TRANSFINITE SEQUENCES OF CONTINUOUS
AND BAIRE CLASS 1 FUNCTIONS

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Abstract. The set of continuous or Baire class 1 functions defined on a metric space $X$ is endowed with the natural pointwise partial order. We investigate how the possible lengths of well-ordered monotone sequences (with respect to this order) depend on the space $X$.

Introduction

Any set $\mathcal{F}$ of real valued functions defined on an arbitrary set $X$ is partially ordered by the pointwise order; that is, $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$. Then, $f < g$ iff $f \leq g$ and $g \not< f$; equivalently, $f(x) \leq g(x)$ for all $x \in X$ and $f(x) < g(x)$ for at least one $x \in X$. Our aim will be to investigate the possible lengths of the increasing or decreasing well-ordered sequences of functions in $\mathcal{F}$ with respect to this order. A classical theorem (see Kuratowski [7], §24.III, Theorem 2') asserts that if $\mathcal{F}$ is the set of Baire class 1 functions (that is, pointwise limits of continuous functions) defined on a Polish space $X$ (that is, a complete separable metric space), then there exists a monotone sequence of length $\xi$ in $\mathcal{F}$ iff $\xi < \omega_1$. P. Komjáth [5] proved that the corresponding question concerning Baire class $\alpha$ functions for $2 \leq \alpha < \omega_1$ is independent of ZFC.

In the present paper we investigate what happens if we replace the Polish space $X$ by an arbitrary metric space.

Section 1 considers chains of continuous functions. We show that for any metric space $X$, there exists a chain in $C(X, \mathbb{R})$ of order type $\xi$ iff $|\xi| \leq d(X)$. Here, $|A|$ denotes the cardinality of the set $A$, while $d(X)$ denotes the density of the space $X$, that is,

$$d(X) = \max(\min\{|D| : D \subseteq X \& \overline{D} = X\}, \omega).$$

In particular, for separable $X$, every well-ordered chain has countable length, just as for Polish spaces.

Section 2 considers chains of Baire class 1 functions on separable metric spaces. Here, the situation is entirely different from the case of Polish spaces, since on some
separable metric spaces there are well-ordered chains of every order type less than \( \omega_2 \). Furthermore, the existence of chains of type \( \omega_2 \) and longer is independent of \( \text{ZFC} + \neg \text{CH} \). Under \( \text{MA} \), there are chains of all types less than \( c^+ \), whereas in the Cohen model, all chains have type less than \( \omega_2 \).

We note here that instead of examining well-ordered sequences, which is a classical problem, we could try to characterize all the possible order types of linearly ordered subsets of the partially ordered set \( \mathcal{F} \). This problem was posed by M. Laczkovich, and is considered in detail in [3].

1. Sequences of continuous functions

**Lemma 1.1.** For any topological space \( X \): If there is a well-ordered sequence of length \( \xi \) in \( C(X, \mathbb{R}) \), then \( \xi < d(X)^+ \).

**Proof.** Let \( \{f_\alpha : \alpha < \xi\} \) be an increasing sequence in \( C(X, \mathbb{R}) \), and let \( D \subseteq X \) be a dense subset of \( X \) such that \( d(X) = \max(|D|, \omega) \). By continuity, the \( f_\alpha|_D \) are all distinct; so, for each \( \alpha < \xi \), choose a \( d_\alpha \in D \) such that \( f_\alpha(d_\alpha) < f_\alpha+1(d_\alpha) \). For each \( d \in D \) the set \( E_d = \{\alpha : d_\alpha = d\} \) is countable, because every well-ordered subset of \( \mathbb{R} \) is countable. Since \( \xi = \bigcup_{d \in D} E_d \), we have \( |\xi| \leq \max(|D|, \omega) = d(X) \). \( \square \)

The converse implication is not true in general. For example, if \( X \) has the countable chain condition (ccc), then every well-ordered chain in \( C(X, \mathbb{R}) \) is countable (because \( X \times \mathbb{R} \) is also ccc). However, the converse is true for metric spaces:

**Lemma 1.2.** If \( (X, \rho) \) is any non-empty metric space and \( \prec \) is any total order of the cardinal \( d(X) \), then there is a chain in \( C(X, \mathbb{R}) \) which is isomorphic to \( \prec \).

**Proof.** First, note that every countable total order is embeddable in \( \mathbb{R} \), so if \( d(X) = \omega \), then the result follows trivially using constant functions. In particular, we may assume that \( X \) is infinite, and then fix \( D \subseteq X \) which is dense and of size \( d(X) \). For each \( n \in \omega \), let \( D_n \) be a subset of \( D \) which is maximal with respect to the property \( \forall d, e \in D_n [d \neq e \rightarrow \rho(d, e) \geq 2^{-n}] \). Then \( \bigcup_n D_n \) is also dense, so we may assume that \( \bigcup_n D_n = D \). We may also assume that \( \prec \) is a total order of the set \( D \). Now, we shall produce \( f_d \in C(X, \mathbb{R}) \) for \( d \in D \) such that \( f_d \prec f_e \) whenever \( d \prec e \).

For each \( n \), if \( c \in D_n \), define \( \varphi^n_c(x) = \max(0, 2^{-n} - \rho(x, c)) \). For each \( d \in D \), let \( \psi^n_d = \sum \{\varphi^n_c : c \in D_n \text{ and } c \prec d\} \). Since every \( x \in X \) has a neighborhood on which all but at most one of the \( \varphi^n_c \) vanish, we have \( \psi^n_d \in C(X, [0, 2^{-n}]) \), and \( \psi^n_d \leq \psi^n_e \) whenever \( d \prec e \). Thus, if we let \( f_d = \sum_{n<\omega} \psi^n_d \), we have \( f_d \in C(X, [0, 2]) \), and \( f_d \prec f_e \) whenever \( d \prec e \). But also, if \( d \in D_n \) and \( d \prec e \), then \( \psi^n_d(d) = 0 < 2^{-n} = \psi^n_e(d) \), so actually \( f_d \prec f_e \) whenever \( d \prec e \). \( \square \)

Putting these lemmas together, we have:

**Theorem 1.3.** Let \( (X, \rho) \) be a metric space. Then there exists a well-ordered sequence of length \( \xi \) in \( C(X, \mathbb{R}) \) iff \( \xi < d(X)^+ \).

**Corollary 1.4.** A metric space \( (X, \rho) \) is separable iff every well-ordered sequence in \( C(X, \mathbb{R}) \) is countable.

2. Sequences of Baire class 1 functions

If we replace continuous functions by Baire class 1 functions, then Corollary 1.4 becomes false, since on some separable metric spaces we can get well-ordered sequences of every type less than \( \omega_2 \). To prove this, we shall apply some basic facts
about $\subset^*$ on $\mathcal{P}(\omega)$. As usual, for $x, y \subseteq \omega$, we say that $x \subseteq^* y$ iff $x \setminus y$ is finite. Then $x \subseteq^* y$ iff $x \setminus y$ is finite and $y \setminus x$ is infinite. This $\subseteq^*$ partially orders $\mathcal{P}(\omega)$.

**Lemma 2.1.** If $X \subseteq \mathcal{P}(\omega)$ is a chain in the order $\subseteq^*$, then on $X$ (viewed as a subset of the Cantor set $2^\omega \cong \mathcal{P}(\omega)$) there is a chain of Baire class 1 functions which is isomorphic to $(X, \subseteq^*)$.

**Proof.** Note that for each $x \in X$,

$$\{y \in X : y \subseteq^* x\} = \bigcup_{m \in \omega} \{y \in X : \forall n \geq m \ [y(n) \leq x(n)]\},$$

which is an $F_\sigma$ set in $X$. Likewise, the sets $\{y \in X : y \supseteq^* x\}$, $\{y \in X : y \subseteq^* x\}$, and $\{y \in X : y \supseteq^* x\}$, are all $F_\sigma$ sets in $X$, and hence also $G_\delta$ sets. It follows that if $f_x : X \to \{0, 1\}$ is the characteristic function of $\{y \in X : y \subseteq^* x\}$, then $f_x : X \to \mathbb{R}$ is a Baire class 1 function. Then $\{f_x : x \in X\}$ is the required chain. \hfill $\square$

**Lemma 2.2.** For any infinite cardinal $\kappa$, suppose that $(\mathcal{P}(\omega), \subseteq^*)$ contains a chain $\{x_\alpha : \alpha < \kappa\}$ (i.e., $\alpha < \beta \rightarrow x_\alpha \subseteq^* x_\beta\}$). Then $(\mathcal{P}(\omega), \subseteq^*)$ contains a chain $X$ of size $\kappa$ such that every ordinal $\xi < \kappa^+$ is embeddable into $X$.

**Proof.** Let $S = \bigcup_{1 \leq n < \omega} \kappa^n$. For $s = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) \in S$, let

$$s^+ = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n + 1).$$

Starting with the $x_\alpha = x_\alpha$, choose $x_s \in \mathcal{P}(\omega)$ by induction on $length(s)$ so that

$$x_s = x_{s^0} \subseteq^* x_{s^0} \subset^* x_{s^0} \subset^* x_{s^0} \subset^* x_s^+$$

whenever $s \in S$ and $0 < \alpha < \beta < \kappa$. Let $X = \{x_s : s \in S\}$. Then, whenever $x, y \in X$ with $x \subseteq^* y$, the ordinal $\kappa$ is embeddable in $(x, y) = \{z \in X : x \subseteq^* z \subseteq^* y\}$. From this, one easily proves by induction on $\xi < \kappa^+$ (using $\text{cf}(\xi) \leq \kappa$) that $\xi$ is embeddable in each such interval $(x, y)$. \hfill $\square$

Since $\mathcal{P}(\omega)$ certainly contains a chain of type $\omega_1$, these two lemmas yield:

**Theorem 2.3.** There is a separable metric space $X$ on which, for every $\xi < \omega_2$, there is a well-ordered chain of length $\xi$ of Baire class 1 functions.

Under $CH$, this is best possible, since there will be only $2^{\omega} = \omega_1$ Baire class 1 functions on a separable metric space, so there could not be a chain of length $\omega_2$. Under $\neg CH$, the existence of longer chains of Baire class 1 functions depends on the model of set theory. It is consistent with $\mathfrak{c} = 2^{\omega}$ being arbitrarily large that there is a chain in $(\mathcal{P}(\omega), \subseteq^*)$ of type $\mathfrak{c}$; for example, this is true under $MA$ (see [2]). In this case, there will be a separable $X$ with well-ordered chains of all lengths less than $\mathfrak{c}^+$. However, in the Cohen model, where $\mathfrak{c}$ can also be made arbitrarily large, we never get chains of type $\omega_2$. We shall prove this by using the following lemma, which relates it to the rectangle problem:

**Lemma 2.4.** Suppose that there is a separable metric space $Y$ with an $\omega_2$-chain of Borel subsets, $\{B_\alpha : \alpha < \omega_2\}$ (so, $\alpha < \beta \rightarrow B_\alpha \subset B_\beta$). Then in $\omega_2 \times \omega_2$, the well-order relation $< \text{is in the } \sigma\text{-algebra generated by the set of all rectangles,} \{S \times T : S, T \in \mathcal{P}(\omega_2)\}$. 

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Proof. Each $B_{\alpha}$ has some countable Borel rank. Since there are only $\omega_1$ ranks, we may, by passing to a subsequence, assume that the ranks are bounded. Say each $B_{\alpha}$ is a $\Sigma^0_\mu$ set for some fixed $\mu < \omega_1$.

Let $J = \omega^\omega$, and let $A \subseteq Y \times J$ be a universal $\Sigma^0_\mu$ set; that is, $A$ is $\Sigma^0_\mu$ in $Y \times J$ and every $\Sigma^0_\mu$ subset of $Y$ is of the form $A^j = \{y : (y, j) \in A\}$ for some $j \in J$ (see [7, §31]). Now, for $\alpha, \beta < \omega_2$, fix $y_\alpha \in B_{\alpha+1} \setminus B_{\alpha}$, and fix $j_\beta \in J$ such that $A^{j_\beta} = B_{\beta}$. Then $\alpha < \beta$ iff $(y_\alpha, j_\beta) \in A$. Thus, $\{(y_\alpha, j_\beta) : \alpha < \beta < \omega_2\}$ is a Borel subset of $\{y_\alpha : \alpha < \omega_2\} \times \{j_\beta : \beta < \omega_2\}$, and is hence in the $\sigma$-algebra generated by open rectangles, so $<$, as a subset of $\omega_2 \times \omega_2$, is in the $\sigma$-algebra generated by rectangles. □

**Theorem 2.5.** Assume that the well-order relation $<$ on $\omega_2$ is not in the $\sigma$-algebra generated by the set of all rectangles. Then no separable metric space can have a chain of length $\omega_2$ of Baire class 1 functions.

Proof. Suppose that $\{f_\alpha : \alpha < \omega_2\}$ is a chain of Baire class one functions on the separable metric space $X$. Let $B_{\alpha} = \{(x, r) : x \in X, r \leq f_\alpha(x)\}$. Then the $B_{\alpha}$ form an $\omega_2$-chain of Borel subsets of the separable metric space $X \times \mathbb{R}$, so we have a contradiction by Lemma 2.4.

Finally, we point out that the hypothesis of this theorem is consistent, since it holds in the extension $V[G]$ formed by adding $\geq \omega_2$ Cohen reals to a ground model $V$ which satisfies $CH$. This fact was first proved in [6]. It also follows from the more general principle $HP_2(\omega_2)$ of Brendle, Fuchino, and Soukup [1]. They define this principle, prove that it holds in Cohen extensions (and in a number of other forcing extensions), and show the following:

**Lemma 2.6.** $HP_2(\kappa)$ implies that if $R$ is any relation on $\mathcal{P}(\omega)$ which is first-order definable over $H(\omega_1)$ from a fixed element of $H(\omega_1)$, then there is no $X \subseteq \mathcal{P}(\omega)$ such that $(X; R)$ is isomorphic to $(\kappa; <)$.

These matters are also discussed in [1], which indicates how such statements are verified in Cohen extensions. Here, $H(\omega_1)$ denotes the set of hereditarily countable sets.

**Lemma 2.7.** $HP_2(\omega_2)$ implies that in $\omega_2 \times \omega_2$, the well-order relation $<$ is not in the $\sigma$-algebra generated by the set of all rectangles, $\{S \times T : S, T \in \mathcal{P}(\omega_2)\}$.

Proof. Suppose that $<$ were in this $\sigma$-algebra. Then we would have fixed $K_n \subseteq \omega_2$ for $n < \omega$ such that $<$ is in the $\sigma$-algebra generated by all the $K_n$. For each $\alpha$, let $u_\alpha = \{n \in \omega : \alpha \in K_n\}$. There is then a formula $\varphi(x, y, z)$ and a fixed $w \in H(\omega_1)$ such that for all $\alpha, \beta < \omega_2$, $\alpha < \beta$ iff $H(\omega_1) \models \varphi(u_\alpha, u_\beta, w)$; here, $w$ encodes the particular countable boolean combination used to get $<$ from the $K_n$. Now, if $X = \{u_\alpha : \alpha < \omega_2\}$, then $\varphi$ defines a relation $R$ on $H(\omega_1)$ such that $(X; R)$ is isomorphic to $(\omega_2; <)$, contradicting Lemma 2.6. □

**References**


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