

A PARTITION RELATION USING STRONGLY COMPACT CARDINALS

SAHARON SHELAH

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ABSTRACT. If κ is strongly compact and $\lambda > \kappa$ and λ is regular (or alternatively $\text{cf}(\lambda) \geq \kappa$), then $(2^{<\lambda})^+ \rightarrow (\lambda + \zeta)_\theta^2$ holds for $\zeta, \theta < \kappa$.

§0. INTRODUCTION

The aim of this paper is to prove the following theorem.

0.1 Theorem. *If κ is a strongly compact cardinal, $\lambda > \kappa$ is regular and $\zeta, \theta < \kappa$, then the partition relation $(2^{<\lambda})^+ \rightarrow (\lambda + \zeta)_\theta^2$ holds.*

0.2 Theorem. *Assume the conditions in Theorem 0.1 hold, with “ λ regular”. Then $\text{cf}(\lambda) > \kappa$ suffices.*

We notice that our argument is valid in the case $\kappa = \omega$. As for the history of the problem we point out that Hajnal proved, in an unpublished work, that $(2^\omega)^+ \rightarrow (\omega_1 + n)_2^2$ holds for every $n < \omega$. Then it was shown in [Sh:26], §6, that for $\kappa > \omega$ regular and $2^{|\alpha|} < \kappa$, the relation $(2^{<\kappa})^+ \rightarrow (\kappa + \alpha)_2^2$ is true. More recently Baumgartner, Hajnal, and Todorčević in [BHT93] extended this to the case when the number of colors is arbitrarily finite. Earlier in [Sh:424], we have $(2^{<\lambda})^{+n} \rightarrow (\lambda \times m)_k^2$ for n large enough (this was complimentary to the main result there that $\aleph_0 < \lambda = \lambda^{<\lambda} + 2^\lambda$ arbitrarily large does not imply $2^\lambda \rightarrow (\lambda \times \omega)_2^2$). Subsequently [BHT93] improves n . We hope that the way the strong compactness was used will be useful elsewhere; see [Sh:666] for a discussion of a possible consistency of failure. I also thank Peter Komjath for improving the presentation.

Notation. If S is a set and κ a cardinal, then $[S]^\kappa = \{a \subseteq S : |a| = \kappa\}$, $[S]^{<\kappa} = \{a \subseteq S : |a| < \kappa\}$. If D is some filter over a set S , then $X \in D^+$ denotes that $S \setminus X \notin D$ and $X \subseteq S$. If $\kappa < \mu$ are regular cardinals, then $S_\kappa^\mu = \{\alpha < \mu : \text{cf}(\alpha) = \kappa\}$, a stationary set. The notation $A = \{x_\alpha : \alpha < \gamma\}_<$, etc., means that A is enumerated increasingly.

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§1. THE CASE OF λ REGULAR

1.1 Lemma. Assume $\mu = \mu^\theta$. Assume that D is a normal filter on μ^+ and $A^* \in D^+$ satisfies $\delta \in A^* \Rightarrow \text{cf}(\delta) > \theta$, and F' is a function with domain $[A^*]^2$ and range of cardinality θ . Then there are a normal filter D_0 on μ^+ extending D , $A_0 \in D_0$ with $A_0 \subseteq A^*$ and $C_0 \subseteq \text{Rang}(F' \upharpoonright [A_0]^2) = C_0$ such that, if $X \in D_0^+$, then $\text{Rang}(F' \upharpoonright [X]^2) \supseteq C_0$.

We first prove a claim.

1.2 Claim. Assume $\mu = \mu^\theta$ and $F' : [S^*]^2 \rightarrow \theta$, D is a normal filter on μ^+ , $S^* \subseteq \mu^+$ belongs to D^+ and $\delta \in S^* \Rightarrow \text{cf}(\delta) > \theta$. There is a set $A \in D^+$ such that $A \subseteq S^*$ and some $C \subseteq \theta$ satisfying $\text{Rang}(F' \upharpoonright [A]^2) = C$ and, if $f : A \rightarrow \mu^+$ is a regressive function, then for some $\alpha < \mu^+$ we have $\text{Rang}(F' \upharpoonright [f^{-1}(\alpha)]^2) = C$ and $f^{-1}(\alpha)$ is a subset of μ^+ from D^+ .

Proof. Toward contradiction assume that no such sets A, C exist. We build a tree T as follows. Every node t of the tree will be of the form

$$\begin{aligned} t &= \langle \langle A_\alpha : \alpha \leq \varepsilon \rangle, \langle f_\alpha : \alpha < \varepsilon \rangle, \langle i_\alpha : \alpha < \varepsilon \rangle \rangle \\ &= \langle \langle A_\alpha^t : \alpha \leq \varepsilon \rangle, \langle f_\alpha^t : \alpha < \varepsilon \rangle, \langle i_\alpha^t : \alpha < \varepsilon \rangle \rangle \end{aligned}$$

for some ordinal $\varepsilon = \varepsilon(t)$ where $\langle A_\alpha : \alpha \leq \varepsilon \rangle$ is a decreasing, continuous sequence of subsets of μ^+ ; for every $\alpha < \varepsilon$, f_α is a regressive function on A_α ; and $\langle i_\alpha : \alpha < \varepsilon \rangle$ is a sequence of distinct elements of θ . It will always be true that if $t <_T t'$, then each of the three sequences of t' extend the corresponding one of t .

To start, we make the node t with $\varepsilon(t) = 0$, $A_0 = S^*$ the root of the tree.

At limit levels we extend (the obvious way) all cofinal branches to a node.

If we are given an element $t = \langle \langle A_\alpha : \alpha \leq \varepsilon \rangle, \langle f_\alpha : \alpha < \varepsilon \rangle, \langle i_\alpha : \alpha < \varepsilon \rangle \rangle$ of the tree and the set A_ε is $= \emptyset \pmod D$, then we leave t as a terminal node. Otherwise, let $C = C_t = \text{Rang}(F' \upharpoonright [A_\varepsilon]^2)$ and notice that by hypothesis, toward contradiction, the pair A_ε, C_t cannot be as required in the Claim. There is, therefore, a regressive function $f = f_t$ with domain A_ε , such that for every $x < \mu^+$ the set $\text{Rang}(F' \upharpoonright [f^{-1}(x)]^2)$ is a proper subset of C_t or $f^{-1}(x)$ is a $= \emptyset \pmod D$ subset of μ^+ . We make as immediate extensions of t the sequences of the form $t_x = \langle \langle A_\alpha : \alpha \leq \varepsilon + 1 \rangle, \langle f_\alpha : \alpha < \varepsilon + 1 \rangle, \langle i_\alpha : \alpha < \varepsilon + 1 \rangle \rangle$ where $A_{\varepsilon+1} = f^{-1}(x)$, $f_\alpha = f_t$ and $i_\varepsilon \in C_t$ is some colour value such that if $A_{\varepsilon+1} \neq \emptyset \pmod D$, then i_ε is not in the range of $F' \upharpoonright [A_\varepsilon]^2$.

Having constructed the tree, observe that every element $x \in S^* \subseteq \mu^+$ belongs to a set $A_{\varepsilon(x)}^{t(x)}$ for some (unique) terminal node $t(x)$ of T . Also, $\varepsilon(x) < \theta^+ (< \mu^+)$ holds by the selection of the i_β 's as $\langle i_\alpha^{t(x)} : \alpha < \varepsilon(x) \rangle$ is a sequence of members of θ with no repetitions while θ , the set of colours, has θ members. For some set $S \subseteq S^*$ of ordinals $x < \mu^+$ which belong to D^+ (by the normality of D), the value of $\varepsilon(x)$ is the same, say ε . For $x \in S$ we let $g_\alpha(x) = f_\alpha^{t(x)}(x)$ where $f_\alpha^{t(x)}$ is the α -th regressive function in the node $t(x) \in T$. Again, by $\mu^\theta = \mu$ and $(\forall \alpha \in S)[\text{cf}(\alpha) > \theta]$ we have that $(\forall x \in S')(\forall \alpha < \varepsilon)g_\alpha(x) = \beta_\alpha$ holds for some sequence $\langle \beta_\alpha : \alpha < \varepsilon \rangle$ and subset $S' \subseteq S$ from D^+ . But then we get that the set S' satisfies $x, y \in S' \Rightarrow (A_\alpha^{t(x)}, f_\alpha^{t(x)}, i_\alpha^{t(x)}) = (A_\alpha^{t(y)}, f_\alpha^{t(y)}, i_\alpha^{t(y)})$ for every $\alpha < \varepsilon$; we can prove this by induction on α . We can then prove that $A_\varepsilon^{t(x)} = A_\varepsilon^{t(y)}$ for $x, y \in S'$.

We can conclude that $x, y \in S' \Rightarrow t(x) = t(y)$, so $S' \subseteq A_{\varepsilon(t)}^t$ for some terminal node t , but this latter set is in D^+ , a contradiction. $\square_{1.2}$

Proof of Lemma 1.1. Apply Claim 1.2 with $S^* = A^*$ to get corresponding (C, A) . Define the ideal I as follows. For $X \subseteq \mu^+$ we let $X \in I$ iff there are a member E of D and a regressive function $f : X \cap A \rightarrow \mu^+$ such that every $\text{Rang}(F' \upharpoonright [f^{-1}(\alpha)]^2)$ is a proper subset of C or $f^{-1}(\alpha)$ is a $= \emptyset \bmod D$ subset of μ^+ .

Now:

1.3 Claim. I is a normal ideal on μ^+ (and $A^* = \mu^+ \bmod I$).

Proof. Straightforward.

Set D_0 to be the dual filter of I , let $A_0 = A$ and let $C_0 = C$; by Claim 1.2 we are done. $\square_{1.1}$

1.4 *Remark.* 1) If Lemma 1.1 holds for some D_0, A_0, C_0 , then it holds for D_1, A_1, C_0 when the normal filter D_1 extends D_0 , and $A_1 \in D_1$ satisfies $A_1 \subseteq A_0$.

2) If D_0, A_0, C_0 satisfy Lemma 1.5, and $X \in D_0^+$, then X contains a homogeneous set of order type $\lambda + 1$ of color ξ for every $\xi \in C_0$.

3) Lemma 1.1 is closely related to the proof in [Sh:26], i.e. 5.1 there.

Proof of Theorem 0.1. Let $\mu = 2^{<\lambda}$, and let $F : [\mu^+]^2 \rightarrow \theta$ be a colouring. We apply Lemma 1.1 for $A^* = S_{\text{cf}(\lambda)}^{\mu^+}$, ($F = F, \theta, \mu = \mu$) and D the club filter. We shall write $F(\alpha, \beta)$ for $F(\{\alpha, \beta\})$ and 0 for $F(\alpha, \alpha)$.

We fix A_0, D_0, C_0 which we get by Lemma 1.1.

1.5 Lemma. *Almost every $\delta \in A_0$ (i.e. for all but a set $= \emptyset \bmod D_0$) satisfies the following: if $s \in [A_0 \cap \delta]^{<\lambda}$ and $\{z_\alpha : \alpha < \gamma\} \subseteq A_0 \cap [\delta, \mu^+)$ with $\gamma < \kappa$, then there is $\{y_\alpha : \alpha < \gamma\} \subseteq A_0 \cap (\text{sup}(s), \delta)$ such that:*

- (a) $F(x, y_\alpha) = F(x, z_\alpha)$ (for $x \in s, \alpha < \gamma$);
- (b) $F(y_\alpha, y_\beta) = F(z_\alpha, z_\beta)$ (for $\alpha < \beta < \gamma$).

Proof. By simple reflection (using the regularity of λ).

1.6 Lemma. *There¹ is $A'_0 \subseteq A_0, A'_0 \in D_0$ such that if $\delta \in A'_0, s \in [\delta]^{<\lambda}$ and $\xi \in C_0$, then there exists a $\delta_1 \in A_0, \delta < \delta_1$ such that:*

- (a) $F(x, \delta) = F(x, \delta_1)$ (for $x \in s$);
- (b) $F(\delta, \delta_1) = \xi$.

Proof. Otherwise, there is some $X \subseteq A_0, X \in D_0^+$ such that for every $\delta \in X$ there are $s(\delta) \in [\delta]^{<\lambda}$ and $\xi(\delta) \in C_0$ such that there is no $\delta_1 > \delta$ satisfying (a) and (b). By normality and $\mu = \mu^{<\lambda}$ we can assume that $s(\delta) = s$ and $\xi(\delta) = \xi$ holds for $\delta \in X$. By Lemma 1.1, that is, the choice of (A_0, D_0, C_0) , there must exist $\delta < \delta_1$ in X with $F(\delta, \delta_1) = \xi$, and this is a contradiction. $\square_{1.6}$

Continuation of the proof of Theorem 0.1. Let $A'_0 \subseteq A_0$ satisfy Lemmas 1.1 and 1.6 and pick some $\delta_1 \in A'_0$. Then let $T = A'_0 \setminus (\delta_1 + 1)$.

¹In fact, if $A_1^* \in D_0^+$, then for some $A'_0 \subseteq A_1 \cap A_0, A_1 \setminus A'_0 = \emptyset$ modulo D_0 and the conclusion holds for every $\delta \in A'_0$.

1.7 Lemma. *There exists a function $G : T \times T \rightarrow C_0$ such that if $s \in [\delta_1]^{<\lambda}$, $\gamma < \kappa$, and $Z = \{z_\alpha : \alpha < \gamma\}_{<} \subseteq T$, then there is $\{y_\alpha : \alpha < \gamma\}_{<} \subseteq (\text{sup}(s), \delta_1)$ such that:*

- (a) $F(x, y_\alpha) = F(x, z_\alpha)$ (for $x \in s, \alpha < \gamma$);
- (b) $F(y_\alpha, y_\beta) = F(z_\alpha, z_\beta)$ (for $\alpha < \beta < \gamma$);
- (c) $F(y_\alpha, z_\beta) = G(z_\alpha, z_\beta)$ (for $\alpha, \beta < \gamma$).

Proof. As κ is strongly compact, it suffices to show that for every $Z \in [T]^{<\kappa}$ there exists a function $G : Z \times Z \rightarrow \theta$ as required. Clauses (a) and (b) are obvious by Lemma 1.5, and it is clear that, if we fix Z , then for every $s \in [\delta_1]^{<\lambda}$ there is an appropriate $G : Z \times Z \rightarrow \theta$. We show that there is some $G : Z \times Z \rightarrow \theta$ that works for every s . Assume otherwise, that is, for every $G : Z \times Z \rightarrow \theta$ there is some $s_G \in [\delta_1]^{<\lambda}$ such that G is not appropriate for s_G . Notice that the number of these functions G is less than κ . Then no G could be right for $s = \bigcup \{s_G : G \text{ a function from } Z \times Z \text{ to } \theta\} \in [\delta_1]^{<\lambda}$, a contradiction. □_{1.7}

Continuation of the proof of Theorem 0.1. We now apply Lemma 1.1 to the colouring $\bar{G}\{x, y\} = \bar{G}(x, y) = \langle F(x, y), G(x, y) \rangle$ for $x < y$ in T and 0 otherwise, and the filter D_0 and the set T to get the normal filter $D_1 \supseteq D_0$, the set $A_1 \subseteq T \subseteq A'_0$ such that $A_1 \in D_1$ and the colour set $C_1 \subseteq \theta \times \theta$. Notice that actually $C_1 \subseteq C_0 \times C_0$. We can also apply Lemmas 1.5 and 1.6 to get some set $A'_1 \subseteq A_1$.

1.8 Lemma. *There is a set $a \in [A'_1]^{<\kappa}$ such that for every decomposition $a = \bigcup \{a_{\bar{\xi}} : \bar{\xi} \in C_1\}$ there is some $\bar{\xi} \in C_1$ such that:*

- (α) *for every $\bar{\varepsilon} \in C_1$ there is an $\bar{\varepsilon}$ -homogeneous subset for the colouring \bar{G} of order type ζ in $a_{\bar{\varepsilon}}$;*
- (β) *similarly for every $\varepsilon \in C_0$ and F .*

Proof. This follows from the strong compactness of κ , as A'_1 itself has this partition property (see Claim 2.8 for more details). □_{1.8}

Continuation of the proof of Theorem 0.1. Fix a set a as in Lemma 1.8.

We now describe the construction of the required homogeneous subset. Let $\delta_2 \in A'_1$ be some element with $\delta_2 > \text{sup}(a)$. For $\bar{\xi} = (\xi_1, \xi_2) \in C_1 \subseteq \theta \times \theta$ let $a_{\bar{\xi}}$ be the following set:

$$a_{\bar{\xi}} = \{x \in a : \bar{G}(x, \delta_2) = \bar{\xi}\}.$$

By Lemma 1.8, there is some $\bar{\xi} = (\xi_1, \xi_2) \in C_1$ for which the statement in that lemma is true and necessarily (as $a \cup \{\delta_2\} \subseteq A'_1 \subseteq A_0$ and $a_{\bar{\xi}} \neq \emptyset$) we have $\xi_1, \xi_2 \in C_0$. Select some $b \subseteq a_{\bar{\xi}}$, $\text{otp}(b) = \zeta$ such that F is constantly ξ_2 on b ; this is possible by clause (β) of Lemma 1.8. This set b will be the ζ part of our homogeneous set of ordinals of order type $\lambda + \zeta$, so we will have to construct a set of order type λ below b . By induction on α we will choose x_α such that the set $\{x_\alpha : \alpha < \lambda\}_{<} \subseteq \delta_1$ satisfies the following conditions:

- (*)₁ $F(x_\beta, x_\alpha) = \xi_2$ (for $\beta < \alpha$),
- (*)₂ $F(x_\alpha, b \cup \{\delta_2\}) = \xi_2$, i.e. $F(x_\alpha, y) = \xi_2$ when $y \in b \cup \{\delta_2\}$.

Assume that we have reached step α , that is, we are given the set of ordinals with $\{x_\beta : \beta < \alpha\}_{<}$ and call this set s . Applying Lemma 1.6 for A_1, A'_1, δ_2 and $s \cup b$ and the colouring \bar{G} here standing for A_0, A'_0, δ, s and the colouring F there (that is, the choice of A'_1) we get that there exists some $\delta_3 > \delta_2$ (standing for δ_1 there)

such that:

- (i) $\delta_3 \in A_1$;
- (ii) $\bar{G}(x, \delta_3) = \bar{G}(x, \delta_2)$ for $x \in s \cup b$;
- (iii) $\bar{G}(\delta_2, \delta_3) = (\xi_1, \xi_2)$.

Hence

- (*)₃ $F(x_\beta, \delta_3) = \xi_2$ (for $\beta < \alpha$).
 [Why? As $F(x_\beta, \delta_3) = F(x_\beta, \delta_2)$ by (ii) and the choice of \bar{G} and $F(x_\beta, \delta_2) = \xi_2$ by (*)₂ from the induction hypothesis.]
- (*)₄ $G(b \cup \{\delta_2\}, \delta_3) = \xi_2$, i.e. $G(y, \delta_3) = \xi_2$ when $y \in b \cup \{\delta_2\}$.
 [Why? If $y \in b$, then by (ii) and the definition of \bar{G} we have $G(y, \delta_3) = G(y, \delta_2)$, but $b \subseteq a_{\bar{\varepsilon}}$ so by the choice of $a_{\bar{\varepsilon}}$ we have $G(y, \delta_2) = \xi_2$. For $y = \delta_2$ use clause (iii), that is, $(\xi_1, \xi_2) = \bar{G}(\delta_2, \delta_3) = (F(\delta_2, \delta_3), G(\delta_2, \delta_3))$.]

By the choice of G this implies that there is some x_α as required; that is, by the choice of \bar{G} (see Lemma 1.7) applied to $Z = \{z_i : i < \gamma\}$, enumerating the set $b \cup \{\delta_2, \delta_3\}$ and s as above, we get $\{y_i : i < \gamma\}$, now necessarily $\delta_3 = z_{\gamma-1}$, and we can choose $y_{\gamma-1}$ as x_α . □_{1.1}

§2. THE CASE OF λ SINGULAR

We prove version 0.2 of the main theorem.

Proof of Theorem 0.2. Let $\sigma = \text{cf}(\lambda)$. Let $\lambda = \sum_{\varepsilon < \sigma} \lambda_\varepsilon$ with $\lambda_\varepsilon > \sigma \geq \kappa > \theta$ strictly increasing. Let $\mu_\varepsilon = 2^{\lambda_\varepsilon}$ and $\mu = \Sigma\{\mu_\varepsilon : \varepsilon < \sigma\} = 2^{<\lambda}$. We also fix $F : [\mu^+]^2 \rightarrow \theta$.

2.1 Claim. *For some \bar{C} we have:*

- (a) $\bar{C} = \langle C_\alpha : \alpha \in S \rangle$;
- (b) $S \subseteq \mu^+, C_\delta \subseteq \delta$;
- (c) $\text{otp}(C_\delta) \leq \sigma$;
- (d) $S^* = \{\delta < \lambda : \text{otp}(C_\delta) = \sigma\}$ is stationary;
- (e) C_δ unbounded in δ if $\text{otp}(C_\delta) = \sigma$;
- (f) $\alpha \in C_\delta \Rightarrow \alpha \in S$ and $C_\alpha = C_\delta \cap \alpha$.

□_{2.1}

Proof. By [Sh:420, §1] as $\sigma^+ < \mu^+, \sigma = \text{cf}(\sigma)$.

Continuation of the proof of Theorem 0.2. Let D_0, A_0, C_0 be as given by Lemma 1.1 with the club filter of μ^+, S^* (from clause (d) of Claim 2.1 above) here standing for D, A^* there, so $A_0 \subseteq S^*$.

Notation. $\varepsilon(\alpha) = \text{otp}(C_\alpha)$.

2.2 Claim. *Let $\chi > 2^\mu, <^*_\chi$ a well ordering of $\mathcal{H}(\chi)$. For any $x \in \mathcal{H}(\chi)$ we can find $\mathfrak{B} = \langle \mathfrak{B}_\alpha : \alpha < \lambda \rangle$ such that:*

- (a) $\mathfrak{B}_\alpha \prec (\mathcal{H}(\chi), \in, <^*_\chi)$;
- (b) $\bar{\lambda}, \mu, F, \langle \lambda_\varepsilon : \varepsilon < \sigma \rangle, \bar{C}, A_0, C_0, D_0$ belong to \mathfrak{B}_α ;
- (c) $\langle \mathfrak{B}_\beta : \beta < \alpha \rangle \in \mathfrak{B}_\alpha$ if $\alpha \notin S^*$;
- (d) $\|\mathfrak{B}_\beta\| = \mu_{\varepsilon(\beta)}$ and $[\mathfrak{B}_\beta]^{\leq \lambda_{\varepsilon(\beta)}} \subseteq \mathfrak{B}_\beta$ and $\mu_{\varepsilon(\beta)} + 1 \subseteq \mathfrak{B}_\beta$ (actually follows);
- (e) $\mathfrak{B}_\alpha = \bigcup \{\mathfrak{B}_\beta : \beta \in C_\alpha\}$ if $\alpha \in S^*$.

Proof. Straightforward.

2.3 *Observation.* 1) We have $\varepsilon(\alpha) < \varepsilon(\beta)$, and $\mathfrak{B}_\alpha \in \mathfrak{B}_\beta$ and $\mathfrak{B}_\alpha \prec \mathfrak{B}_\beta$ if $\alpha \in \mathcal{C}_\beta$.

2.4 **Claim.** *There is a set $A'_0 \subseteq A_0$ such that:*

- (α) $A'_0 \in D_0$ and $\alpha < \delta \in A'_0 \Rightarrow \sup(\mathfrak{B}_\alpha \cap \mu^+) < \delta$;
- (β) if $\xi \in C_0$ and $\delta \in A'_0$ and $s \in \bigcup\{[\delta \cap \mathfrak{B}_\alpha]^{\leq \lambda_{\varepsilon(\alpha)}} : \alpha \in \mathcal{C}_\delta\}$, then there is $\delta_1 \in A_0$ such that $\delta < \delta_1$ and
 - (a) $F(x, \delta) = F(x, \delta_1)$ for $x \in s$,
 - (b) $F(\delta, \delta_1) = \xi$.

Proof. Requirement (α) holds for all but a nonstationary set of $\delta \in A_0$. Requirement (β) is proved as in Lemma 1.6. □_{2.4}

Now fix $A'_0 \subseteq A_0$ as in Claim 2.4, and fix $\delta_1 \in A'_1$ and let $T = A'_0 \setminus (\delta_1 + 1)$. Recall $\delta_1 \in A'_0 \subseteq S^* = \{\delta : \text{otp}(\mathcal{C}_\delta) = \sigma, \delta = \sup(\mathcal{C}_\delta)\} \subseteq \{\delta < \mu^+ : \text{cf}(\delta) = \sigma\}$.

2.5 **Claim.** *There is a function $G_\varepsilon : T \times T \rightarrow C_0$ such that:*

- if $s \in [\delta \cap \mathfrak{B}_\alpha]^{\leq \lambda_\varepsilon}$ and $\varepsilon = \varepsilon(\alpha)$ and $\alpha \in \mathcal{C}_{\delta_1}$ and $\gamma < \kappa$ and $Z = \{z_\beta : \beta < \gamma\} < \subseteq T$, then there is $\{y_\beta : \beta < \gamma\} < \subseteq \delta \cap \mathfrak{B}_\alpha = \mu^+ \cap \mathfrak{B}_\alpha, y_0 > \sup(s)$ such that:
 - (a) $F(x, y_\beta) = F(x, z_\beta)$ for $x \in s, \beta < \delta$;
 - (b) $F(z_{\beta_1}, y_{\beta_2}) = G(y_{\beta_1}, y_{\beta_2})$;
 - (c) $F(z_{\beta_1}, z_{\beta_2}) = F(y_{\beta_1}, y_{\beta_2})$ for $\beta_1 < \beta_2 < \gamma$.

Proof. As in Claim 1.7.

2.6 **Claim.** *There exists a function $G : T \times T \rightarrow C_0$ such that if $s \in [T]^{< \kappa}$, then for arbitrarily large $\varepsilon < \sigma$ we have $G \upharpoonright (s \times s) = G_\varepsilon \upharpoonright (s \times s)$.*

Proof. Let D^* be a uniform κ -complete ultrafilter on σ and define G by $G(\alpha, \beta)$ is the unique $\xi \in C_0$ such that $\{\varepsilon < \sigma : G_\varepsilon(\alpha, \beta) = \xi\} \in D^*$. □_{2.6}

Continuation of the proof of Theorem 0.2. Now we apply Lemma 1.1 to the colouring \bar{G} where $\bar{G}\{x, y\} = \bar{G}(x, y) = (F(x, y), G(x, y))$ for $x < y$ in T and zero otherwise and to the filter D_0 and the set T . We get a normal filter D_1 and a set $A_1 \subseteq T \subseteq A'_0$ and a set of colours C_1 . As $A_1 \subseteq A_0$ necessarily $C_1 \subseteq C_0 \times C_0$.

2.7 **Claim.** *There is $A'_1 \subseteq A_1$ such that:*

- (α) $A_1 \setminus A'_1 = \emptyset \text{ mod } D_1$;
- (β) if $\delta \in A'_1, \alpha \in \mathcal{C}_\delta$ and $s \in [\delta \cap \mathfrak{B}_\alpha]^{\leq \lambda_{\varepsilon(\alpha)}}$ and $\bar{\xi} \in C_1$, then for some δ_* we have $\delta < \delta_* \in A_1$ and
 - (a) $\bar{G}(x, \delta) = \bar{G}(x, \delta_*)$ for every $x \in s$,
 - (b) $\bar{G}(\delta, \delta_*) = \bar{\xi}$.

Proof. As in the proof of Lemma 1.6. □_{2.7}

2.8 **Claim.** *There is a set $a \in [A'_1]^{< \kappa}$ such that:*

- for every decomposition of a as $\bigcup\{a_{\bar{\xi}} : \bar{\xi} \in C_1\}$ there is $\bar{\xi} \in C_1$ such that:
 - (α) for every $\bar{\varepsilon} \in C_1$ there is $b \subseteq a_{\bar{\xi}}$ of order type ζ such that $\bar{G} \upharpoonright [b]^2$ is constantly $\bar{\varepsilon}$;
 - (β) for every $\varepsilon \in C_0$ there is $b \subseteq a_{\bar{\xi}}$ of order type ζ such that $F \upharpoonright [b]^2$ is constantly ε .

Proof. The claim holds since A'_1 has this property and κ is strongly compact. If $A'_1 = \cup\{a_{\bar{\xi}} : \bar{\xi} \in C_1\}$ for some $\xi, a_{\bar{\xi}} \in D_1^+$ hence clause (α) holds by the choice of D_1, C_1 ; and clause (β) holds as $D_1^+ \subseteq D_0^+$ (as $D_0 \subseteq D_1$) and the choice of D_0, C_0 . □_{2.8}

Continuation of the proof of Theorem 0.2. Now choose $\delta_2 \in A'_1$ such that $\delta_2 > \text{sup}(a)$ and for $\bar{\xi} = (\xi_1, \xi_2) \in C_1 \subseteq \theta \times \theta$ define $a_{\bar{\xi}}$ as

$$\bar{a}_{\bar{\xi}} = \{x \in a : \bar{G}(x, \delta_2) = \bar{\xi}\}.$$

Clearly $\langle a_{\bar{\xi}} : \bar{\xi} \in C_1 \rangle$ is a decomposition of a and so there is $\bar{\xi} = (\xi_1, \xi_2) \in C_1$ as guaranteed by □ of Claim 2.8. In particular, there is $b \subseteq a_{\bar{\xi}}$ of order type ζ such that $F \upharpoonright [b]^2$ is constantly ξ_2 (note that $(\xi_1, \xi_2) \in C_1 \subseteq C_0 \times C_0$ so $\xi_2 \in C_0$). Now let $E = \{\varepsilon < \sigma : G_\varepsilon(\alpha, \delta_2) = G(\alpha, \delta_2) \text{ for every } \alpha \in b\}$. By the definition of G this is an unbounded subset of σ and clearly

(*) if $\varepsilon \in E$ and $\alpha \in b$, then $G_\varepsilon(\alpha, \delta_2) = G(\alpha, \delta_2) = (\xi_1, \xi_2)$.

For $\alpha < \lambda$ let $\Upsilon(\alpha) = \text{Min}\{\varepsilon \in E : \alpha < \lambda_\varepsilon\}$ and let $C_{\delta_1} = \{\gamma(\Upsilon) : \Upsilon < \sigma\}_<$. Now we try to choose by induction on $\alpha < \lambda$ a element x_α satisfying

- (*)₀ $x_\alpha < \delta_1$ and moreover $x_\alpha \in \delta_1 \cap \mathfrak{B}_{\gamma(\Upsilon(\alpha))}$, and $\beta < \alpha \Rightarrow x_\beta < x_\alpha$,
- (*)₁ $F(x_\beta, x_\alpha) = \xi_2$ for $\beta < \alpha$,
- (*)₂ $F(x_\alpha, \beta) = \xi_2$ for $\beta \in b \cup \{\delta_2\}$.

At step α , by Claim 2.7, that is, by the choice of A'_1 applying clause (β) there with $\{x_\beta : \beta < \alpha\} \cup b, \delta_2, \bar{\xi}$ here standing for $s, \delta, \bar{\xi}$ there, we can find δ_3 satisfying the requirement there on δ_1 , so

- (i) $\delta_2 < \delta_3 \in A_1$,
- (ii) $\bar{G}(x, \delta_3) = \bar{G}(x, \delta_2)$ for $x \in s \cup b$,
- (iii) $\bar{G}(\delta_2, \delta_3) = (\xi_1, \xi_2)$.

Now

- (*)₃ $F(x_\beta, \delta_3) = \xi_2$ for $\beta < \alpha$.
[Why? By (ii) we have $\bar{G}(x_\beta, \delta_3) = \bar{G}(x_\beta, \delta_2)$, hence $F(x_\beta, \delta_3) = F(x_\beta, \delta_2)$, but the latter by (*)₂ is equal to ξ_2 .]
- (*)₄ $G(\beta, \delta_3) = \xi_2$ for $\beta \in b$.
[Why? By (ii) and as $\beta \in b \Rightarrow \bar{G}(\beta, \delta_2) = (\xi_1, \xi_2) \Rightarrow G(\beta, \delta_2) = \xi_2$.]
- (*)₅ $G(\delta_2, \delta_3) = \xi_2$.
[Why? By clause (iii).]
- (*)₆ $\{x_\beta : \beta < \alpha\}$ is a subset of $\delta_1 \cap \mathfrak{B}_{\gamma(\Upsilon(\alpha))}$.

Let $\langle y_i : i < \zeta + 2 \rangle$ list $b \cup \{\delta_2, \delta_3\}$ increasing order.

Now we use the choice of $G_{\Upsilon(\alpha)}$ to choose an increasing sequence $\langle z_i : i < \zeta + 2 \rangle$ in $\delta_1 \cap \mathfrak{B}_{\gamma(\Upsilon(\alpha))}$, $z_0 > x_\beta$ for $\beta < \alpha$ such that $F(z_i, y_j) = G(y_i, y_j)$ for $i, j < \zeta + 2$ and $F(x_\beta, z_i) = F(x_\beta, y_i)$ for $i < \zeta + 2$. Let $x_\alpha = z_{\zeta+1}$ so $x_\alpha \in \delta_1 \cap \mathfrak{B}_{\gamma(\Upsilon(\alpha))}$ is $> x_\beta$ for $\beta < \alpha$.

Also x_α satisfies (*)₀ of the recursive definition. Now $\beta < \alpha \Rightarrow F(x_\beta, x_\alpha) = F(x_\beta, z_{\zeta+1}) = F(x_\beta, y_{\zeta+1}) = F(x_\beta, \delta_3)$ which is ξ_2 by (*)₃ above, so for our choice of x_α , (*)₁ holds. Next if $\beta \in b \cup \{\delta_2\}$, then $F(x_\alpha, x_\beta) = F(x_\beta, z_{\zeta+1}) = G(x_\beta, \delta_3)$ which is ξ_2 by (*)₄ or (*)₅. So x_α is as required. □_{0.2}

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INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY, JERUSALEM, ISRAEL – AND – DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY
E-mail address: `shelah@math.huji.ac.il`

²References of the form `math.XX/...` refer to the `xxx.lanl.gov` archiv.