_1, \delta_2)$ . Since  $\xi_{j,n}$  is a simple zero of  $\lambda \mapsto f_0(\xi'_j, \lambda)$ , we can choose  $0 < \rho < \delta/4$  so that  $|f_0(\xi'_j, \lambda)| > 0$  for  $0 < |\lambda - \xi_{j,n}| \leq \rho$ . Thus

$$a := \inf\{|f_0(\xi'_j, z_n)| : |z_n - \xi_{j,n}| = \rho\} > 0.$$

Hence we can choose  $0 < \delta_3 < \delta/2$  such that

$$\inf\{|f_0(\zeta', \lambda)| : |\zeta' - \xi'_j| \leq \delta_3, |z_n - \xi_{j,n}| = \rho\} \geq \frac{a}{2}.$$

Now note that for  $0 < t \leq 1$ ,  $|\zeta' - \xi'_j| \leq \delta_3$ , and  $|z_n - \xi_{j,n}| = \rho$ , the estimate (6) implies

$$|g(t\zeta', tz_n)| \leq Ct^{m+1} |(\zeta', z_n)|^{m+1} \leq C(|\xi'_j| + |\xi_{j,n}| + \delta)^{m+1} t^{m+1},$$

while

$$|f_0(t\zeta', tz_n)| = t^m |f_0(\zeta', z_n)| \geq \frac{a}{2} t^m.$$

These estimates show the existence of  $0 < t_0 \leq 1$  such that for each  $0 < t \leq t_0$  and each  $\zeta' \in \mathbb{C}^{n-1}$  with  $|\zeta' - \xi'_j| < \delta_3$  we can apply Rouché's Theorem to see that the equation  $f(t\zeta', z_n) = 0$  has exactly one solution  $\zeta_n \in \mathbb{C}$  satisfying  $|\zeta_n - t\xi_j| < t\rho$ . Since  $f$  is real for real arguments and the neighborhood in which the root is unique is symmetric with respect to complex conjugation, it follows that  $V$  is 1-hyperbolic with respect to  $\xi_j$ . Since the same arguments apply to  $-\xi_j$ , the proof is complete.  $\square$

**Example 14.** The sufficient condition in Theorem 10 is not necessary. In fact, for

$$P(s, w_1, w_2) := (s^2 - w_1^2)^2 - w_2(w_1^2 - w_2^2)(w_1^2 + w_2^2)$$

the variety  $V(P)$  satisfies  $\text{RPL}_{\text{loc}}(0)$ , although the hypotheses of Theorem 10 are violated. This example is an adaptation of Example 4.10 of Bainbridge [2]. For the proof we refer to [7].

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